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Deflection of a hyperbolic shear deformable microbeam under a concentrated load

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Abstract

Deflection analysis of a simply supported microbeam subjected to a concentrated load at the middle is investigated on the basis of a shear deformable beam theory and non-classical theory. Effects of shear deformation and small size are taken into consideration by hyperbolic shear deformable beam theory and modified strain gradient theory, respectively. The governing differential equations and corresponding boundary conditions are obtained by implementing minimum total potential energy principle. Navier-type solution is employed to achieve an analytical solution for deflections of simply supported homogeneous microbeams. The effects of shear deformation, material length scale parameter and slenderness ratio on the bending response of microbeams are investigated in detail.

Keywords: Bending, hyperbolic shear deformation theory, modified strain gradient theory, size dependency.

1. Introduction

As a result of the great advances in technology (especially in nanotechnology), the applications of the miniaturized structures increase in micro- and nano-electro mechanical systems (MEMS and NEMS) [1-3]. Microbeam is one of the basic structures for MEMS/NEMS such as micro-resonators [4], Atomic Force Microscopes [5], micro-actuators [6], micro-switches [7]. The dimensions of the microbeams are on the order of microns and sub-microns and it is demonstrated by some experimental studies that the mechanical deformation behavior of these structures affected by small size [8-11]. Consequently, size effects should be taken into consideration on the determination of the mechanical characteristics of such structures. However, the continuum models evaluated by conventional elasticity theory fail to predict the mechanical responses of micro- and nano-sized structures due to the lack of any additional material length scale parameters. Subsequently, various non-classical continuum theories have been developed like couple stress theory [12-14], micropolar theory [15], nonlocal elasticity theory [16, 17] and strain gradient theories [18-21].

One of the most popular higher-order continuum theories is modified strain gradient theory elaborated by Lam et al. [10]. Unlike in the classical continuum mechanics, the total deformation energy density is not only a function of first-order deformation gradient but also a function of second-order deformation gradients in this theory. A number of studies have been performed to investigate mechanical responses of homogeneous microbars [22–25] and microbeams [26–32]. Approximate solutions for static and dynamic analyses of microbeams were also carried out by finite element method based on Bernoulli-Euler and Timoshenko beam theories, respectively [33, 34]. Buckling behavior of boron nitride nanotube surrounded by an elastic matrix is investigated by discrete singular convolution method [35]. Nonlocal continuum and discrete rod models are developed for the size-effects in the torsional and axial vibration responses of microtubules [36].

In order to determine the mechanical characteristics of beams, several beam theories have been introduced such as Bernoulli-Euler (BET), Timoshenko, parabolic [37,38], trigonometric (sinusoidal) [39], hyperbolic (HBT) [40], exponential [41] and a general exponential [42] beam theories. BET is useful for slender beams with a large aspect ratio due to the lack of shear deformation effects. Effects of shear deformation can be taken into account by TBT.

However, the distributions of transverse shear stress and strain are assumed as uniform along the thickness of the beam in this theory. Consequently, TBT needs a shear correction factor due to there are no transverse shear stress and strain at the top and bottom surfaces of the beam. On the other hand, the transverse shear stress and strain vary throughout the height of the beam and equal to zero at the upper and lower surfaces of the beam in the other shear deformation beam theories mentioned above. Recently, several size-dependent shear deformation beam models have been developed to investigate the static and dynamic analyses of microstructures [43-56].

In the present study, bending response of a simply supported microbeam under a point load at the middle is investigated on the basis of hyperbolic shear deformable beam theory and modified strain gradient theory. The governing differential equations and corresponding boundary conditions are derived by implementing minimum total potential energy principle. Navier-type solution is employed to obtain an analytical solution for deflections of simply supported homogeneous microbeams. A parametric study is performed to determine the effects of shear deformation, material length scale parameter and slenderness ratio on the bending response of microbeams.

