



Dynamical Behavior of a Rigid Body with One Fixed Point (Gyroscope). Basic Concepts and Results. Open Problems: a Review

Svetoslav Nikolov^{1,2}, Nataliya Nedkova¹

¹Department of Mechanics, University of Transport
Geo Milev Str. No. 158, Sofia, 1574, Bulgaria, S.Nikolov@imbm.bas.bg.

²Department of Biomechanics, Institute of Mechanics-BAS
Acad. G Bonchev Str. Bl. 4, Sofia, 1113, Bulgaria.

Received March 07 2016; revised April 04 2016; accepted for publication April 06 2016.
Corresponding author: Svetoslav Nikolov, S.Nikolov@imbm.bas.bg

Abstract

The study of the dynamic behavior of a rigid body with one fixed point (gyroscope) has a long history. A number of famous mathematicians and mechanical engineers have devoted enormous time and effort to clarify the role of dynamic effects on its movement (behavior) – stable, periodic, quasi-periodic or chaotic. The main objectives of this review are: 1) to outline the characteristic features of the theory of dynamical systems and 2) to reveal the specific properties of the motion of a rigid body with one fixed point (gyroscope). This article consists of six sections. The first section addresses the main concepts of the theory of dynamical systems. Section two presents the main theoretical results (obtained so far) concerning the dynamic behavior of a solid with one fixed point (gyroscope). Section three examines the problem of gyroscopic stabilization. Section four deals with the non-linear (chaotic) dynamics of the gyroscope. Section five is a brief analysis of the gyroscope applications in engineering. The final section provides conclusions and generalizations on why the theory of dynamical systems should be used in the study of the movement of gyroscopic systems.

Keywords: Gyroscopic systems, theory of dynamical systems, dynamical behavior.

1. Dynamical Systems. Basic Concepts

Let us consider the following system of differential equations

$$\dot{x}_i \equiv \frac{dx_i}{dt} = f_i(x_1, x_2, \dots, x_n, t), \quad i = 1, 2, \dots, n, \quad (1.1)$$

where the functions f are continuous for $-\infty < x_j < +\infty$ and satisfy the Lipschitz condition in an arbitrary area D , i.e.

$$\sum_{i=1}^n |f_i(x_1, x_2, \dots, x_n, t) - f_i(y_1, y_2, \dots, y_n, t)| < const \sum_{i=1}^n |x_i - y_i|, \quad (1.2)$$

for arbitrary $(x_1, x_2, \dots, x_n) \in D$, $(y_1, y_2, \dots, y_n) \in D$. Such systems (of the (1.1) type), are called *dynamical*. The term “dynamical system” first appeared in mechanics, where it referred to a mechanical system with a finite number of degrees of freedom [8]. The states of such a system are usually characterized by the coordinate and the speed of its variation, and the law of motion determines the speed of change in the system’s state [6, 10, 75, 77, 89].

In the simplest case, the state of the dynamical system can be characterized by the quantities x_1, x_2, \dots, x_n ,

which can assume arbitrary real values in D . We should note that the different states correspond to two different sets (x_1, x_2, \dots, x_n) and (y_1, y_2, \dots, y_n) , and vice versa. The proximity of all x_i to y_i means proximity of the corresponding states. Then, the law of motion could be written in the form

$$\dot{x}_i = f_i(x_1, x_2, \dots, x_n). \quad (1.3)$$

The system (1.3) is called autonomous if, unlike system (1.1), which is called non-autonomous, it explicitly depends on time.

The division of the dynamical systems into autonomous and non-autonomous is in a certain sense conventional. And indeed, through the introduction of a new independent variable x_{n+1} , such that $x_{n+1} = t$, it is possible to substitute a non-autonomous system of n^{th} order (1.1) with its equivalent system of $(n+1)^{\text{st}}$ order.

If we consider the values x_1, x_2, \dots, x_n as coordinates of point x in the n -dimensional space, then we can represent geometrically the state of the dynamical system by means of this point x . Then, x is called a phase point, and the space – *phase space* of the dynamical system. The change in the state in time is displayed as a motion of the phase point along a certain curve, called *phase trajectory*.

All dynamical systems can be separated into two main groups: conservative and dissipative. In case the system is described by differential equations, (1.3) then it can be shown (according to the divergence theorem) that the variation of its phase volume dV during a time dt is

$$dV = dt \int \left(\frac{d\dot{x}_1}{dx_1} + \frac{d\dot{x}_2}{dx_2} + \dots + \frac{d\dot{x}_n}{dx_n} \right) dx_1 dx_2 \dots dx_n = dt \int \text{div} \dot{x} dx, \quad (1.4)$$

where \dot{x} is a vector with components $\dot{x}_1, \dot{x}_2, \dots, \dot{x}_n$. Hence, the sufficient condition for the conservation of the phase volume has the form

$$\text{div} \dot{x} = 0. \quad (1.5)$$

Similarly, the sufficient condition for the decrease of the phase volume is

$$\text{div} \dot{x} < 0. \quad (1.6)$$

In nature all systems are dissipative [62], yet if dissipation is very small (for a limited time) then such systems behave as conservative. Dissipative systems can also be sub-classified into passive and active ones.

Systems are called passive if they do not contain any energy sources. On the other hand, the systems are called active if they contain a constant (or alternating) energy source.

Active systems can be: amplifiers and generators. For amplifiers the evolution of the dynamical variables is completely determined by the external action arriving at the input. In case that action is not available for a long time, the signal at the output of the amplifier has to be unavailable. However, if the amplification factor is big enough, then a signal at the output of the amplifier actually exists in the case where no external action arrives at the input. If the amplification factor is big enough, fluctuations (both external and internal) that are considerably amplified manifest themselves in the output signal.

Generators are those active systems in which the motion is spontaneously excited, without any external action (force). Here we note that an amplifier is necessary but insufficient component of any generator [62]. An amplifier becomes a generator if it is included in a feedback loop, so that a part of the signal from the amplifier output is supplied to its input.

Often the term “self-oscillatory system” meaning “generator” is used. Historically, the term (and also definition) “self-oscillations” refers only to autonomous systems [4]. Later, this definition is nowadays extended and generalized to non-autonomous systems by Landa [62].

1.1 Asymptotic Behavior of the Dynamical Systems

Asymptotical, is called the behavior of a dynamical system, under an unlimited increase in the time argument. The study of the asymptotic behavior of the solutions of the dynamical systems is related to the definition of the property – stability [8, 15, 30, 48].

There exist different definitions of stability and different kinds of this property corresponding to them. One of the most often used is – stability by Lyapunov. The phase trajectory $x = x(x_0, t)$ is called stable by Lyapunov, if for each positive arbitrarily small number ε , there exists a positive number δ , such that for an arbitrary trajectory $y = y(y_0, t)$, with an initial condition y_0 , the condition $d(x_0, y_0) < \delta$ (where $d(\cdot)$ is the distance in the phase space) and the inequality $d(x, y) < \varepsilon$, is valid, i.e. for an arbitrary moment t , the phase point $y(t)$ must be situated at a distance, smaller than ε from another point of the phase trajectory $x = x(x_0, t)$, regardless of the time.

Equilibrium states can serve as attractors of the dynamical systems, i.e. all trajectories in their neighbourhood can be asymptotically attracted at $t \rightarrow \infty$. At $t \rightarrow -\infty$, the corresponding sets are called repellers [8, 48].

1.2 Structural Stability (Robustness) of the Dynamical Systems

The concept of structural stability (together with results, obtained at the application of this concept) for a two-dimensional dynamical system is formulated by Andronov and Pontryagin in 1937 in [5], which is afterwards generalised in [66] by Leontovich and Mayer. It should be noted here that Andronov and Pontryagin use the term – robustness as equivalent to structural stability. In the literature in English, the equivalent term is structural stability, but sometimes robustness is also used. The latter, however, is also used in biology in a wider (and mathematically indefinite) sense, as the capability of the biological system to preserve a sufficient number of its properties determining its identity, under the influence of a great number of external impacts. A similar understanding of robustness, could also be designated as a capability for structural and functional preservation of the biological system. This is, however, so wide that it is not formulated in mathematical language up until now. Still, it is evident that the terms stability and robustness (structural stability) are particular cases of the wider concept robustness, and their investigation is a necessary (albeit insufficient) stage on the way to the clarification of their meaning in the robustness context. For that reason, we should also adduce here the mathematical definition of the robustness property (i.e. structural stability) [30, 48]:

Definition: The dynamical system (1.1) is **robust**, if there exists an infinitely small positive number δ , such that the phase trajectories of the perturbed dynamical systems,

$$\dot{x}_i = X_i(x_1, x_2, \dots, x_n) + \xi_i(x_1, x_2, \dots, x_n), \tag{1.7}$$

where the perturbation $\xi_i(x_1, x_2, \dots, x_n)$ is sufficiently small, have equal topological structure.

Theorem. The system (1.1) is **robust** in the area $D \subset R^2$, if, and only if:

1. The equilibrium state is simple and is not of “centre” type;
2. The limit cycles are stable or unstable;
3. There are no separatrices, starting and ending in one and a same saddle, or separatrices, connecting two saddles.

It follows from the given definition and theorem that robust dynamical systems are these, whose phase space structure, is not changed at small perturbations (changes) in the right hand side of the differential equations describing them. The requirement for robustness of two-dimensional autonomous dynamical systems essentially simplifies the possible number of structures in the phase plane [24]. Each of these structures is determined by a finite number of specific phase trajectories: equilibrium states, separatrix curves of saddle equilibrium states and closed phase trajectories, called limit cycles.

The robustness of the dynamical system can be considered as stability of the structure of separate parts or of the whole phase space in relation to small perturbations. It is precisely in view of this circumstance that the term robustness is often replaced with its equivalent expression – structural stability [70, 84].

The use of the term robustness, for cases of multidimensional systems, is related with a number of difficulties. From the studies [2, 25, 79, 80, 95], it becomes clear that robust systems can be quite complex and, most importantly, there can be whole areas in the parametric space of the multidimensional systems, in which they are not robust, i.e. they are *structurally unstable*.

1.3 Structural Stability of Multidimensional Dynamical Systems

Let us consider the linear system

$$\dot{x} = Ax, \tag{1.8}$$

where $x \in R^n$, and A is a constant matrix. If we choose the coordinate system x in such a way that the matrix A has a Jordan form, then each zero solution will have one of the following types of behavior:

Table 1. Types of behavior of the phase trajectories.

Type 1	$\ x(t)\ \rightarrow \infty$ at $t \rightarrow \pm\infty$	saddle, saddle-focus
Type 2	$\ x(t)\ \rightarrow 0$ at $t \rightarrow +\infty$, $\ x(t)\ \rightarrow \infty$ at $t \rightarrow -\infty$	attractor
Type 3	$\ x(t)\ \rightarrow 0$ at $t \rightarrow -\infty$, $\ x(t)\ \rightarrow \infty$ at $t \rightarrow +\infty$	repeller
Type 4	$\ x(t)\ $ и $\ x(t)\ ^{-1}$ - are unlimited	centre

We have solutions of type 4, when A has eigenvalues (proper numbers) with a zero real part. If the eigenvalue

(proper number) is zero, then the whole straight line, generated by the corresponding proper vector, consists of equilibrium states. For a couple of purely imaginary eigenvalues (proper numbers), there exists an equilibrium state in the origin of the coordinate system, and a periodical solution. If the number of the imaginary couples is more than one, then there may exist **quasiperiodical solutions**.

