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Periodic Solutions of the Duffing Harmonic Oscillator by He's Energy Balance Method

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Abstract

Duffing harmonic oscillator is a common model for nonlinear phenomena in science and engineering. This paper presents He´s Energy Balance Method (EBM) for solving nonlinear differential equations. Two strong nonlinear cases have been studied analytically. The analytical results of the EBM are compared with the solutions obtained by applying He´s Frequency Amplitude Formulation (FAF) and numerical solutions using Runge-Kutta method. The results show that this method is potentially presented to solve high nonlinear oscillator equations.

Keywords: Energy balance method, Frequency amplitude formulation, Duffing harmonic oscillator, Periodic solutions.

1. Introduction

The study of given nonlinear problems is of crucial importance not only in all areas of physics but also in engineering and other disciplines, since most phenomena in our world are essentially nonlinear and are described by nonlinear equations. It is very difficult to solve nonlinear problems, and in general it is often more difficult to extract an analytic approximation from a numerical one for a given nonlinear problem. There are many analytical approaches to solve nonlinear differential equations. One of the widely used techniques is perturbation [1-4], whereby the solution is expanded in powers of a small parameter. However, for the nonlinear conservative systems, the generalizations of some of the standard perturbation techniques overcome this limitation. Several approaches have been proposed for dealing with the Duffing harmonic oscillator; for example, the harmonic balance method [5-9], the energy balance method [10-12], the Hamiltonian approach [13], the homotopy perturbation method [14-16], the parameter expansion method [17, 18], the frequency amplitude formulation [19, 20], the variational iteration method [21-23] and other new methods [24-30].

In this paper, He's energy balance method is employed to solve the Duffing harmonic nonlinear oscillator problem and to compare the results with the numerical solution.

2. Basic Idea of He's Energy Balance Method

Let us consider a general nonlinear oscillator in the following formula

$$u'' + f(u) = 0, \tag{1}$$

in which u and t are the generalized dimensionless displacement and time variables, respectively. Its variational principle can be easily obtained as

$$J(u) = \int_0^t \left(\frac{1}{2} u'^2 + F(u) \right) dt \tag{2}$$

where

$$F(u) = \int f(u) du. \tag{3}$$

Its Hamiltonian, therefore, can be written as

$$H = \frac{1}{2} u'^2 + F(u) = F(A), \tag{4}$$

or

$$R(t) = \frac{1}{2} u'^2 + F(u) - F(A) = 0. \tag{5}$$

The Oscillation system contains two important physical parameters, i. e., the frequency ω and the amplitude of oscillation A. Therefore, let us consider such initial conditions as:

$$u(0) = A, \qquad u'(0) = 0. \tag{6}$$

Assume that its initial approximate guess can be expressed as

$$u = A\cos(\omega t). \tag{7}$$

Substituting Eq. (7), into u term of Eq. (5), yields

$$R(t) = \frac{1}{2}\omega^2 A^2 \sin^2 \omega t + F(A\cos\omega t) - F(A) = 0. \tag{8}$$

Since Eq. (7) is only an approximation to the exact solution, Eq. (8) cannot be made zero everywhere. Collocation at $\omega = \frac{\pi}{4}$ gives

$$\omega = \frac{2}{A}\sqrt{F(A) - F\left(\frac{A}{\sqrt{2}}\right)}. \tag{9}$$

Its period can be determined using the relation $T = \dfrac{2\pi}{\omega}$ as

$$T = \frac{2\pi}{\frac{2}{A}\sqrt{F(A) - F\left(\frac{A}{\sqrt{2}}\right)}}. \tag{10}$$

3. Applications

In order to assess advantages and the accuracy of He's energy balance method, we should consider the following two examples.

