



## Jeffery Hamel Flow of a non-Newtonian Fluid

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### Abstract

This paper presents the Jeffery Hamel flow of a non-Newtonian fluid namely Casson fluid. A suitable similarity transform is applied to reduce governing nonlinear partial differential equations to a much simpler ordinary differential equation. The variation of Parameters Method (VPM) is then employed to solve the resulting equation. The same problem is solved numerically by using Runge-Kutta order 4 method. A comparison between both solutions is carried out to check the efficiency of VPM. The effects of emerging parameters are demonstrated for both diverging and converging channels using graphical simulation.

**Keywords:** Jeffery-Hamel flow, Casson fluid, Variation of Parameters Method (VPM), Converging and diverging channels, Runge-Kutta order 4 method.

### 1. Introduction

The flow through tapered walls has gained much interest due to the wide range of applications in the aerospace, chemical, civil, environmental, mechanical and bio-mechanical engineering. The flow in rivers and channels is also closely related to these applications. In the human body where capillaries and arteries are linked to each other, blood exhibits such a kind of flow.

Mathematical formulation of the flows through nonparallel walls was firstly accomplished by Jeffery and Hamel [1-2]; therefore, these types of flows are called the Jeffery-Hamel flows. As the results of this pioneering work, such problems have gained a considerable attention. These are the exact similarity solution of conservation equations in the special case of two-dimensional flow through a channel with inclined plane walls meeting at a vertex with a source or sink. Jeffery-Hamel flows have been studied and discussed extensively by several authors in many textbooks [3-8].

Due to the complex rheological nature of non-Newtonian fluids, there is no single model that can be used to simulate the whole class. Therefore, many non-Newtonian flow models have been presented over the years. Casson fluid is one of these models that exhibit blood type behavior [9-10]. Some of the studies related to Casson fluid can be found in [11-13] and references therein.

In most of the cases, equations describing a physical problem are inherently in the form of a non-linearity. Therefore, an exact solution is unlikely; however, different approximation techniques are developed to solve these complex equations including Adomian's Decomposition Method (ADM), Homotopy Analysis Method (HAM), Homotopy Perturbation Method (HPM) and Variational Iteration Method (VIM) [14-21]. Either most of these techniques are very difficult to employ and require huge amount of computational work, or we have very low level of accuracy which not only affects the results but also is unreliable in certain cases. To sort this problem out, we have employed another solution scheme using a well-known technique called the Variation of Parameters Method (VPM). The main advantage of using this method is that, it is free from round off errors, calculation of the so-called Adomian's polynomials, perturbation, linearization or discretization and uses only the initial conditions which are easier to be implemented. It reduces the computational work while still maintaining a higher level of accuracy. Different studies are available that prove the accuracy and effectiveness of this technique [22-27].

In this article, VPM is used to solve a rather complex problem that governs the flow of Casson fluid in converging and diverging channels. The convergence of the solutions is presented coupled a comparison of a numerical that is also carried out to verify the results obtained by VPM.

## 2. Mathematical Formulation:

Consider the flow of a Casson fluid initiated due to source or sink at the intersection of two rigid plane walls. The angle between the walls is taken as  $2\alpha$ . The Flow is assumed to be symmetric and purely radial. The rheological equation describing an incompressible Casson fluid is [11-13]

$$\tau_{ij} = 2 \left( \mu_B + \frac{p_y}{\sqrt{2\pi}} \right) e_{ij}, \quad \pi \neq \pi_c, \quad (1)$$

where  $\pi = e_{ij}e_{ij}$  with  $e_{ij}$  being the  $(i, j)$ th component of the deformation rate,  $\pi$  is the product of the deformation rate with itself,  $\pi_c$  denotes a critical value of this product based on a non-Newtonian model,  $\mu_B$  denotes the plastic dynamic viscosity of non-Newtonian fluids and  $p_y$  is the yield stress of the fluid.