If we have the first three types of solutions (see Table 1), the equilibrium state in the origin of the coordinate system is called hyperbolic.

The structurally stable multidimensional dynamical systems possess only hyperbolic equilibrium states and limit cycles. There can be only three types of robust equilibrium states in two-dimensional systems: **a node, a focus, and a saddle**. In the n -dimensional dynamical systems, there can be separated $(n + 1)$ types of specific points [8].

The **Morse-Smale** systems are the simplest among multidimensional dynamical systems. These are systems, satisfying axiom A, the strict condition for transversality, and have a finite non-fuzzy set. From the latter requirement, it follows that all basic sets of the Morse-Smale systems are trivial, i.e. each of them is a periodic trajectory. Only **the equilibrium states and the limit cycles** can be attractors in the Morse-Smale systems.

The class of multidimensional dynamical systems, characterised by the existence of an even number of hyperbolic periodical solutions and an odd number of fuzzy trajectories is another class of structurally stable systems, different from the Morse-Smale systems. Such class of systems is called **untrivially hyperbolic systems** [7, 85, 87, 89].

1.4 Bifurcation Theory

This section of the theory of dynamical systems, which is especially dedicated to learning how to perform qualitative change in the behavior of phase trajectories at a change of one or several parameters, is called bifurcation theory [77, 88, 89].

1.4.1. Bifurcations of Stationary Solutions in One-parameter Dynamical Systems

All actually occurring models have coefficients or external parameters, the numerical values of which are not exactly known. Their values are selected so that, with the help of models, to describe better the existing empirical data. In the theory of dynamical systems we are interested in the *qualitative changes in the dynamics* at a change in the parameter values.

The most simple and best studied is the bifurcation of the equilibrium state in the presence of one parameter [13, 59, 75]. More complex bifurcations are described (examined) in the theory of bifurcations: bifurcation of the equilibrium state in the presence of more than one parameter in the system; bifurcation of periodic motion; relationship between the equilibrium and / or limit cycles; bifurcations of the more complex basis sets. We may refer to them the study on how, at a change of the parameters of systems of Morse-Smale type, these systems are transformed into (or are born from) systems with non-trivial hiperbolicity that satisfy axiom A.

As fully complete can be considered the theory of bifurcation of the equilibrium state in a single-parameter dynamical systems [46, 86]

$$\dot{x} = f(x, \mu), \quad (1.9)$$

where μ is a parameter. We'll consider that for values of the parameter μ belonging to some interval (as, without losing the overall picture, this interval may be considered to be in the vicinity of zero), this system has an equilibrium state $x = 0$. For the study of its stability, we must consider the linearized system in variations and determine the signs of the real parts of the proper numbers of the parts on the right hand side of the matrix of this system. If all roots of the real parts are negative, the linearized system is asymptotically stable and the examined equilibrium state is steady.

If, among the real parts of the roots, there is only one that is positive, at some initial conditions the solutions of the system in variations grow exponentially, and this means that the equilibrium state is unstable.

If in some cases the real part of the roots is zero, the examining of variations in the linearized system is insufficient to address the issue of the stability of the stationary solution. Such conditions are *not grossly degenerate*. The degree (**co-dimensionality**) of degeneracy is equal to the unit at one zero root or two purely imaginary roots. Co-dimensionality two appears at two zero roots or at one zero root and two purely imaginary roots, etc. Small disturbances in the right parts of the dynamic system can ensure the absence of degeneracy. In one-parameter dynamical systems (at selected parameter values) degeneracies of co-dimensionality one appear irreversibly, and at two-parameter dynamical systems – of co-dimensionality two, etc. [1, 59].

These degenerate cases, which are persistent for a certain group of dynamical systems (with one, two or more parameters) at small disturbances, appear to be critical for this family (group). If the system (1.9) gradually evolves, i.e. the value of the parameter μ is amended, the state of equilibrium will vary smoothly: $x_{st} = x_{st}(\mu)$, as it remains stable, until at some important parameter value, for example $\mu = 0$, one of the roots of the characteristic equation becomes zero or a complex conjugate couple – purely imaginary. Upon further gradual variation of the bifurcation parameter (for example $\mu \rightarrow +\infty$), the equilibrium state loses its stability. Even if there appears another zero root, it can completely disappear. The theory of bifurcations must answer the question: how does the loss of stability occur –

hardly or softly? In other words, what the limit of stability is – *dangerous* or *safe*. In a hard loss of stability when crossing the border of stability in a close vicinity to this limit, the asymptotic behavior of the dynamical system completely changes. In the second case – soft loss of stability, the arisen changes are smaller than the less impaired limit of stability. The answer to this question lies in the following: if at a critical value of the parameter, the considered equilibrium state remains asymptotically stable, then the limit of stability is safe, and if the equilibrium state does not remain stable (that is, if it is unstable), then the limit of stability is dangerous.

Bifurcation theory considers *degenerate cases* with small co-dimensionality [43, 89]. One reason is that the number of different degenerate cases steeply increases with the increase in the co-dimensionality. For example, at a bifurcation of equilibrium state with co-dimensionality one, there exist two different critical cases. If the co-dimensionality is two, the critical cases become five. For co-dimensionality three, the cases are thirteen, as one of them precludes explicit (in the form of formulas) criteria for stability. In co-dimensionality four or more, the number of special cases cannot be calculated. Therefore, the study of such high degenerations in theory is inappropriate.

When a critical event is separate, bifurcation theory offers usually that the system be brought to a particular (usually normal) form, which is most convenient to study its stability. Right parts of dynamical systems, if they are brought in the form of polynomials, are in their normal form.

Usually, it is sufficient to store the polynomial to a certain low grade (4-5) degree. Thus, in general, for nondegenerate systems, in determining the stability of their equilibria, it is sufficient to take only linear members. If the co-dimensionality of degeneracy is one, in the critical cases have to be taken the square and the cubic members, etc. Therefore, the general appearance of the normal form is:

$$\dot{x} = \Lambda x + P(x), \tag{1.10}$$

where, in critical cases, some of the eigenvalues of the Jordan matrix Λ are zeros or purely imaginary, and $P(x)$ is a polynomial of degree not less than two:

$$P(x) = P_2(x) + \dots + P_n(x). \tag{1.11}$$

To be stable in the critical case, the stationary solution of the system (1.10) is sufficient to have fulfilled inequalities

$$P_2(x) < 0, \dots, P_n(x) < 0. \tag{1.12}$$

Each critical case is determined primarily by the degree of degeneracy of the linear part of the normal form, i.e. how many zero and purely imaginary eigenvalues the Jordan matrix Λ has. Further, the degeneracies in the nonlinear members may be supplemented. For example, finding some coefficients of the polynomials $P_k(x)$, $k = 2, \dots, n$.

1.4.2 Bifurcations of Periodic Motions

We consider bifurcations of periodic movements in one-parameter dynamical systems. To make the examination of these bifurcations more visual and closer to the bifurcations of the equilibrium state, we shall consider successive images on a plane, intersecting the periodic trajectory. The image of the periodic trajectory on the intersecting plane will be a fixed point, the stability of which is determined by the values of the multipliers of the linearized equation in variations relating to the periodic trajectory [13, 46, 59, 86].

A bifurcation occurs when the absolute magnitude of one or more multipliers is equal to one. In one-parameter family of dynamical systems, cases of generality occur when:

- 1) one of the multipliers is equal to plus one;
- 2) one of the multipliers is equal to negative one;
- or we have
- 3) a pair of complex conjugate multipliers having module plus one.

The case when at least one multiplier equals plus one is similar to the bifurcations of the equilibrium state. The consecutive image (Poincaré image) on the intersecting plane can be written as:

$$\begin{aligned} x_{i+1} &= x_i + R(x_i, \mu) + f(x_i, y_i, \mu)y_i, \\ y_{i+1} &= [A(\mu) + g(x_i, y_i, \mu)]y_i \end{aligned} \tag{1.13}$$

where x is the coordinate of the turning into a unit multiplier, y are the other $(n-1)$ phase coordinates,

$R(0,0) = 0$, $\frac{\partial R(0,0)}{\partial x} = 0$, $A(\mu)$ is a matrix whose proper numbers' are smaller than module of one for all values of the bifurcation parameter.

In one-parameter systems, the nonremovable situation is

$$l_2 = \frac{\partial^2 R(0,0)}{\partial x^2} > 0. \tag{1.14}$$

It leads to a dangerous border of the area of stability, as at $\mu = 0$ the periodic movement is described by the already existing at $\mu < 0$ saddle limit cycle. At the time of bifurcation a particular point of *saddle-node* type occurs. This is very similar (there is an analogy) to the bifurcation, where the stationary solution of a given dynamical system disappears. Certainly, there is also a significant difference. The closing of the unstable multitude of the complex limit cycle of a saddle-node type can only be of the following three types:

- *It can be self-limiting*, i.e. forming a non-rough homoclinic structure and then an even set of periodic movements must be present in its vicinity;
- *It can be a smooth or a non-smooth torus* (and a Klein bottle). On the torus are reeled phase trajectories that remain in the place of the missing cycle of a saddle-node type. If the invariant torus is smooth, the new attractor is fully defined. If a non-smooth torus is born after the time of bifurcation, the new limit set will contain an even multitude of saddle periodic trajectories, homoclinic trajectories, Poisson stable continuous trajectories, as well as stable limit cycles with small areas of attraction in phase space;
- *It can remain stable* in very small intervals of variation of the parameter. Moreover, for arbitrarily small values of change of the bifurcation parameter, sequences of bifurcations will be observed. Therefore, under certain values of the parameter, it is impossible to show the new attractor.

The case, where one of the multipliers of the limit cycle at the time of bifurcation is equal to minus one, has no analogue with the bifurcations of the equilibrium state. Here the Poincaré map can be written in the form:

$$\begin{aligned} x_{i+1} &= \rho(\mu)x_i + a_2(\mu)x_i^2 + a_3(\mu)x_i^3 + \dots + f(x_i, y_i, \mu)y_i, \\ y_{i+1} &= [A(\mu) + g(x_i, y_i, \mu)]y_i, \end{aligned} \quad (1.15)$$

where $\rho(0) = -1$, $|\rho(\mu)| < 1$ at $\mu < 0$ and $|\rho(\mu)| > 1$ at $\mu > 0$. In this case, the first Lyapunov quantity has the form:

$$L(0) = -2a_3(0) - 2(a_2(0))^2. \quad (1.16)$$

If $L(0) < 0$, the phase trajectories for which $y_i = 0$, at some point in time, form an invariant multitude – **Möbius strip**, in the middle of which lies the considered periodical movement $x_i = 0$. At $\mu > 0$, it produces a *stable periodic movement with a period close to two*. The image of this new movement on the intersecting plane will be a pair of stable fixed points in the point of origin that correspond to the limit cycle (that has lost its stability). The phase trajectories of the cycle with period two will consecutively pass through these two points. Therefore, this bifurcation is called "flip- light" and the loss of stability here is **soft**.