2. Preliminaries

Lam et al. [10] simplified the strain gradient theory proposed by Fleck and Hutchinson [18, 21] and introduced a relatively new modified strain gradient theory in which there are three additional material length scale parameters related to the dilatation gradient vector, deviatoric stretch and symmetric rotation gradient tensors in addition two classical ones for linear elastic materials. The strain energy U based on this theory can be written as [10, 26]

$$U = \frac{1}{2}\int_0^L \int_A \left(\sigma_{ij}\varepsilon_{ij} + p_i\gamma_i + \tau_{ijk}^{(1)}\eta_{ijk}^{(1)} + m_{ij}^s\chi_{ij}^s\right) dA\, dx \tag{1}$$

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{1}$$

$$\gamma_i = \varepsilon_{mm,i} \tag{3}$$

$$\eta_{ijk}^{(1)} = \frac{1}{3}\left(\varepsilon_{jk,i} + \varepsilon_{ki,j} + \varepsilon_{ij,k}\right) - \frac{1}{15}\left[\delta_{ij}\left(\varepsilon_{mm,k} + 2\varepsilon_{mk,m}\right) + \delta_{jk}\left(\varepsilon_{mm,i} + 2\varepsilon_{mi,m}\right) + \delta_{ki}\left(\varepsilon_{mm,j} + 2\varepsilon_{mj,m}\right)\right] \tag{4}$$

$$\chi_{ij}^s = \frac{1}{2}\left(\theta_{i,j} + \theta_{j,i}\right) \tag{5}$$

$$\theta_i = \frac{1}{2}e_{ijk}u_{k,j} \tag{6}$$

where u_i, θ_i, ε_{ij}, γ_i, $\eta_{ijk}^{(1)}$ and χ_{ij}^s represent the components of the displacement vector $\mathbf{u}$, the rotation vector $\boldsymbol{\theta}$, the strain tensor $\boldsymbol{\varepsilon}$, the dilatation gradient vector $\boldsymbol{\gamma}$, the deviatoric stretch gradient tensor $\boldsymbol{\eta}^{(1)}$ and the symmetric rotation gradient tensor $\boldsymbol{\chi}^s$, respectively. Also, δ and e_{ijk} are the Kronecker delta and the permutation symbols, respectively. On the other hand, the components of the classical stress tensor $\boldsymbol{\sigma}$ (conjugated with the strain tensor) and the higher-order stress tensors $\mathbf{p}$, $\boldsymbol{\tau}^{(1)}$ and $\mathbf{m}^s$ (conjugated with the higher-order deformation gradient tensors) can be expressed by [10]

$$\sigma_{ij} = \lambda\varepsilon_{mm}\delta_{ij} + 2\mu\varepsilon_{ij} \tag{7}$$

$$p_i = 2\mu l_0^2\gamma_i \tag{8}$$

$$\tau_{ijk}^{(1)} = 2\mu l_1^2\eta_{ijk}^{(1)} \tag{9}$$

$$m_{ij}^s = 2\mu l_2^2\chi_{ij}^s \tag{10}$$

where l_0, l_1, l_2 are additional material length scale parameters related to dilatation gradients, deviatoric stretch gradients and rotation gradients, respectively. Furthermore, λ and μ are the Lamé constants defined as follows

$$\lambda = \frac{Ev}{(1+v)(1-2v)}, \quad \mu = \frac{E}{2(1+v)} \tag{11}$$

3. Hyperbolic shear deformable microbeam model

In this section, the equilibrium equation and corresponding boundary conditions of a hyperbolic shear deformable beam model are derived. The displacement components of an initially straight beam on the basis of hyperbolic beam theory (HBT) can be expressed as [40]

$$u_1(x,z) = -z\frac{\mathrm{d}w(x)}{\mathrm{d}x} + H(z)\left(\frac{dw}{dx} - \varphi\right)$$

$$u_2(x,z) = 0$$

$$u_3(x,z) = w(x) \tag{12}$$

in which

$$H(z) = h\sinh\left(\frac{z}{h}\right) - z\cosh\left(\frac{1}{2}\right) \tag{13}$$