These assumptions mean that the velocity field is of the type  $\mathbf{V} = [u(r, \theta), 0, 0]$ . Equations of continuity and momentum for this problem are:

$$\nabla \cdot \mathbf{V} = 0, \quad (2)$$

$$\rho \left[ \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{V} \right] = -\nabla p + \nabla \cdot \tau_{ij}. \quad (3)$$

Under aforementioned assumptions, Eqs.(1) to (3) reduce to

$$\frac{1}{r} \frac{\partial}{\partial r} (ru) = 0, \quad (4)$$

$$u \frac{\partial u}{\partial r} = -\frac{1}{\rho} \frac{\partial p}{\partial r} + \nu \left( 1 + \frac{1}{\beta} \right) \left[ \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} - \frac{u}{r^2} \right], \quad (5)$$

$$-\frac{1}{\rho r} \frac{\partial p}{\partial \theta} + \left(1 + \frac{1}{\gamma}\right) \frac{2\nu}{r^2} \frac{\partial u}{\partial \theta} = 0, \tag{6}$$

where  $\nu$  is kinematic viscosity,  $p$  is the pressure,  $\beta = \mu_B \frac{\sqrt{2\pi}}{p_y}$  the Casson fluid parameter and  $k$  denotes the porous medium permeability.

The auxiliary conditions are as follows

$$\begin{aligned} u_r = U, \quad \frac{\partial u}{\partial \theta} = 0, \quad \text{at} \quad \theta = 0, \\ u_r = 0, \quad \theta = \alpha. \end{aligned} \tag{7}$$

From continuity equation (4), we may derive

$$f(\theta) = ru. \tag{8}$$

To make the problem dimensionless we have

$$F(\eta) = \frac{f(\theta)}{rU}, \quad \eta = \frac{\theta}{\alpha}. \tag{9}$$

Eliminating  $p$  from Eqs.(5) , (6) and using Eqs. (8), (9) we get a nonlinear ordinary differential equation for the normalized velocity profile  $F(\eta)$  as follows

$$\left(1 + \frac{1}{\beta}\right) F''(\eta) + 2\alpha \operatorname{Re} F(\eta) F'(\eta) + 4\alpha^2 \left(1 + \frac{1}{\beta}\right) F'(\eta) = 0. \tag{10}$$

Similarly boundary conditions become

$$F(0) = 1, \quad F'(0) = 0, \quad F(1) = 0. \tag{11}$$

Re is Reynolds number given by

$$\operatorname{Re} = \frac{Ur\alpha}{\nu} \left( \begin{array}{l} \text{Divergent Channel : } \alpha > 0, U > 0 \\ \text{Convergent Channel : } \alpha < 0, U < 0 \end{array} \right), \tag{12}$$

$U$  is the velocity at the center line of the channel i.e. at  $r = 0$ . It is pertinent to mention that  $\beta = \infty$  corresponds to the case of a simple viscous fluid.

Values for the skin friction coefficient can be obtained by using

$$C_f = \frac{\tau_{r\theta} \text{ at } \eta = 1}{U^2} = \frac{1}{\operatorname{Re}} \left(1 + \frac{1}{\beta}\right) F'(1).$$

### 3. Solution of the Problem

Following the standard procedure proposed for VPM [22-27], to solve Eq. (10) recurrence relation can be written as:

$$F_{n+1}(\eta) = A_1 + A_2\eta + A_3 \frac{\eta^2}{2} - \int_0^\eta \left( \frac{\eta^2}{2!} - \eta s + \frac{s^2}{2!} \right) \left( 2\alpha \operatorname{Re} \left( \frac{\beta}{1+\beta} \right) F_n(\eta) F_n'(\eta) + 4\alpha^2 F_n'(\eta) \right) ds, \tag{13}$$

with  $n = 0, 1, 2, \dots$

Using boundary conditions in Eq. (11) we get  $A_1 = 1, A_2 = 0$ ; further, we suppose  $A_3 = \sigma$ , hence Eq. (13) can now be written as