If $L(0) > 0$, then the already existing at $\mu < 0$ limit cycle of saddle type with a period close to twice the period of the periodic movement at issue, "bites" in the periodical movement at the moment of the bifurcation. Only the unstable periodic motion, with its unstable multitude – Möbius strip, remains after the bifurcation.

The third possible case of bifurcation of the periodic movement is similar to the bifurcations of the equilibrium state. This occurs when, at the time of the bifurcation, the two multipliers become equal to $\exp[i\phi(0)]$, where $\phi(0) \neq 0, \pi/2, 2\pi/3$ and π .

Then, if $g(0) < 0$, the limit of stability is safe, i.e. a two-dimensional stable torus is born at the point of bifurcation, which corresponds to the occurrence of an oscillation at a second frequency. The amplitude of the oscillation at the new frequency grows in proportion to the root of the post-critical parameter values.

When $g(0) > 0$, all proceeds analogously to the case of the hard birth of an auto(self)-oscillation, i.e. the previously unstable torus "comes from infinity" and, at the moment of the bifurcation, it merges with the considered cycle, transmitting to it its instability.

Despite all similarity between the considered bifurcations and these at steady-state of a focus type, there is also a number of *essential differences and complicated features*. For example, the invariant torus is a set, not a single trajectory. If we look at the image of the torus on the intersecting plane, it will be a closed curve. The full study of the dynamics of the phase trajectories on a torus is limited to examining *the mapping of a circle on itself*. The studying of the image of a circle was taken up by Poincaré, who introduced an important feature called **number of returns** $\rho(f)$ for

$$f : S^1 \rightarrow S^1, \quad (1.17)$$

where $f - C^2$ is a smooth image of the circle on itself. If the number of returns is *an irrational number* (this is the general case), then the image *will not have periodic orbits*. Such image is topologically conjugated and causes rotation of the circle at an angle $2\pi\rho(f)$. If the number of returns is *a rational number*, for example equal to m/n , where m and n are mutually prime numbers, then *there is a periodic orbit* with period n in the torus. The fixed points corresponding to this orbit on the intersecting plane are located so that the stable and the saddle points alternate. The number of such orbits is not determined by the number of returns. Furthermore, the smoothness of the saddle separatrices coming from the saddle to the node is terminal (to some extent) and is the smaller, the farther away from

the bifurcation value of the parameter it is considered. In amending the bifurcation parameter, the number of returns will be amended irreversibly and, also as in the one-parameter dynamical systems, cases of irrational and rational returns will occur sequentially. The distinction between these cases is virtually impossible. The reduction in smoothness at the increase in the post-critical values of the bifurcation parameter ultimately leads to the destruction of the two-dimensional torus. For the bifurcation where the two-dimensional torus is terminated, it is known that a three dimensional torus is born there, which may be considered to be an exceptional phenomenon. To understand best what is happening in the case of a rational number of returns, a two-parameter perspective should be used, i.e. it should be considered how the nature of the phase trajectories is amended at the simultaneous change of both m and n .

On the plane of these parameters, areas of existence of periodic trajectories can be separated, twisting at n turnovers about the primarily concerned cycle, making m cross turnovers. This phenomenon in the theory of dynamical systems is called **resonance**. More specifically, the resonance is called a random proportion of the form:

$$a_j = \sum_{i=1}^n K_i a_i, \quad (1.18)$$

where a_i are natural numbers. The number $q = \sum_{i=1}^n K_i$ is called order of the resonance. In particular, if $a_j = 0$, then we have resonances of random order, as it is in the present case. In resonance of a high order ($q > 4$), each area of existence of periodic solutions goes out of the unit circle of the complex plane in the points $\exp(i2\pi^{m/n})$. The distance l , reached by these points is

$$l^{\frac{q-2}{2}}, \quad (1.19)$$

i.e. it is very small. Therefore, in general, the loss of stability of a limit cycle for a one-parameter dynamical system is the birth and disappearance of an infinite number of cycles with a large period.

An exception, are the cases of *strong resonance* when $q \leq 4$, $\varphi(0) = 0, \pi/2, 2\pi/3$ and π .

1.4.3 Global Bifurcations

We shall examine more sophisticated bifurcations for the understanding of which it is insufficient to consider small local surroundings near the equilibrium states or the limit cycles [1, 23, 43, 46, 59, 81, 86, 88].

These bifurcations are **global**. They lead to qualitative change in the stable and the unstable manifolds of the states of equilibrium and / or the limit cycles. One of these situations is the occurrence of a loop of the separatrices of one and the same saddle (steady state). It is here that we will deal in more detail with the bifurcations emerging in this situation.

Let $\lambda_1, \dots, \lambda_n$ are the roots of the characteristic equation in the equilibrium state of saddle type and $\text{Re } \lambda_i < 0$ for $i = 1, \dots, (n-1)$, and $\lambda_n > 0$. The saddle values are

$$\sigma_i = \text{Re } \lambda_i + \lambda_n < 0. \quad (1.20)$$

Let at some (bifurcation) value of the parameter, one of the branches emerging from the saddle unstable separatrices goes back to the saddle and forms a *loop (closed loop)*. After the bifurcation, one or more periodical trajectories of stable or saddle type occur in place of the loop. These periodic trajectories ride away from the trap of the separatrices. If the parameter is modified in the opposite direction, they merge into the separatrix loop. The case where the closed loop is composed of separatrices of several saddles can cause the birth of periodic movements of the type considered just now.

Another bifurcation of this kind can occur when an equilibrium state of saddle-node type appears in the site of the stable limit cycle existing until the bifurcation. For post-critical parameter values, this loop, which until the bifurcation is stable, breaks with the formation of two equilibrium states – a saddle and a node in the vicinity of the saddle-node. They are bound in a closed loop by the *unstable separatrices* of the saddle. This loop remains stable because the bifurcation border is *safe*, i.e. the distance between the saddle and the node grows smoothly in proportion to the post-critical values of the bifurcation parameter.

1.4.4 Sequence of Bifurcations; the Occurrence of Chaos in the Dynamical Systems

The bifurcations in the dynamical systems can be divided into three types [38, 46, 59]:

- 1) bifurcations not originating from the Morse-Smale class of systems;
- 2) bifurcations in a class of systems with non-trivial hyperbolicity;
- 3) bifurcations associated with the transition from Morse-Smale to systems with non-trivial hyperbolicity.

The first two bifurcations are well studied, resulting in the proof that systems of Morse-Smale type and systems with non-trivial hyperbolicity (having stochastic (chaotic) behavior) can form open areas in the parameter space in which areas the systems are *structurally stable*. **Bifurcations of a new type** that lead to transition from one class to

another class of systems are observed on the boundaries of those areas. Along with this, there may be areas in which there are no structurally stable systems in the parameter space of the systems. In these areas, bifurcations occur with the slightest modification of the parameters (close to zero). Among these bifurcations, there may exist such that lead to transition from Morse-Smale type to chaotic and vice versa.

Bifurcations leading to transition from Morse-Smale systems to systems with non-trivial hyperbolicity are of particular interest because they can explain the mechanisms of emergence of chaotic properties in deterministic dynamical systems, in particular the onset of turbulence in a fluid. The motion on an n dimensional torus (if n is big enough) seems as turbulent.

From the theory of non-linear asymptotic methods in mechanics, it is known that multi-periodical movements forming *stable toroidal manifolds* are possible.

As stated, in the bifurcation of periodic movement, an invariant torus can be born possessing extreme smoothness and, in the general case, at big enough post-critical parameter values, it is destroyed. Ruelle and Takens [81] demonstrated that usually, instead of the birth of a three dimensional torus (in the beginning their theorem has been demonstrated for torus of dimension not less than four), the occurrence of a strange attractor following sequence of bifurcations in systems of the Morse-Smale type, passing into systems with non-trivial hyperbolicity with unlimited number of frequencies, should be anticipated.

Modern bifurcation theory allows addressing fully the issue of determination of the main types of boundaries separating the areas with structurally stable systems of Morse-Smale from the areas of structurally stable systems with non-trivial hyperbolicity in parameter space.

The cases related to the occurrence (or disappearance) of homoclinical structures are studied very well. The time of occurrence of the homoclinical structure in Morse-Smale systems precedes the appearance of non-rough, i.e. having purely imaginary roots equilibrium state or periodic movement with multipliers on the unit circle. Usually, only one equilibrium state of saddle type enters into the homoclinical contour, and the transversality of the intersection of the stable and unstable manifolds of the periodic movements (existing in the vicinity of the homoclinical loop) is broken only in one direction.

The limit at issue may be **attainable** and **unattainable**. We shall explain what this means. Let the point z_0 of parameter space of a dynamic system and is part of a hard (with co-dimensionality one) limit H^1 , i.e. $z_0 \in H^1$. This boundary in parameter space divides Morse-Smale systems from stochastic systems. This means that a sphere U with the center z_0 can be found, which divides the boundary H^1 into two parts U^+ and U^- . The bifurcation boundary H^1 is called attainable in point $z_0 \in H^1$ from the areas U^\pm , if there exists a small vicinity $U_0 \subset U$, $z_0 \in U_0$ such that for an arbitrary curve in the parameter space of the parameters μ, z_μ , for $0 \leq \mu \leq \mu_0$, then $z_{\mu=0} = z_0$ and $z_\mu \in U_0 \cap U^\pm$ at $\mu = 0$, the systems z_0 and z_μ are topologically equivalent for each μ_1 and μ_2 in this interval for μ , i.e. $z_{\mu_1} \rightarrow z_{\mu_2}$ at $\mu_1 \rightarrow \mu_2$. The bifurcation boundary is called **unattainable** if topologically non-equivalent systems can be found on a randomly chosen curve. In other words, if the limit is attainable, a transition from the field of the structurally stable systems of Morse-Smale to the areas with structurally stable systems with non-trivial hyperbolicity is carried out. However, if the limit is unattainable, then at an infinitesimal variation of the parameters, an infinite multitude of bifurcations and dynamical systems with stochastic behavior that are structurally unstable, emerge.

The bifurcation surfaces associated with the appearance of homoclinical structures are attainable [23, 86].