3.1 Example 1

As for the first example, let us consider the following nonlinear oscillator [31] which is an example of conservative nonlinear oscillatory systems having a rational form for the restoring force as

$$u'' + u^3 + \frac{u}{1+u^2} = 0, \tag{11}$$

with the initial conditions

$$u(0)=A, \qquad \frac{du}{dt}(0)=0. \tag{12}$$

For this problem,

$$f(u)=u^3+\frac{u}{1+u^2} \quad \text{and} \quad F(u)=\frac{1}{4}u^4+\frac{1}{2}\ln(1+u^2).$$

Its variational formulation can be easily established as

$$J(u)=\int_0^t\left\{-\frac{1}{2}u'^2+\frac{u^4}{4}+\frac{1}{2}\ln\left(1+u^2\right)\right\}dt. \tag{13}$$

Its Hamiltonian, therefore, can be written as follows:

$$R(t)=\frac{1}{2}u'^2+\frac{u^4}{4}+\frac{1}{2}\ln\left(1+u^2\right)-\frac{A^4}{4}-\frac{1}{2}\ln\left(1+A^2\right)=0. \tag{14}$$

We use the trial function (7), as used in the method of energy balance, to determine the angular frequency ω

$$u=A\cos\omega t. \tag{15}$$

Substituting Eq. (15) into Eq. (14), yields

$$R(t)=\frac{1}{2}A^2\omega^2\sin^2\omega t+\frac{A^4\cos^4\omega t}{4}+\frac{1}{2}\ln\left(1+A^2\cos^2\omega t\right)-\frac{A^4}{4}-\frac{1}{2}\ln\left(1+A^2\right)=0 \tag{16}$$

If we collocate at $\omega t=\frac{\pi}{4}$ we obtain

$$\frac{1}{4}A^2\omega^2+\frac{A^4}{16}+\frac{1}{2}\ln\left(1+\frac{A^2}{2}\right)-\frac{A^4}{4}-\frac{1}{2}\ln\left(1+A^2\right)=0 \tag{17}$$

or

$$\omega=\frac{\sqrt{\frac{3}{4}A^4+2\ln(1+A^2)-2\ln(1+\frac{A^2}{2})}}{A}, \qquad T=\frac{2\pi}{\omega}. \tag{18}$$

In order to compare He's energy balance's result with He's frequency amplitude, we write Fan's result [31]

$$\omega=\sqrt{\frac{3}{4}A^2+\frac{1}{1+\frac{3}{4}A^2}}. \tag{19}$$

Substituting Eq. (18) into Eq. (15), we can obtain the approximate solution as

$$u=a\cos\frac{\sqrt{\frac{3}{4}A^4+2\ln(1+A^2)-2\ln(1+\frac{A^2}{2})}}{A}t \tag{20}$$

Table 1: Comparison of He's Energy balance solution with He's frequency amplitude formulation

A	ω_{EBM}	ω_{FAF} [31]
0.01	1	1
0.1	1.0004	1.0003
0.2	1.00045	1.00044
0.4	1.0066	1.0064
0.6	1.02894	1.0283
0.8	1.07626	1.07502
1	1.15124	1.14953
5	4.33618	4.33597
10	8.66104	8.66101
100	86.6025	86.6025