$$F_{n+1}(\eta) = 1 + \sigma \frac{\eta^2}{2} - \int_0^\eta \left( \frac{\eta^2}{2!} - \eta s + \frac{s^2}{2!} \right) \left( 2\alpha \operatorname{Re} \left( \frac{\beta}{1+\beta} \right) F_n(\eta) F_n'(\eta) + 4\alpha^2 F_n'(\eta) \right) ds, \quad n = 0, 1, 2, \dots \tag{14}$$

where  $\sigma$  can be calculated by using the condition  $F(1) = 0$ . The first few iterations of the solution are as follows:

$$\begin{aligned} F_1(\eta) &= 1 + \sigma \frac{\eta^2}{2} - \frac{1}{6} \left( \alpha^2 \sigma + \frac{1}{2} \operatorname{Re} \alpha \sigma \right) \eta^4 - \frac{1}{120} \operatorname{Re} \alpha \left( \frac{\beta}{1+\beta} \right) \sigma^2 \eta^6, \\ F_2(\eta) &= 1 + \sigma \frac{\eta^2}{2} - \frac{1}{6} \left( \alpha^2 \sigma + \frac{1}{2} \operatorname{Re} \alpha \sigma \right) \eta^4 \\ &\quad + \left( \frac{\beta}{1+\beta} \right) \left\{ \frac{1}{45} \left( \frac{1}{4} \operatorname{Re}^2 \alpha^2 \sigma + \operatorname{Re} \alpha^3 \sigma + \alpha^4 \sigma \right) - \frac{1}{120} \operatorname{Re} \alpha \sigma^2 \right\} \eta^6 \\ &\quad + \frac{1}{280} \left( \frac{\beta}{1+\beta} \right) \left( \frac{1}{2} \operatorname{Re}^2 \alpha^2 \sigma + \alpha^3 \operatorname{Re} \sigma^2 \right) \eta^8 \\ &\quad + \left( \frac{\beta}{1+\beta} \right) \left( \frac{1}{10800} \alpha^2 \operatorname{Re}^2 \sigma^3 - \frac{1}{12960} \operatorname{Re}^3 \alpha^3 \sigma^2 \right) \eta^{10} \\ &\quad - \frac{1}{47520} \left( \frac{\beta}{1+\beta} \right) \left( \operatorname{Re}^2 \alpha^4 \sigma^3 + \frac{1}{2} \operatorname{Re}^3 \alpha^3 \sigma^3 \right) \eta^{12} - \frac{1}{2620800} \operatorname{Re}^3 \alpha^3 \sigma^4 \eta^{14}. \end{aligned} \tag{15}$$

In a similar fashion, we can determine further approximation for the solution  $F(\eta)$ .

#### 4. Results and discussions

This section describes the influence of different emerging parameters on the velocity profile graphically. For this purpose, Figs. 1-6 are plotted. In Figs. 1-3, the effects of channel angle  $\alpha$ , Reynolds number  $\operatorname{Re}$  and the Casson fluid parameter  $\beta$  are demonstrated for a diverging channel. All these parameters affect the velocity profile in an almost similar fashion; That is, increase in  $\alpha$ ,  $\operatorname{Re}$  and  $\beta$  decreases the velocity profile. From Fig. 2 it can also be obtained that by increasing  $\beta$ , the velocity of the fluid decreases quite significantly in comparison with the other two parameters. For all these parameters, maximum velocity is observed near the center of channel. It is pertinent to mention that  $\beta = \infty$  gives the velocity to a simple Newtonian fluid.

For the converging channel, the effects of the same parameters are depicted in Figs. 4-6. A quite opposite behavior is observed in this case; increase in the velocity is observed for  $\alpha$ ,  $\operatorname{Re}$  and  $\beta$ . As in the previous case, the influence of  $\beta$  is more significant in comparison with  $\alpha$  and  $\operatorname{Re}$ .