A typical sequence of bifurcations associated with attainable bifurcation boundary and transition from systems of Morse-Smale to stochastic systems is the following: at value μ_1 of the bifurcation parameter, a complex special point of saddle-node type occurs from the hardened phase trajectory, which breaks up in two rough specific points – a stable and a saddle one. Then at $\mu_2 > \mu_1$, a stable limit cycle is born from the stable state of equilibrium with a doubled period. Further, for $\mu_3 > \mu_2$, the limit cycle loses stability and a limit cycle with doubled period is born in its vicinity. In a further change in the parameter, sequences of bifurcations of doubling of the period are observed, as at some μ_{cr} there will exist an even number of saddle periodic movements with periods $T, 2T, 2^2T, \dots, 2^n T$ for random positive integers n .

Another possible scenario for transition to systems with non-trivial hyperbolicity is connected with the destruction of an invariant two-dimensional torus. In examining the image of the torus on an intersecting plane (in that image has the appearance of a circle), the sequence of bifurcations can sometimes be reduced to the already observed one, i.e., a fixed point first with period two, then with period 2^2 , etc. appears in the image of a circle on itself.

As a conclusion of this part, we shall emphasize on the principal difference between stability and structural stability:

In the *stability* case, it is investigated whether at small perturbations (which are expressed with variations of the initial conditions of the differential equations) the phase trajectory remains in a limited area (not necessarily small) in the region of equilibrium states, limit cycles or other trajectories, limited in space, including chaotic trajectories.

In the *structural stability* case, we are interested whether at small perturbations in the right hand sides of the differential equations, the mutual phase space disposition of the specific trajectories (equilibrium states, limit cycles, separatrices, etc.), related with the change in their character (stable or unstable), as well as with their disappearance, or with the appearance of new specific trajectories, is altered.

For the time being, in greatest detail and precision is studied the structural stability of two-dimensional dynamical systems. In the literature, the corresponding task is known as *investigation of the qualitative behavior of a two-dimensional dynamical system, depending on the variation of one parameter*. There also exist results, concerning the problem of the *dependence of the behavior upon two parameters*, which generates expectations for a possible advance in the application of the notions of stability and robustness in a rigid body with one point fixed (gyroscope).

2. Dynamical Behavior of Rigid Body with One Fixed Point (Gyroscope). Historical Chronology and Basic Theoretical Results

It is well-known that one of the simplest and most important types of dynamical systems is a rigid body. In the case that a rigid body has one fixed point (and the particles in the rigid body are not all collinear), three coordinates are required to specify the configuration (orientation) of the body. It follows that if such a body is acted on by a known set of forces, three independent equations of motion are required to determine the motion. Hence, a sufficient set of equations is provided by the vector equation

$$E(0) = \dot{\Psi}(0), \quad (2.1)$$

where $E(0)$ is the net external torque with respect to the fixed point O of rigid body rotation, $\Psi(0)$ is the angular momentum with respect to the point O , and the dot represents the time rate of change as noted by an observer fixed in an inertial frame [31, 33]. If a rigid body has no fixed point (and the particles are not collinear), six coordinates are required to specify the configuration of the body: three to specify the position of a point in the body, and three to specify the orientation of the body.

If the particles in a rigid body are collinear, one less coordinate is required to specify the configuration than is required for a rigid body in which the particles are not all collinear. Thus, in this special case, if no point is fixed, five coordinates are required.

2.1 Forces

In mechanics and control theory the problem of force influences over the dynamics of stationary (autonomous) and unstationary systems is important. We will employ potential forces F_p , dissipative forces F_d and gyroscopic forces F_g . The first two forces take care of convergence to the target point and the gyroscopic force handles the obstacle avoidance [26]. Mathematically, the three forces F_p, F_d and F_g can be written in the following form:

$$F_p = -\nabla U(q), \quad F_d = -D(q, \dot{q})\dot{q}, \quad F_g = S(q, \dot{q})\dot{q}, \quad (2.2)$$

where U is a (potential) function, the matrix D is symmetric and positive-definite, the matrix S is skew-symmetric, q is the position and \dot{q} is a velocity vector.

Gyroscopic forces have two useful perspectives in the dynamics of mechanical systems: (i) they create coupling between different degrees of freedom, just like mechanical couplings; (ii) they rotate the velocity vector just like magnetic field acting on a charged particle. The first interpretation regards the matrix S in (2.2) as an interconnection matrix and the second interpretation considers S as an infinitesimal rotation. Note that gyroscopic forces are very useful in the stabilization of dynamical systems, because they are unpotential forces with zero power.

2.2 Lagrange Formalism for the Description of the State of a Dynamical (Mechanical) System

It is well-known that formalism of classical mechanics underlies a number of powerful mathematical methods [32]. In the frameworks of classical and quantum theories, the Hamiltonian and Lagrangian formulations can be used as the first one gives a somewhat clearer geometric picture of classical dynamics.

Around year 1790, Lagrange introduced generalized coordinates (q^1, \dots, q^n) , their velocities $(\dot{q}^1, \dots, \dot{q}^n)$, and the so-called Lagrangian $L(q^i, \dot{q}^i)$ (which is often the kinetic energy minus the potential energy) to describe the state of a mechanical system. The proposed equations of motion have the form

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = 0, \quad (2.3)$$

and now they are called Euler-Lagrange equations. The Lagrangian method was introduced as a powerful alternative to the Newtonian method for deriving equations of motion for complex dynamical system [16, 21, 31]. Forty years later, Hamilton realized how to obtain these equations from a variational principle

$$\delta \int_a^b L(q^i(t), \dot{q}^i(t)) dt = 0, \tag{2.4}$$

called the principle of critical action, in which the variation is over all curves with two fixed endpoints and with a fixed time interval $[a, b]$.

It is well-known that the collection of pairs (q, \dot{q}) may be thought as of elements of the tangent bundle TQ of configuration space Q . Some authors call TQ as “the velocity phase space” [21, 70]. For mechanical systems like the rigid body, coupled structures etc., it is essential that Q be taken to be a manifold and not just Euclidian space.

The Legendre transform (that is, change the variables to the cotangent bundle T^*Q) is

$$p_i = \frac{\partial L}{\partial \dot{q}^i}. \tag{2.5}$$

Hence, the Hamiltonian can be defined by

$$H(q^i, p_i) = p_i \dot{q}^i - L(q^i, \dot{q}^i). \tag{2.6}$$

Then the Euler-Lagrange equations (2.3) become Hamilton’s equations

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}^i = -\frac{\partial H}{\partial q_i}, \quad (i = 1, \dots, n). \tag{2.7}$$

It is important to note that the symmetry in these equations leads to a rich geometric structure.

The Hamiltonian function $H(q, p)$ defines the system and, in the absence of constraining forces and time dependence, is the total energy of the system.

Recall that the set of all possible spatial positions of bodies in the system is their configuration space Q . For example, Q for a three dimensional rigid body moving freely in space is $SE(3)$, the six dimensional group of Euclidean (rigid) transformations of three-dimensional space, that is, all possible rotations and translations. If translations are ignored and only rotations are considered, then the configuration space is $SO(3)$.

The description of the body’s orientation is space in terms of three angles between axes that are either fixed in space or are attached to symmetry planes of the body’s motion have been formulated by Euler [21, 37, 99]. The three Euler angles, ψ, φ and θ are generalized coordinates for the problem and form a coordinate chart for $SO(3)$. It is important to note here that Euler angles are not observed, and they are imaginary construction which can be in another form. For example, we can use the so-called Krilov-Bulgakov angles, where the sequence of rotations is: 1-2-3 (i.e. the first rotation is around first axes; second rotation is around second axes; etc.). For Euler’s angles we have: 3-1-3. Generally, the sequence of rotations can be:

Class I	1-2-3,	3-1-2,	2-3-1;
Class II	1-3-2,	2-1-3,	3-2-1;
Class III	1-2-1,	2-3-2,	3-1-3;
Class IV	1-3-1,	3-2-3,	2-1-2.

According to [21], we present the element $A \in SO(3)$ giving the configuration of the body as a map of reference configuration $B \subset R^3$ to the current configuration $A(B)$. For a rigid body in motion, the matrix A becomes time dependent and the velocity of a point of the body is $\dot{x} = \dot{A}X = \dot{A}A^{-1}x$, where $X \in B$ is the label point. Since A is an orthogonal matrix, we can write

$$\dot{x} = \dot{A}A^{-1}x = \omega \times x, \tag{2.8}$$

which defines the spatial angular velocity vector ω . The corresponding body angular velocity is defined by

$$\Omega = A^{-1}\omega, \tag{2.9}$$

where Ω is the angular velocity.

The kinetic energy has the expression

$$T = \frac{1}{2} \int_B \rho(X) \|\dot{A}X\|^2 d^3X, \tag{2.10}$$

where ρ is the mass density. Since

$$\|\dot{A}X\| = \|\omega \times x\| = \|A^{-1}(\omega \times x)\| = \|\Omega \times X\|, \tag{2.11}$$

the kinetic energy is a quadratic function of Ω . Writing

$$T = \frac{1}{2} \Omega^T \mathbf{I} \Omega, \tag{2.12}$$

defines the time independent moment of inertia tensor \mathbf{I} , which, if the body does not degenerate to a line, is a positive definite 3×3 matrix, or better, a quadratic form which can be diagonalized. Its eigenvalues are called the principal moments in inertia which are distinct and uniquely define the principal axes. In this basis $\mathbf{I} = \text{diag}(I_1, I_2, I_3)$.

The body angular momentum is defined as

$$\Pi = \mathbf{I} \Omega, \quad \mathbf{I} = \begin{vmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{vmatrix}, \tag{2.13}$$

so that in principal axes

$$\Pi = (\Pi_1, \Pi_2, \Pi_3) = (I_1 \Omega_1, I_2 \Omega_2, I_3 \Omega_3). \tag{2.14}$$

Assuming that no external moments act on the body, the spatial angular momentum vector $\pi = A \Pi$ is considered in time.

The Euler equations of motion for rigid body dynamics are given by

$$\dot{\Pi} = \Pi \times \Omega, \tag{2.15}$$

where $\Pi = \mathbf{I} \Omega$ is the body angular momentum and Ω is the body angular velocity.

In the presence of external forces F_i , the equations (2.3) are

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = F_i, \quad (i = 1, \dots, n). \tag{2.16}$$

Note that if forces F_i are derivable from a potential U in the sense that forces can be incorporated into the Lagrangian by adding $-U$ to the Lagrangian.