where u_1, u_2 and u_3 are the $x-$, $y-$ and $z-$ components of the displacement vector, and also w is the transverse displacement, φ is the angle of rotation of the cross-sections about y −axis of any point on the mid-plane of the beam, respectively. $H(z)$ is a function which depends on z and provide zero-shear stress and strain conditions at the top and bottom surfaces of the beam. It is notable that the displacement components for BET can be directly achieved by setting $H(z)$ in Eq. (12) equal to zero. Substituting Eqs. (12) and (13) into Eq. (2), we obtain the non-zero strain components as

$$\varepsilon_{11} = -z\frac{d^2w}{dx^2} + H\left(\frac{d^2w}{dx^2} - \frac{d\varphi}{dx}\right),\ \gamma_{13} = 2\varepsilon_{13} = \frac{dH}{dz}\left(\frac{dw}{dx} - \varphi\right) \tag{14}$$

and the non-zero components of higher-order gradients are determined by implementing Eqs. (12)-(14) in Eqs. (3)-(5)

$$\gamma_1 = -z\frac{d^3w}{dx^3} + H\left(\frac{d^3w}{dx^3} - \frac{d^2\varphi}{dx^2}\right),\quad \gamma_3 = -\frac{d^2w}{dx^2} + \frac{dH}{dz}\left(\frac{d^2w}{dx^2} - \frac{d\varphi}{dx}\right) \tag{15}$$

$$\begin{gathered}
\eta_{111}^{(1)} = \frac{1}{5}\left[2\left(-z\frac{d^3w}{dx^3} + H\left(\frac{d^3w}{dx^3} - \frac{d^2\varphi}{dx^2}\right)\right) - \frac{d^2H}{dz^2}\left(\frac{dw}{dx} - \varphi\right)\right],\\
\eta_{113}^{(1)} = \eta_{131}^{(1)} = \eta_{311}^{(1)} = -\frac{4}{15}\left[\frac{d^2w}{dx^2} - 2\frac{dH}{dz}\left(\frac{d^2w}{dx^2} - \frac{d\varphi}{dx}\right)\right],\\
\eta_{122}^{(1)} = \eta_{212}^{(1)} = \eta_{221}^{(1)} = -\frac{1}{5}\left[\left(-z\frac{d^3w}{dx^3} + H\left(\frac{d^3w}{dx^3} - \frac{d^2\varphi}{dx^2}\right)\right) + \frac{1}{3}\frac{d^2H}{dz^2}\left(\frac{dw}{dx} - \varphi\right)\right],\\
\eta_{133}^{(1)} = \eta_{313}^{(1)} = \eta_{331}^{(1)} = -\frac{1}{5}\left[\left(-z\frac{d^3w}{dx^3} + H\left(\frac{d^3w}{dx^3} - \frac{d^2\varphi}{dx^2}\right)\right) - \frac{4}{3}\frac{d^2H}{dz^2}\left(\frac{dw}{dx} - \varphi\right)\right],\\
\eta_{223}^{(1)} = \eta_{232}^{(1)} = \eta_{322}^{(1)} = \frac{1}{3}\eta_{333}^{(1)} = \frac{1}{15}\left[\frac{d^2w}{dx^2} - 2\frac{dH}{dz}\left(\frac{d^2w}{dx^2} - \frac{d\varphi}{dx}\right)\right]
\end{gathered} \tag{16}$$

$$\begin{gathered}
\chi_{12}^{s} = \chi_{21}^{s} = -\frac{1}{2}\left[\frac{d^2w}{dx^2} - \frac{1}{2}\frac{dH}{dz}\left(\frac{d^2w}{dx^2} - \frac{d\varphi}{dx}\right)\right],\\
\chi_{23}^{s} = \chi_{32}^{s} = \frac{1}{4}\frac{d^2H}{dz^2}\left(\frac{dw}{dx} - \varphi\right)
\end{gathered} \tag{17}$$