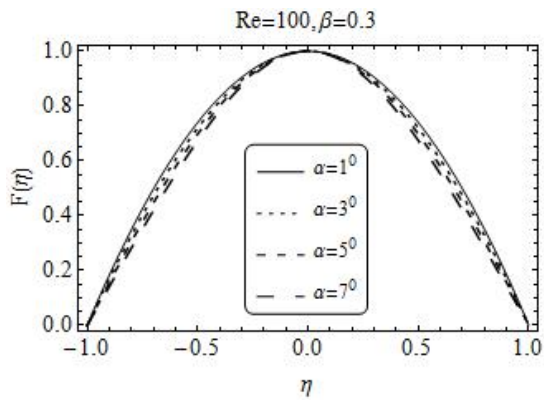


Fig. 1. Effects of  $\alpha$  on the diverging channel

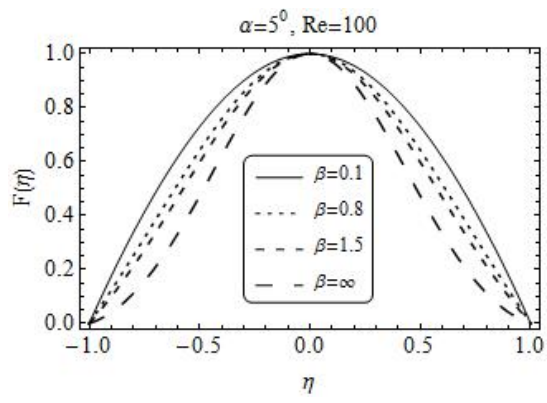


Fig. 2. Effects of  $\beta$  on the diverging channel

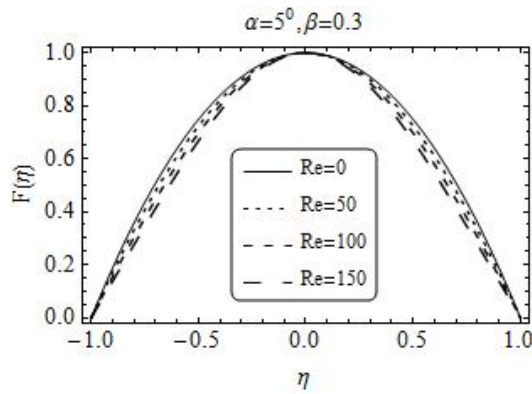


Fig. 3. Effects of  $Re$  on the diverging channel

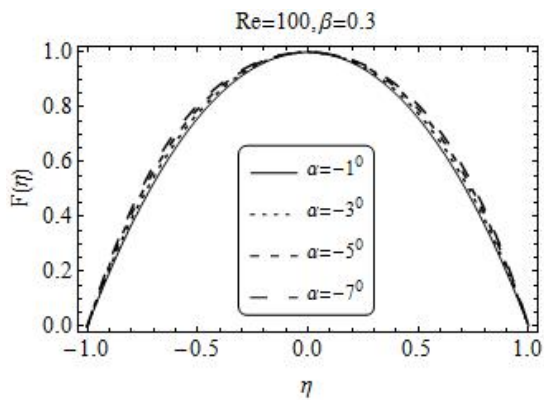


Fig. 4. Effects of  $\alpha$  on the converging channel

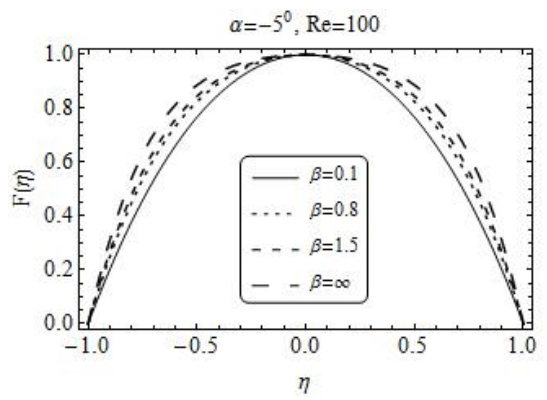


Fig. 5. Effects of  $\beta$  on the converging channel

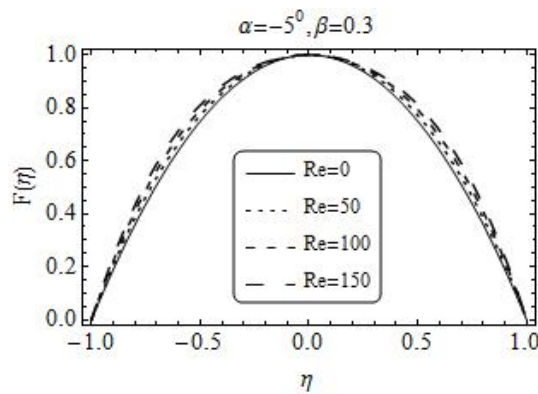


Fig. 6. Effects of  $Re$  on the converging channel

It is important to check the convergence of the analytical solution. For this purpose, values of the unknown constant  $\sigma$  are tabulated in Table1. It is observed that VPM solution converges at 6<sup>th</sup> approximation for both diverging and converging channels.

Aimed for comparison, same problem is solved by using a well-known numerical technique Runge-Kutta order 4 method coupled with the shooting technique. Table 2 displays comparison results in which an excellent agreement between the solutions, both for converging and diverging channels, is found. Table 3 provides the effects of different parameters on the skin friction coefficient. It is observed that the magnitude of the skin friction coefficient increases with an increase in  $\alpha$  and Re, while a decrease in the magnitude of the skin friction coefficient is observed for increasing values of  $\beta$ .

**Table 1.** Convergence of VPM solution for Re=100,  $\alpha = 5^\circ$  and  $\beta = 0.3$

Order of Approximations	For Diverging Channel	For Converging Channel
	$\sigma = -F'(0)$	$\sigma = -F'(0)$
1	2.662463	1.561663
2	2.615743	1.533493
4	2.61788	1.53224
6	2.617881	1.532239
8	2.617881	1.532239