2.3 Principal Axes of Inertia

In general, the center of mass inertia tensor \mathbf{I} can be made into a diagonal tensor with components given by the eigenvalues I_1, I_2 and I_3 of inertia tensor [21]. These components (known as principal moments of inertia) are the three roots of cubic polynomial

$$I^3 - \text{Tr}(\mathbf{I})I^2 + \text{Ad}(\mathbf{I})I - \text{Det}(\mathbf{I}) = 0, \tag{2.17}$$

obtained from $\text{Det}(\mathbf{I} - I \underline{\mathbf{1}}) = 0$, with coefficients

$$\begin{aligned} \text{Tr}(\mathbf{I}) &= I_{11} + I_{22} + I_{33}, \\ \text{Ad}(\mathbf{I}) &= ad_{11} + ad_{22} + ad_{33}, \\ \text{Det}(\mathbf{I}) &= I_{11}ad_{11} - I_{12}ad_{12} + I_{13}ad_{13}, \end{aligned} \tag{2.18}$$

where ad_{ij} is the determinant of the two-by-two matrix obtained from \mathbf{I} by removing the i^{th} -row and j^{th} -column from the inertia matrix \mathbf{I} , and $\underline{\mathbf{1}}$ is the unit tensor (i.e., in Cartesian coordinates, $\underline{\mathbf{1}} = \hat{x}_1 \hat{x}_1 + \hat{x}_2 \hat{x}_2 + \hat{x}_3 \hat{x}_3$). Note that \hat{x}_1, \hat{x}_2 and \hat{x}_3 are the Cartesian center of mass unit vectors. By denoting as \mathbf{I}' the diagonal inertia tensor calculated in the body frame of reference (along the principal axes), we find

$$\mathbf{I}' = \mathbf{R} \mathbf{I} \mathbf{R}^T = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}, \tag{2.19}$$

where \mathbf{R}^T is the transpose of the rotation matrix \mathbf{R} , i.e. $(\mathbf{R}^T)_{ij} = R_{ji}$. In the body frame, the inertia tensor is, therefore, expressed in dyadic form as

$$\mathbf{I}' = I_1 \hat{e}_1 \hat{e}_1 + I_2 \hat{e}_2 \hat{e}_2 + I_3 \hat{e}_3 \hat{e}_3, \tag{2.20}$$

and the rotational kinetic energy is

$$K' = \frac{1}{2} \omega \cdot I' \cdot \omega = \frac{1}{2} (I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2) \tag{2.21}$$

Here $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ are unit vectors which form a new frame of reference known as the body frame.

A rigid body can be classified into one of three different categories [21]. First, a rigid body can be said to be a *spherical top* if its three principal moments of inertia are equal ($I_1 = I_2 = I_3$) i.e., the three roots of the cubic polynomial (2.17) are triply degenerate. Next, a rigid body can be said to be a *symmetric top* if two of its principal moments of inertia are equal ($I_1 = I_2 \neq I_3$), i.e. I_3 is a single root and $I_1 = I_2$ are doubly-degenerate roots of the cubic polynomial (2.17). Lastly, when the three roots ($I_1 \neq I_2 \neq I_3$) are all single roots of the cubic polynomial (2.17), a rigid body is said to be an *asymmetric top*.

The rate of change of rotational kinetic energy is expressed as

$$\frac{dK'_{rot}}{dt} = \omega \cdot I' \cdot \dot{\omega} \tag{2.22}$$

According to [16, 18, 21, 63, 99] the rigid body relative equilibria correspond to steady rotational motions about their principal axes. Using the so-called energy – Casimir method, it is easy to show that the uniform rotation of a free rigid body about the longest and shortest principal axes is stable motion, while motion about intermediate axis is unstable.

2.4 Gyroscope. General States

Definition: Dynamically symmetric body with a fixed point and big enough own kinetic moment is called a **gyroscope**.

It is known that the movement of a rigid body with a fixed point (gyro) in uniform force field is described by equations of the Euler-Poisson type having the form:

$$\begin{cases} I\dot{\omega} + \omega \times I\omega = \sigma \times \gamma, \\ \dot{\gamma} = \gamma \times \omega, \end{cases} \tag{2.23}$$

where $\omega = (\omega_1, \omega_2, \omega_3)$ is the vector angular velocity, $r = (r_1, r_2, r_3)$ is the radius vector of the center of mass, γ is the single orth of the vertical axes of the coordinate system associated invariably with the solid and starting from the fixed point O , $I = \text{diag}(I_1, I_2, I_3)$ is the tensor of inertia with reference to O and the same coordinate axes, and $\sigma = mg$ is the force of gravity (Fig. 1).

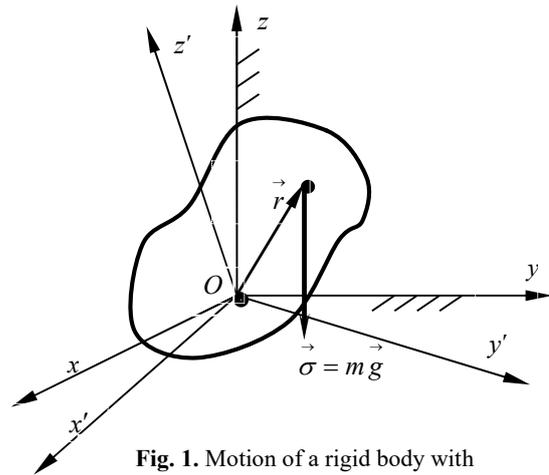


Fig. 1. Motion of a rigid body with one fixed point in a homogeneous force field.

Through the kinematic moment $M = I\omega$, equations (2.23) can be represented in the Hamiltonian form, i.e.

$$\dot{M}_i = \{M_i, H\}, \quad \dot{\gamma}_i = \{\gamma_i, H\}, \quad i = 1, 2, 3, \tag{2.24}$$

and through the Poisson parentheses they have the form:

$$\{M_i, M_j\} = -\varepsilon_{ijk} M_k, \quad \{M_i, \gamma_j\} = -\varepsilon_{ijk} \gamma_k, \quad \{\gamma_i, \gamma_j\} = 0. \tag{2.25}$$

H is the Hamiltonian representing the total energy of the body, i.e.

$$H = \frac{1}{2} (AM, M) - \sigma(r, \gamma), \tag{2.26}$$

where $A = I^{-1} = \text{diag}(a_1, a_2, a_3)$.

The Lie-Poisson bracket (2.25) is degenerate, as it has two Casimir functions, which have the form:

$$F_1 = (M, \gamma), \quad F_2 = \gamma^2. \tag{2.27}$$

In vector form equations (2.24) can be written as:

$$\begin{cases} \dot{M} = M \times \frac{\partial H}{\partial M} + \gamma \times \frac{\partial H}{\partial \gamma}, \\ \dot{\gamma} = \gamma \times \frac{\partial H}{\partial M}. \end{cases} \quad (2.28)$$

Functions F_1 and F_2 are integrals of equation (2.28) at a random Hamiltonian H , as for the equations of Euler-Poisson (2.23) they have a real physical and geometrical meaning. The integral F_1 is a projection of the kinetic moment on the fixed vertical axis and it is called the integral of the area in the dynamics of the rigid body. It is connected with the symmetry of rotation about a fixed vertical axis. The origin of the integral $F_2 = const.$ is purely geometric. It represents the square of the module of the single orth of the vertical. In actual movements, the value of this integral is one, i.e. $F_2 = \gamma^2 = 1$.

There is an analogy between the Euler-Poisson equations and the equations describing the equilibrium of the infinitely thin, flexible cylinder (rod) [18, 63]. Kirchhoff first noticed this analogy, which in a sense allows interpreting the dynamics of the rigid body in space. More precisely, the study of the evolution of the system over time is replaced with an analysis of the shape of the elastic rod.

The Liouville integrability of the system (2.23) (and the system (2.28) and the Hamiltonian (2.26)) requires the existence of an additional integral. Many famous mathematicians and mechanical engineers (especially in the 19th century) have lost a lot of effort and time in search of that integral. *After all, until today, its general form has not been found.*

It turns out that there are real dynamical effects that prevent the integrability of these equations in the general case. There are several cases of integrability. The cases of Euler, Lagrange and Kowalevski are general cases of integrability [54], i.e. the additional first integral exists under restrictions set for the parameters (the matrix A and the vector r) for all initial conditions. The case of Goryachev-Chaplygin is a special case of integrability, as an equality to zero of the constant area, i.e. $F_1 = 0$ is necessary for the existence of the additional integral. The Hess case is connected with the existence of a linear invariant relation of M as regards $F = 0$, for which $\dot{F} = \lambda F = 0$. There are also a few particular solutions that are periodic and asymptotic movements.

In the general nonintegrable cases the body can accomplish periodic, quasi-periodic and complex chaotic motions. *The study of these movements in phase space by using the methods of nonlinear chaotic dynamics is one of the main tasks of modern nonlinear mechanics* [44].

3. Gyroscopic Stabilization

The study of the stability of the movement of mechanical systems under the action of various forces has a long history. Until now, significant results were obtained for autonomous systems; in [29, 71] the theorems of Thomson (Kelvin) -Tait-Chetayev are used. Practically, the stability of non-autonomous systems influenced by gyroscopic and dissipative forces is not studied well [73, 74].

Linear conservative gyroscopic systems have the form:

$$M \ddot{x} + G \dot{x} + Kx = 0, \quad \text{or} \quad \left(\ddot{x} + \frac{G}{M} \dot{x} + \frac{K}{M} x = 0 \right), \quad (3.1)$$

where the vector x represents the generalized coordinates, M is the mass matrix, G describes the gyroscopic forces and K potential forces. Also, M, G and K are real $n \times n$ matrices with $M^T = M > 0$ (positive definite), $G^T = -G$ (a skew-symmetric matrix) and $K^T = K$ (symmetric matrix). The linear equation (3.1) is typical for small oscillations of a dynamical system in the region of an equilibrium point ($x = 0$). The results on the problem of the stability of equilibrium can be found in [22].

It is well-known that gyroscopic forces can stabilize the unstable conservative systems, while they cannot destabilize a stable conservative system. In [29, 91], is shown that an unstable conservative system

$$M \ddot{x} + Kx = 0, \quad K \geq 0, \quad (3.2)$$

can be stabilized by gyroscopic forces if and only if the number of unstable degrees of freedom is even. Hence, when $K < 0$, then the dimension n must be even. Later, for this case in [60], Lakhadanov obtained that suitable stabilizing matrices are $G = g_0 G_0$, where $\det G_0 \neq 0$ and g_0 is a sufficiently large number. In the general case (with an indefinite K), an unstable conservative system can be decoupled by choosing modal coordinates, as we only need to stabilize the subsystem which has a negative definite stiffness matrix. In the generic case, all the eigenvalues of the matrix polynomial $M\lambda^2 + gG\lambda - K$ are purely imaginary and simple. In the critical case, at $g = g_0$, there exists a pair of double purely imaginary eigenvalues $\lambda = \pm i\rho_0$ with a Jordan chain of length 2, while the remaining eigenvalues are simple and lie on the imaginary axis [83]. For $n = 2$ in (3.1), the exact critical

(bifurcation) value of the gyroscopic parameter is

$$g_0 = g_{bif} = \sqrt{\text{Tr}(K) + \sqrt{\det(K)}} = \sqrt{\lambda_1} + \sqrt{\lambda_2}, \quad (3.3)$$

where $\lambda_1(K)$ and $\lambda_2(K)$ are eigenvalues of the matrix $K > 0$, when $\det G = 1$. Here, it can be noted that the determination of g_0 for $n > 2$ is much more complicated and modern geometric methods must be used [83]. For $n > 2$ it is an open question in which cases the Lyapunov's stability conditions can be skipped [53].