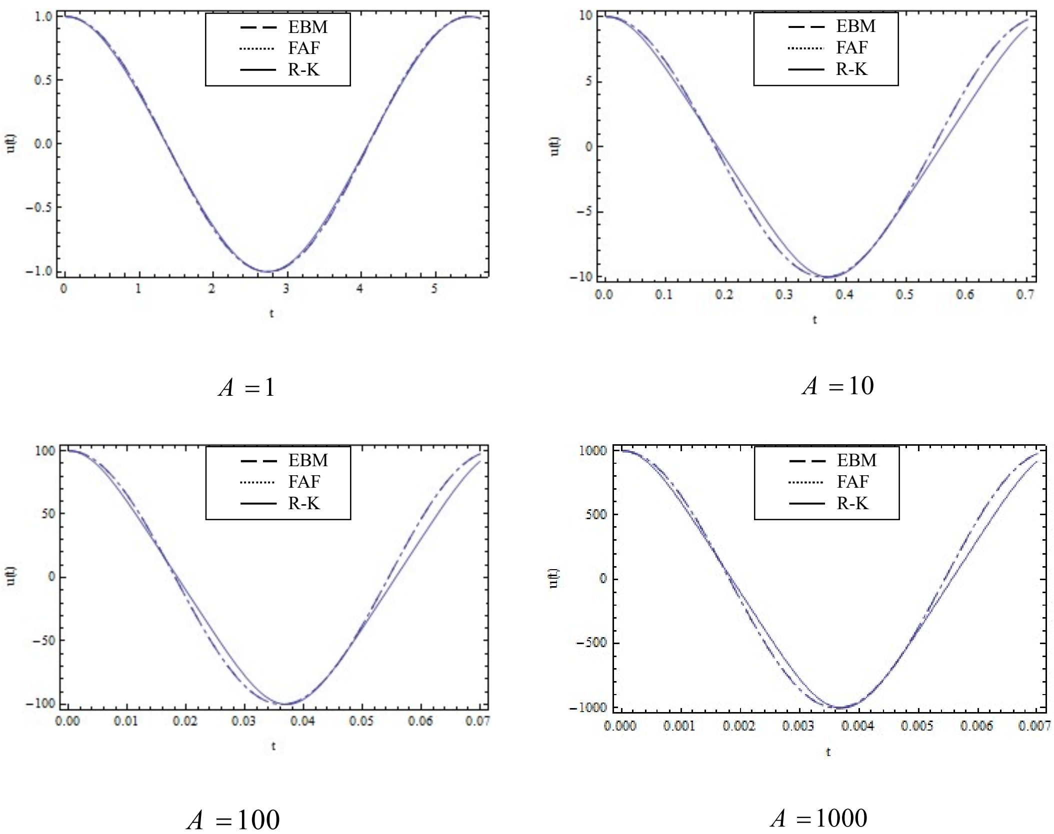

Fig. 1. Comparison of analytical solutions with the numerical solution

The above results are in good agreement with the results obtained by He's frequency amplitude formulation [31] and the numerical integration results obtained by using the Runge-Kutta method as illustrated in Table 1 and Fig. 1.

3.2 Example 2

This is an example of a conservative nonlinear oscillator system having an irrational elastic item [31]:

$$u'' + u + \frac{u}{1+u^2} = 0, \tag{21}$$

with the initial conditions

$$u(0) = A, \qquad \frac{du}{dt}(0) = 0, \tag{22}$$

where

$$f(u) = u + \frac{u}{1+u^2}.$$

Its variational formulation is

$$J(u) = \int_0^t \left\{ -\frac{1}{2}u'^2 + \frac{u^2}{2} + \frac{1}{2}\ln\left(1+u^2\right) \right\} dt. \tag{23}$$

By a similar manipulation, as illustrated in previous example, we obtain

$$R(t) = \frac{1}{2}A^2\omega^2 \sin^2 \omega t + \frac{A^2 \cos^2 \omega t}{2} + \frac{1}{2}\ln\left(1 + A^2\cos^2 \omega t\right) - \frac{A^2}{2} - \frac{1}{2}\ln\left(1+A^2\right) = 0. \tag{24}$$

From Eq. (24) and with $\omega t = \frac{\pi}{4}$ we have

$$\omega = \frac{\sqrt{A^2 + 2\ln(1+A^2) - 2\ln(1+\frac{A^2}{2})}}{A}, \quad T = \frac{2\pi}{\omega}. \tag{25}$$

In order to compare with He's frequency amplitude, we write Fan's result [31] as

$$\omega = \sqrt{1 + \frac{1}{1+\frac{3}{4}A^2}}. \tag{26}$$

Then, we can obtain the following approximate periodic solution

$$u = A\cos\frac{\sqrt{A^2 + 2\ln(1+A^2) - 2\ln(1+\frac{A^2}{2})}}{A}t. \tag{27}$$

Table 2. Comparison of He's Energy balance solution with He's frequency amplitude formulation