10	2.617881	1.532239
12	2.617881	1.532239
Numerical	2.617881	1.532239

**Table 2.** Comparison of VPM and numerical solution for Re=100,  $\alpha = 5^\circ$  and  $\beta = 0.3$

$\eta$	For Diverging Channel		For Converging Channel	
	VPM	Numerical	VPM	Numerical
0	1	1	1	1
0.1	0.986954	0.986954	0.992313	0.992313
0.2	0.948339	0.948339	0.968947	0.968947
0.3	0.885656	0.885656	0.928987	0.928987
0.4	0.801219	0.801219	0.870919	0.870919
0.5	0.697894	0.697894	0.792662	0.792662
0.6	0.578809	0.578809	0.691628	0.691628
0.7	0.447036	0.447036	0.564845	0.564845
0.8	0.305319	0.305319	0.409171	0.409171
0.9	0.155833	0.155833	0.221645	0.221645
1	0	0	0	0

**Table 3.** Numerical values of the skin friction coefficient for different values of parameters

$\alpha$	Re	$\beta$	$C_f$ Re
1°	50	0.3	-8.491011
3°			-8.132965
5°			-7.765948
-1°			-8.840078
-3°			-9.180186
-5°			-9.511377
5°	100		-6.879197
	150		-5.989193
	200		-5.101456
-5°	100		-10.363046
	150		-11.197414
			-12.012961
5°	50	0.1	-21.068544
		0.5	-5.104082
		0.9	-3.328600
-5°	50	0.1	-22.814960
		0.5	-6.848243
		0.9	-5.070275

## 5. Conclusions

In this paper, Jeffery Hamel flow of a non-Newtonian fluid is presented. The governing nonlinear equation is solved using an efficient technique known as Variation of Parameters Method (VPM). A numerical solution is also obtained for the purpose of comparison. An exceptional agreement is found between both solutions. The Graphs are presented to study the behavior of emerging parameters. It is observed that for an increase in values of parameters, the behavior of the velocity in the narrowing channel is opposite to the one in the widening channel. Numerical values for the skin friction coefficient are also tabulated.

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