In [52, 56-58], the influence of dissipative and nonconservative positional forces on gyroscopic stabilization is considered. It is obtained that the stability is extremely sensitive to the choice of a perturbation, while the balance of forces leading to asymptotic stability is not obvious.

Gyroscopic stabilization in the case $K \leq 0$ was dealt in [41].

In [83], the mechanisms of transition between divergence, flutter and stability for several conservative gyroscopic systems with parameters were obtained, as in section 3 (of [83]) the authors proved a theorem, which states a sufficient condition for gyroscopic stabilization for conservative systems with an even dimension and with $K > 0$.

For reduction to the non-autonomous case of (3.1) we make the replacement of variables, i.e.

$$x = A(t)z, \quad A = \exp(-\Gamma t/2), \quad (3.4)$$

where $\Gamma = \frac{G}{M}$. Thus, in the new coordinates z , Eq. (3.1) takes the form

$$\ddot{z} + Q(t)z = 0, \quad Q = A^{-1}(P - \Gamma^2/4)A, \quad (3.5)$$

where $P = \frac{K}{M}$. It is seen that after this replacement, Lagrange's function

$$L = \frac{(\dot{z}, \dot{z})}{2} - \frac{(Qz, z)}{2} \quad (3.6)$$

depends explicitly on time t . Note that after making the inverse change (3.4) the Lagrangian (3.6) becomes the function

$$L = \frac{(\dot{x}, \dot{x})}{2} + \frac{(\dot{x}, \Gamma x)}{2} - \frac{(Px, x)}{2}, \quad (3.7)$$

i.e. Lagrange's equation with this Lagrangian is obviously identical with (3.1). Since the matrix Γ is skew-symmetric, the matrices A and A^{-1} are orthogonal. Thus, the problems of stability of the trivial solution of (3.1) and (3.5) are equivalent [55]. Assuming that the matrix $Q(t)$ is periodic with respect to time, in [17, 50] the classical Thomson (Lord Kelvin) result on the impossibility of gyroscopic stabilization of the equilibrium $x = 0$ for an odd degree of instability is derived from Hill's formula, which relates the multiplier of the zeroth periodic solution of system (3.5) with its Morse index. In this connection, in [55] one useful extended Hill's formula to the more general case (when the elements of the matrix Q depend conditionally-periodically on time) is derived.

When applied separately, dissipative and nonconservative positional forces destroy gyroscopic stabilization of a linear autonomous nonconservative system with even number for freedom [29, 49, 60]. On the other hand, their combination can make system asymptotically stable. It is well-known that the complexity of the choice of such a combination is associated with a Whitney umbrella singularity existing on the boundary of the gyroscopic stabilization domain of the nonconservative system. In [52], an approximation to the boundary of the asymptotic stability domain near the singularity is explicitly found and an analytical estimate of the critical gyroscopic parameter is obtained.

4. Nonlinear (Chaotic) Dynamics of a Gyroscope

The rotation of a rigid body with one fixed point in the absence of external forces, but in the presence of a viscous medium, is described by the following differential equation:

$$\dot{M} = M \times AM + BM, \quad (4.1)$$

where M is the vector of the kinetic moment of the selected coordinate system associated with the body, $A = I^{-1} = \text{diag}(a_1, a_2, a_3)$, and B is a constant matrix. Such a formulation is acceptable for small angular speeds and simple geometric shape of the bodies; this condition does not cause any vortices. In the case of dissipation ($\text{div} < 0$),

B fulfills the condition $TrB < 0$. In all other cases it determines a gyroscopic or control action. The above staging is studied in [64], as under certain parameter values in (4.1), two attractors coexist. The system (4.1) also has some special cases: 1) of Greenhill and 2) of Klein & Sommerfeld. Their trajectories lie on integrable surfaces. In more complex cases, (4.1) has two strange (chaotic) attractors [84].

A number of recent studies of the behavior of gyroscopic systems use the achievements of chaotic [19, 20, 28, 40, 76] and control theories [27, 35, 36, 39, 51, 96]. This allows to determine the critical (bifurcation) system parameters as well as to reveal new mechanisms in its behavior.

The system consisting of two (non-bearing and bearing) axis-symmetric rotors is subjected to a particular attention in research [3, 9, 14, 34, 75, 93]. This system has the following property: the mass distribution in space does not change and the tensor of the system remains constant at the relative motion of the non-bearing rotor. Such a system of rigid bodies is called *gyrostat* – see Figure 2. Under certain assumptions it can be shown that this system is close to the chaotic system of Lorenz, which was discovered in meteorology [40, 74].

A more general definition of a gyrostat was provided in [81] by Rumiantsev, i.e.

Definition: A gyrostat is a mechanical system S which consists of a solid body S_1 and other bodies S_2 which are connected to it. These other bodies are either variable or solid, but their motion relative to the body S_1 does not alter the geometry of the mass system S .

According to [81], S_1 is the carrier or platform, and S_2 is a collection of axisymmetric rotors. Note that for solid gyrostats the rotors S_2 must be exactly axisymmetric (i.e. the inertia tensor of S is independent of time) and for liquid gyrostats the fluid must be inviscid and of uniform density. The analogy is based on the Volterra equations of motion, a generalization of the Euler equations for rigid body motion about a fixed point. In rescaled variables, the Volterra equations of motion for a gyrostat have the form [42, 67, 74, 76, 92]:

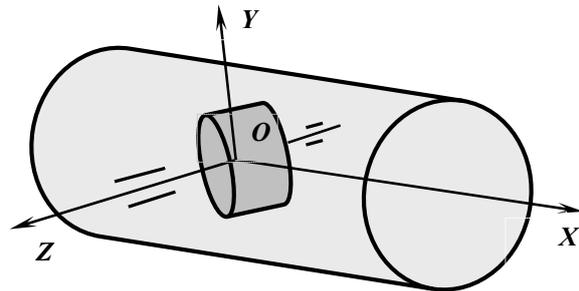


Fig. 2. Two bodies - a lifting and lifted axisymmetric rotor.

$$\begin{aligned}\dot{v}_1 &= -cv_2 + bv_3 + pv_2v_3, \\ \dot{v}_2 &= cv_1 - av_3 + qv_1v_3, \\ \dot{v}_3 &= -bv_1 + av_2 + rv_1v_2,\end{aligned}\tag{4.2}$$

where a, b, c, p, q and r constants characterizing the geometry of the gyrostat, with the constant that

$$p + q + r = 0,\tag{4.3}$$

and v_1, v_2 and v_3 are state variables that describe the dynamics of the system. It is seen that (4.2) (like the Navier-Stokes) equations) is quadratically nonlinear, and it possesses two quadratic integrals of motion and also conserves phase space volume. According to [92] (and reference there in), the Volterra gyrostat can be used as a “building block” for building higher order dynamical systems, by coupling together multiple gyrostats.

Taking into account Figure 1, the system (4.1) can be written in the form

$$\begin{aligned}\dot{M}_1 &= c_1M_1 + M_2 + c_2M_2M_3, \\ \dot{M}_2 &= -M_1 - c_3M_2 + c_4M_1M_3, \\ \dot{M}_3 &= c_5M_3 - c_6M_1M_2,\end{aligned}\tag{4.4}$$

where $c_1 - c_6$ are constants. The system (4.4) enjoys the natural symmetry $(M_1, M_2, M_3) \rightarrow (-M_1, -M_2, M_3)$. The M_3 -axis is invariant. All trajectories, which start on the M_3 -axis remain on it.

The divergence of the flow (4.4) is

$$D_3 = \frac{\partial \dot{M}_1}{\partial M_1} + \frac{\partial \dot{M}_2}{\partial M_2} + \frac{\partial \dot{M}_3}{\partial M_3} = c_5 - (c_1 + c_3).\tag{4.5}$$

The system (4.4) is dissipative, when $D_3 < 0$, i.e. $c_5 < c_1 + c_3$. It means that a volume element $D_3(t_0)$ is contracted by the flow into a volume $D_3(t) = D_3(t_0) \times e^{-(c_1+c_3-c_5)t}$. That is, as $t \rightarrow +\infty$, each volume containing the system trajectories shrinks to zero at an exponential rate (independently of the system states) $D_3 = -(c_1 + c_3 - c_5)$. Hence, all system trajectories will be confined to a specific subset of zero volume.

In [9, 45], the chaotic dynamics of an unbalanced gyrostat and of a torque-free rigid body in going from

minor axis to major axis spin under the influence of viscous damping and nonautonomous perturbations is investigated. Using the Melnikov's function of the perturbed system for heteroclinic separatrix orbits the chaotization of the motion is proved.

5. Gyroscope Applications

Gyroscopic instruments are devices that use the properties of the gyroscope [72]. The main elements of each gyroscopic device are one or more gyroscopes with two or three degrees of freedom. The composition of the gyroscopic instruments also includes auxiliary devices correcting the position of the axis of the gyroscope or measuring the angles of their diversion.

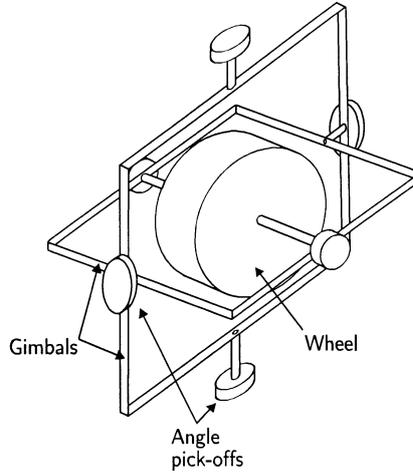


Fig. 3. A convention mechanical gyroscope (source: [98]).

The purpose of the gyroscopic instruments is quite diverse. A large group are the devices for gyroscopic stabilization (used in automatic control) of airplanes, ships, military torpedoes and missiles. As a rule, these devices contain an indicator (free gyro) registering the object deviation from the set course and a tracking power system that captures the signal indicator, amplifies it, and then transmits it to the power device. The power device returns the object to the specified course. Furthermore, the instrument may comprise a feedback system that reduces / enhances the degree of impact strength. The latter stabilization system is called an “indicator”. Such a system can be used to reduce the impact of sudden disruptions on the accuracy of the firing of cannons, set of vessels, aircraft or tanks, as well as spatial stabilization of artificial earth satellites [3, 98].

Depending on the type of materials and systems that are used in the manufacturing of the gyroscopes, the latter can be divided roughly into [65, 78, 98]:

- Mechanical (conventional) - see Figure 3;
- Optical (fiber optic gyroscope (FOG)) - see Figure 4;
- MEMS (micro-machined electro-mechanical system).