A	ω_{EBM}	ω_{FAF} [31]
0.01	1.41419	1.41419
0.1	1.41158	1.41158
0.2	1.40389	1.40388
0.4	1.37595	1.37581
0.6	1.33743	1.33694
0.8	1.2955	1.29448
1	1.25514	1.25357
5	1.02588	1.025
10	1.00681	1.00656
100	1.00007	1.00007
1000	1	1

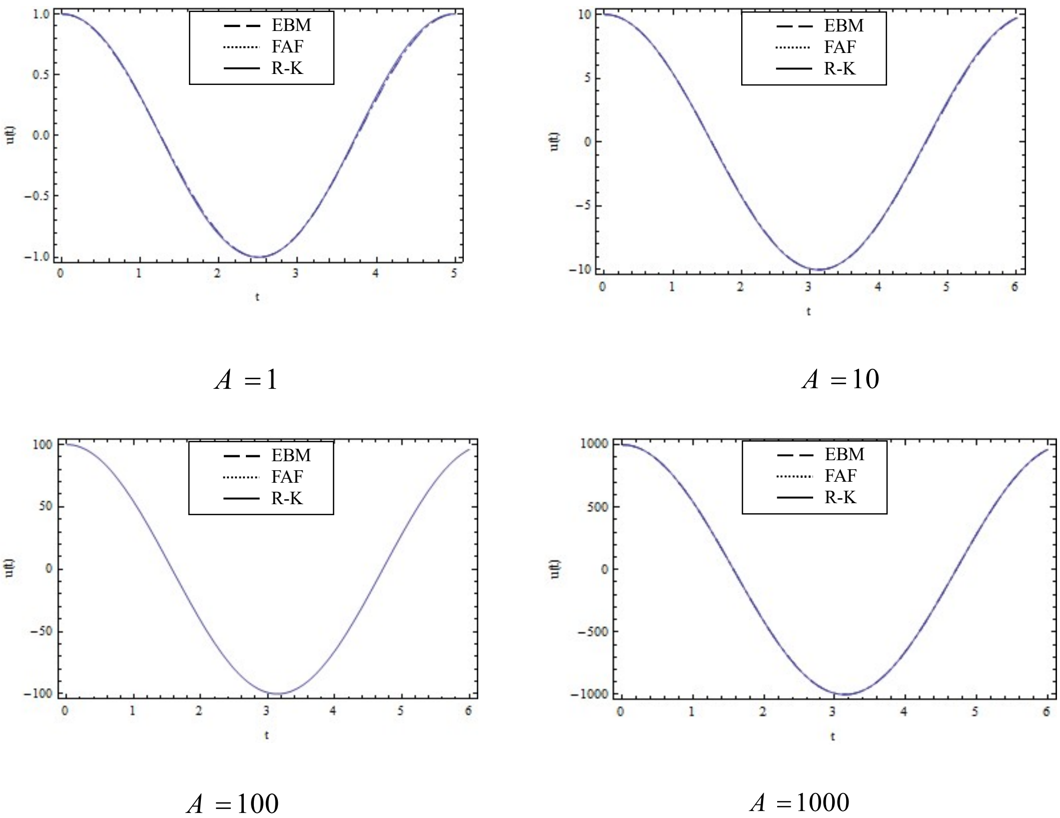

Fig. 2. Comparison of analytical solutions with the numerical solution

Table 2 and Fig. 2. Show an excellent agreement between the results obtained from the energy balance method, He's frequency amplitude formulation [31] and numerical results using Runge-Kutta method.

4. Conclusion

An analytical method called energy balance method has been successfully used to found approximate periods for strongly nonlinear Duffing harmonic oscillator. The approximate periods for such nonlinear problems show a good agreement with the numerical solutions. In comparison with the previously published methods, the determination of solutions is straightforward and simple. To sum up, we can say that the energy balance method applied in this paper to determine approximate periods for a Diffing harmonic oscillator can be considered as an efficient alternative of the previously proposed methods.

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