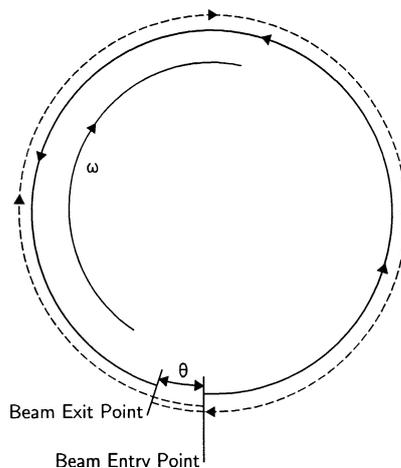


Fig. 4. The Sagnac effect (if the sensor is undergoing a rotation then the beam travelling (dashed line) in the direction of rotation will experience a longer path to the other end of the fiber than the beam travelling against the rotation (solid line)). θ is the angle through which the gyroscope turns whilst the beams are in flight (source: [98]).

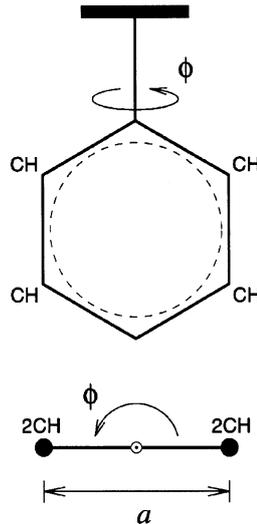


Fig. 5. Schematic representation of a molecular gyroscope $C_a C_6 H_5$ reduced to a model of a two-particle rotor of diameter a . ϕ is the revolution angle of the gyroscope moment of inertia (source: [11]).

In recent years, to determine the effect of magnetic fields on living matter (cells, tissues, physiological systems or whole organisms) the principles of the gyroscopic movement in the so-called *molecular gyroscopes* are used (see Figure 5) [11]. In [90], a superfluid gyroscope to explore rotational effects at the level of single quanta of circulation is proposed. On the other hand, in [65] the ultrahigh sensitivity of slow-light gyroscope is investigated. Slow light has been generated using electromagnetically induced transparency.

In [68, 97], the authors reported for a new dynamic test of the spin-spin coupling between a gyroscope and the Earth.

In papers [11, 47, 61, 69, 94], the problems of nonlinear dynamics, stability and control of different gyroscope systems are investigated. For example, in [12], the orbital stability of pendulum-like motion of a rigid body in the Bolyev-Steklov case is considered. In the case of oscillations with small amplitudes as well as in the case of rotations with high angular velocities the analytical results are obtained on the basis of a nonlinear analysis.

6. Conclusion

Gyroscopic effects play a major role on the dynamic behavior of various systems in physics, biology and the engineering sciences. The movement of a rigid body with one fixed point (gyroscope) can be pointed out as a typical example. By their nature, gyroscopic effects (forces) have a highly nonlinear character, which determines the difficulty (or in some cases the impossibility) of their modeling and approximation and calculation (study) using mechanical and mathematical models and trivial approaches. Our view on this issue is that a number of problems related to the investigation of the behavior and the study of the dynamic features of the gyroscopic models can be solved successfully by using the theory of dynamical systems. Therefore, our main goal in this review article was on the one hand, the disclosure of the characteristic features of the theory of dynamical systems, and the description of the specific properties of the movement (behavior) of a rigid body with one fixed point (gyroscope), on the other. From this examination, it becomes clear that a great part of gyroscopic systems can be regarded as dynamical.

For gyroscopic systems, it is important to find the conditions (parameters) that stabilize their movement (behavior), the intervals of change of these parameters and, not least, the bifurcation values determining the limits of their stability. Significant results have been attained in the case of linear (with arbitrary dimensionality (degrees of freedom)) and conservative nonlinear low dimensional systems. The same cannot be said for dissipative nonlinear gyroscopic systems. There are still no in-depth qualitative and quantitative analysis of their dynamic behavior – transitions from stable (unstable) to unstable (stable) condition, ways of occurrence of homoclinic and heteroclinic cycles, the route to chaos and more.

Acknowledgements

This work was supported by the grant No 1616 / 22.04.2014 of the University of Transport “T. Kableshkov” Sofia.

References

- [1] Afraimovich, V., Gonchenko, S., Lerman, L., Shilnikov, A. and Turaev, D., “Scientific Heritage of L.P. Shilnikov”, Regular and Chaotic Dynamics, Vol. 19, No. 4, pp. 435-460, 2014.
- [2] Alligood, K., Sauer, T. and Yorke, J., Chaos. An Introduction to Dynamical Systems, Springer, New York, 1996.
- [3] Anchev, A., “On the Stability of Permanent Rotations of a Heavy Gyrostat”, J. of Appl. Math. and Mech., Vol. 26, No. 1, pp. 22-28, 1962.
- [4] Andronov, A., Witt, A. and Chaikin, S., Theory of Oscillations, Addison-Wesley, Reading, MA, 1966.
- [5] Andronov, A. and Pontryagin, L., “Systemes grossieres”, DAN USSR, Vol. 14, pp. 247-251, 1937.
- [6] Arnold, V., Ordinary Differential Equations. Fourth Ed., Igevs, 2000 (in Russian).
- [7] Arnold, V., Afraimovich, V., Iliaschenko, Yu. and Shilnikov, L., Bifurcation Theory. Nauka, Moscow, 1986 (in Russian).
- [8] Arrowsmith, D. and Place, C., Dynamical Systems: differential equations, maps and chaotic behaviour, Chapman & Hall, London, 1992.
- [9] Aslanov, V. and Doroshin, A., “Chaotic Dynamics of an unbalanced Gyrostat”, J. of Applied Mathematics and Mechanics, Vol. 74, pp. 524-535, 2010.
- [10] Bachvarov, S., Mechanics. Part I, Stand. Print, Sofia, 2001 (in Bulgarian).
- [11] Banhi, V. and Savin, A., “Molecular Gyroscopes and Biological Affects of Weak Extremely Low-frequency Magnetic Fields”, Physical Review E, Vol. 65, pp. 051912, 2002.
- [12] Bardin, B., “On the Orbital Stability of Pendulum Like Motions of Rigid Body in the Bobylev-Steklov Case”, Regular and Chaotic Dynamics, Vol. 15, No. 6, pp. 704-716, 2010.
- [13] Barreira, L. and Valls, C., Dynamical Systems: An Introduction, Springer, London, 2013.
- [14] Basak, I., “Explicit Solution of the Zhukovski-Volterra Gyrostat”, Regular and Chaotic Dynamics, Vol. 14, No. 2, pp. 223-236, 2009.
- [15] Bautin, N. and Leontovich, E., Methods and Approaches for Qualitative Investigation of Two Dimensional Dynamical Systems, Nauka, Moscow, 1989 (in Russian).
- [16] Bloch, A., Nonholonomic Mechanics and Control. Springer, New York, 2003.
- [17] Bolotin, S., “The Hill Determinant of a Periodic Trajectories”, Matematika, Mechanika, Vol. 3, pp. 30-34, 1988.
- [18] Borisov, A. and Mamaev, I., Dynamics of Rigid Body, Moscow-Ijevsk, 2001.
- [19] Borisov, A., Kilin, A. and Mamaev, I., “Absolute and Relative Choreographies in Rigid Body Dynamics”, Regular and Chaotic Dynamics, Vol. 13, No. 3, pp. 204-220, 2008.
- [20] Borisov, A., Kilin, A. and Mamaev, I., “Chaos in a Restricted Problem of Rotation of a Rigid Body with a Fixed Point”, Regular and Chaotic Dynamics, Vol. 13, No. 3, pp. 221-233, 2008.
- [21] Brizard, A., An Introduction to Lagrangian Mechanics, World Scientific, Singapore, 2008.
- [22] Bulatovic, R., “The Stability of Linear Potential Gyroscopic Systems in Cases when the Potential Energy has a Maximum”, Prikl. Mat. Mekh., Vol. 61, No. 3, pp. 385-389, 1997.
- [23] Burra, L., Chaotic Dynamics in Nonlinear Theory, Springer, New York, 2014.
- [24] Butenin, N., Neimark, Yu. and Fufaev, N., Introduction in Theory of Nonlinear Oscillations, Nauka, Moscow 1976 (in Russian).
- [25] Carlson, J. and Doyle, J., “Highly Optimized Tolerance: robustness and design in complex systems”, Phys. Rev. Lett., Vol. 84, pp. 2529-2532, 2000.
- [26] Chang, D. and Marsden, J., Gyroscopic forces and collision avoidance with convex obstacles. In: New Trends in Nonlinear Dynamics and Control and Their Application, Springer, Berlin, Vol. 295, pp. 145-159, 2003.
- [27] Chen, H., “Chaos and Chaos Synchronization of a Symmetric Gyro with Linear Plus Cubic Damping”, J. of Sound and Vibrations, Vol. 255, No. 4, pp. 719-740, 2002.
- [28] Chen, H. and Ge, Zh., “Bifurcations and Chaos of a Two-degree of Freedom Dissipative Gyroscope”, Chaos, Solitons and Fractals, Vol. 24, pp. 125-136, 2005.
- [29] Chetayev, N., The Stability of Motion, Pergamon Press, New York 1961.
- [30] Coddington, E. and Levinson, N., Theory of Ordinary Differential Equations, McGraw-Hill Inc, London, 1987.
- [31] Coutinho, M.: Guide to Dynamics Simulations of Rigid Bodies and Particle Systems. Springer, London (2013)
- [32] Deriglazov, A., Classical Mechanics: Hamiltonian and Lagrangian formalism, Springer, New York, 2010.
- [33] Desloge, E., Classical Mechanics, Vol. I, John Wiley & Sons, New York, 1982.
- [34] Elipe, A., Arribas, M. and Riaguas, A., “Complete Analysis of Bifurcations in the Axial Gyrostat Problem”, J. Phys. A: Math. Gen., Vol. 30, pp. 587-601, 1997.
- [35] Elmandouh, A., “New Integrable Problems in the Dynamics of Particle and Rigid Body”, Acta Mech., Vol. 226, No. 11, pp. 3749-3762, 2015.
- [36] Elmandouh, A., “New Integrable Problems in Rigid Body Dynamics with Quartic Integrals”, Acta Mech., Vol. 226, No. 8, pp. 2461-2472, 2015.
- [37] Eueliri, L., “Theoria Motus Corporum Solidorum seu Rigidorum”. Griefswald, A. F. Rose, 1785; or Eueleri, L., Opera Omnia Ser. 2 Teubner, 3, 1948 and 4, 1950.
- [38] Fan, Y. and Chay T., “Crisis and topological entropy”, Physical Review E, Vol. 51, pp. 1012-1019, 1995.
- [39] Farivar, F., Shoorehdeli, M., Nekoui, M. and Teshnehlab, M., “Chaos Control and Generalized Projective Synchronization of Heavy Symmetric Chaotic Gyroscope Systems via Gaussian Radial Basis Adaptive Variable

- Structure Control”, *Chaos, Solitons and Fractals*, Vol. 45, pp. 80-97, 2012.
- [40] Ge, Z., Chen, H. and Chen, H., “The Regular and Chaotic Motions of Symmetric Heavy Gyroscope with Harmonic Excitation”, *J. of Sound and Vibration*, Vol. 198, No. 2, pp. 131-147, 1996.
- [41] Gluhovsky, A., “Nonlinear Systems that are Superpositions of Gyrostats”, *Sov. Phys. Dokl.*, Vol. 27, pp. 823-825, 1982.
- [42] Gluhovsky, A., “The structure of Energy Conserving Low-order Models”, *Physics of Fluids*, Vol. 11, No. 2, pp. 334-343, 1999.
- [43] Gonchenko, S. and Ovsyannikov, I., “On Bifurcations of Three-dimensional Diffeomorphisms with a Non-transversal Heteroclinic Cycle Containing Saddle-foci”, *Nonlinear Dynamics*, Vol. 6, No. 1, pp. 61-77, 2010 (in Russian).
- [44] Gradwell, G., Khonsari, M. and Ram, Y., “Stability Boundaries of a Conservative Gyroscopic System”, *J. of Applied Mechanics*, Vol. 70, pp. 561-567, 2003.
- [45] Gray, G., Kammer, D., Dobson, I. and Miller, A., “Heteroclinic Bifurcations in Rigid Bodies Containing Internally Moving Parts and a Viscous Damper”, *ASME Applied Mechanics*, Vol. 66, 720-728, 1999.
- [46] Guckenheimer, J. and Holmes, Ph., *Nonlinear Oscillations, Dynamical systems, and Bifurcations of Vector Fields*, Springer, New York, 1992.
- [47] Han, Zh. and Wang, S., “Multiple Solutions for Nonlinear Systems with Gyroscopic Terms”, *Nonlinear Analysis*, Vol. 75, pp. 5756-5764, 2012.
- [48] Hartman, Ph., *Ordinary Differential Equations*, John Wiley & Sons, London, 1964.
- [49] Hauger, W., “Stability of a Gyroscopic Nonconservative System”, *J. of Applied Mechanics*, Vol. 42, pp. 739-740, 1975.
- [50] Hill G., “On the Part of the Motion of the Lunar Perigee which is a Function of the Mean Motion of the Sun and Moon”, *Acta Math.*, Vol. 8, pp. 1-36, 1886.
- [51] Idowu, B., Vincent, U. and Njah, A., “Synchronization of Chaos in Non-identical Parametrically Excited Systems”, *Chaos, Solitons and Fractals*, Vol. 39, pp. 2322-2331, 2009.
- [52] Kirillov, O., “Gyroscopic Stabilization in the Presence of Nonconservative Forces”, *Dokl. Acad. Nauk*, Vol. 416, pp. 451-456, 2007.
- [53] Kliem, W. and Pommer, Ch., “Indefinite Damping in Mechanical Systems and Gyroscopic Stabilization”, *Z. Angew. Math. Phys. (ZAMP)*, Vol. 60, pp. 785-795, 2009.
- [54] Kowalevski, S., “Sur le Probleme de la Rotation d’un corps Solide Antor d’un Point Fixe”, *Acta Math.*, Vol. 12, pp. 177-232, 1889.
- [55] Kozlov, V., “Gyroscopic Stabilization and Parametric Resonance”, *J. of Applied Maths and Mechs.*, Vol. 65, No. 5, pp.715-721, 2001.
- [56] Krechetnikov, R. and Marsden J., “On Destabilizing Effects of Two Fundamental Nonconservative Forces”, *Physica D*, Vol. 214, pp. 25-32, 2006.
- [57] Krechetnikov, R. and Marsden J., “Dissipation Induced Instabilities in Finite Dimensions”, *Reviews of Modern Physics*, Vol. 79, pp. 519-553, 2007.
- [58] Krechetnikov, R. and Marsden J.” Dissipation Induced Instability Phenomena in Infinite Dimensional Systems”, *Arch. Rational Mech. Anal.*, Vol. 194, pp. 611-668, 2009.
- [59] Kuznetsov, Yu., *Elements of Applied Bifurcation Theory*, 2 ed., Springer, New York, 1998.
- [60] Lakhadanov, V., “On Stabilization of Potential Systems”, *PMM*, Vol. 39, No. 1, pp. 45-50, 1975.
- [61] Lancaster, P., “Stability of Linear Gyroscopic Systems: A review”, *Linear Algebra and its Applications*, Vol. 439, pp. 686-706, 2013.
- [62] Landa, P., *Nonlinear Oscillations and Waves in Dynamical Systems*, Kluwer Academic Publishers, Dordrecht, 1996.
- [63] Leimanis, E., *The General Problem of Motion of Coupled Rigid Bodies about a Fixed Point*, Springer, Berlin, 1965.
- [64] Leipnik, R. and Newton, T., “Double Strange Attractor in Rigid Body Motion with Linear Feedback Control”, *Phys. Lett.*, Vol. 86, No. 2, pp. 63-67, 1981.
- [65] Leonhardt, U. and Piwnicki, P., “Ultrahigh Sensitivity of Slow-light Gyroscope”, *Physical Review A*, Vol. 62, pp. 055801, 2000.
- [66] Leontovich, E. and Mayer, A., “On Trajectories which Define the Qualitative Separation of Sphere of Trajectories”, *DAN USSR*, Vol. 14, pp. 251-254, 1937.
- [67] Liu, Y. and Rimrott, F., “Global Motion of a Dissipative Asymmetric Gyrostat”, *Archive of Applied Mechanics* Vol. 62, pp. 329-337, 1992.
- [68] Luo, J., Nie, Y., Zhang, Y. and Zhou, Z., “Null Result for Violation of the Equivalence Principle with Free-fall Rotating Gyroscopes”, *Physical Review D*, Vol. 65, pp. 042005, 2002.
- [69] Markeev, A., “The Dynamics of a Rigid Body Colliding with a Rigid Surface”, *Regular and Chaotic Dynamics*, Vol. 13, No. 2, pp. 96-129, 2008.
- [70] Marsden, J., *Lectures on Mechanics*, Cambridge University Press, London Math. Society Lecture Notes Series 174., 1992.
- [71] Merkin D., *Gyroscopic Systems*, Nauka, Moscow, 1924 (in Russian).
- [72] Metelicin, I., *Theory of Gyroscope. Theory of stability*, Nauka, Moscow, 1977 (in Russian).

- [73] Morozov, V. and Kalenova, V., “Stability of Non-autonomous Mechanical Systems with Gyroscopic and Dissipative Forces”, XII Russian Control Conference, Moscow 16-19 June 2014, pp. 1888-1894, 2014.
- [74] Neimark, Yu., Method of Points Map in Theory of Nonlinear Oscillations, Nauka, Moscow, 1972 (in Russian).
- [75] Neimark, Yu. and Landa, P., Stochastic and Chaotic Oscillations, Kluwer Acad. Publishers, Singapore, 1992.
- [76] Nikolov, S., and Nedkova, N., “Gyrostat Model Regular and Chaotic Behaviour”, J. of Theoretical and Applied Mechanics, Vol. 45, No 4, pp. 15-30, 2015.
- [77] Panchev, S.: Theory of Chaos. Sofia, Acad. Publ. “prof. Marin Drinov”, Sofia, 2001 (in Bulgarian).
- [78] Pakniyat, A., Salarieh, H. and Alasty, A., “Stability Analysis of a New Class of MEMS Gyroscopes with Parametric Resonance”, Acta Mech. Vol. 223, No. 6, pp. 1169-1185, 2012.
- [79] Pali, J. and Smale, S., “Structure Stability Theorems”, Mathematics, Vol. 13, No. 2, pp. 145-155, 1969.
- [80] Poston, T. and Stewart, I., Catastrophe Theory and its Applications, PITMAN, London, 1978.
- [81] Rumiantsev, V., “On the Stability of Motion of Gyrostat”, J. Appl. Math. Mech., Vol. 25, pp. 9-19, 1961.
- [82] Ruelle, D. and Takens, F., “On the Nature of Turbulence”, Commun. in Math. Phys., Vol. 20, pp. 167-192, 1971.
- [83] Seyranian, A., Stoustrup, J. and Kliem, W., “On Gyroscopic Stabilization”, Zangew Math Phys (ZAMP), Vol. 46, pp. 255-267, 1995.
- [84] Sheu, L., Chen, H., Chen, J., Tam, L., Chen, W. and et al., Chaos in the Newton-Leipnik System with Fractional Order”, Chaos, Solitons and Fractals, Vol. 36, pp. 98-103, 2008.
- [85] Shilnikov, L., “On a New Type of Bifurcation of Multi-dimensional Dynamical Systems”, DAN USSR, Vol. 189, No 1, pp. 59-62, 1969.
- [86] Shilnikov, L., On a New Type Bifurcation of Multidimensional Dynamical Systems”, Sov. Math., Vol. 10, pp. 1368-1371, 1969.
- [87] Shilnikov, L., “A Contribution to the Problem of the Structure of an Extended Neighborhood of a Rough Equilibrium State of Saddle-focus Type”, Math. USSR Sbornik, Vol. 81(123), pp. 92-103, 1970.
- [88] Shilnikov, L., Shilnikov, A., Turaev, D. and Chua, L., Methods of Qualitative Theory in Nonlinear Dynamics. Part II, World Scientific, Singapore, 2001.
- [89] Sonechkin, D., Stochasticity in the Model of General Circulation Atmosphere, Gidrometeoizdat, Leningrad, 1984 in Russian.
- [90] Stringari, S., “Superfluid Gyroscope with Cold Atomic Gases”, Physical Review Letters, Vol. 86, No. 21, pp. 4725-4728, 2001.
- [91] Thomson, W. and Tait, P., Treatise on Natural Philosophy. vol. I, part I, Cambridge University Press, 1879.
- [92] Tong, Chr. and Gluhovsky, A., “Gyrostatic Extensions of the Howard-Krishnamutri Model of the Thermal Convection with Shear”, Nonlin. Processes Geophys., Vol. 15, pp. 71-79, 2008.
- [93] Tong, Chr., “Lord Kelvin’s Gyrostat and its Analogs in Physics, Including the Lorenz Model”, Am. J. Phys. Vol. 77, No. 6, pp. 526-537, 2009.
- [94] Tsai, N., Wu, B., “Nonlinear Dynamics and Control for Single-axis Gyroscope Systems”, Nonlinear Dyn., Vol. 51, pp. 355-364, 2008.
- [95] Uitni, H., “Peculiarities of Map in Euclidian Space”, Mathematics, Vol. 13, No. 2, pp. 105-123, 1969.
- [96] Wang, Ch. and Yau, H., “Nonlinear Dynamic Analysis and Sliding Mode Control for a Gyroscope System”, Nonlinear Dyn., Vol. 66, pp. 53-65, 2011.
- [97] Will, C., “Covariant Calculation of General Relativistic Effects in an Orbiting Gyroscope Experiment”, Physical Review D, Vol. 67, pp. 062003, 2003.
- [98] Woodman, O.: An Introduction to Internal Navigation. University of Cambridge Press (2007)
- [99] Zhuravlev, V., Grounding in Theoretical Mechanics, Fizmatlit, Moscow, 2001 (in Russian).