By implementing Eq. (14) in Eq. (7), the non-zero components of classical stress tensor $\boldsymbol{\sigma}$ can be written as

$$\begin{gathered}
\sigma_{11} = E\eta\left(-z\frac{d^2w}{dx^2} + H\left(\frac{d^2w}{dx^2} - \frac{d\varphi}{dx}\right)\right),\qquad \sigma_{13} = \mu\frac{dH}{dz}\left(\frac{dw}{dx} - \varphi\right)\\
\sigma_{22} = \sigma_{33} = \frac{Ev}{(1+v)(1-2v)}\left(-z\frac{d^2w}{dx^2} + H\left(\frac{d^2w}{dx^2} - \frac{d\varphi}{dx}\right)\right)
\end{gathered} \tag{18}$$

in which

$$\eta = \frac{(1-v)}{(1+v)(1-2v)} \tag{19}$$

By employing above related equations into Eqs. (8)-(10), the non-zero components of higher-order stress tensors are achieved as

$$\begin{gathered}
p_1 = 2\mu l_0^2\left(-z\frac{d^3w}{dx^3} + H\left(\frac{d^3w}{dx^3} - \frac{d^2\varphi}{dx^2}\right)\right),\\
p_3 = -2\mu l_0^2\left(\frac{d^2w}{dx^2} - \frac{dH}{dz}\left(\frac{d^2w}{dx^2} - \frac{d\varphi}{dx}\right)\right)\\
\tau_{111}^{(1)} = \frac{2}{5}\mu l_1^2\left[2\left(-z\frac{d^3w}{dx^3} + H\left(\frac{d^3w}{dx^3} - \frac{d^2\varphi}{dx^2}\right)\right) - \frac{d^2H}{dz^2}\left(\frac{dw}{dx} - \varphi\right)\right],\\
\tau_{113}^{(1)} = \tau_{131}^{(1)} = \tau_{311}^{(1)} = -\frac{8}{15}\mu l_1^2\left[\frac{d^2w}{dx^2} - 2\frac{dH}{dz}\left(\frac{d^2w}{dx^2} - \frac{d\varphi}{dx}\right)\right],\\
\tau_{122}^{(1)} = \tau_{212}^{(1)} = \tau_{221}^{(1)} = -\frac{2}{5}\mu l_1^2\left[\left(-z\frac{d^3w}{dx^3} + H\left(\frac{d^3w}{dx^3} - \frac{d^2\varphi}{dx^2}\right)\right) + \frac{1}{3}\frac{d^2H}{dz^2}\left(\frac{dw}{dx} - \varphi\right)\right],
\end{gathered} \tag{20}$$

$$\tau_{133}^{(1)}=\tau_{313}^{(1)}=\tau_{331}^{(1)}=-\frac{2}{5}\mu l_1^2\left[\left(-z\frac{d^3w}{dx^3}+H\left(\frac{d^3w}{dx^3}-\frac{d^2\varphi}{dx^2}\right)\right)-\frac{4}{3}\frac{d^2H}{dz^2}\left(\frac{dw}{dx}-\varphi\right)\right],$$
$$\tau_{223}^{(1)}=\tau_{232}^{(1)}=\tau_{322}^{(1)}=\frac{1}{3}\tau_{333}^{(1)}=\frac{2}{15}\mu l_1^2\left[\frac{d^2w}{dx^2}-2\frac{dH}{dz}\left(\frac{d^2w}{dx^2}-\frac{d\varphi}{dx}\right)\right] \tag{21}$$

$$m_{12}^s=m_{21}^s=-\mu l_2^2\left[\frac{d^2w}{dx^2}-\frac{1}{2}\frac{dH}{dz}\left(\frac{d^2w}{dx^2}-\frac{d\varphi}{dx}\right)\right],$$
$$m_{23}^s=m_{32}^s=\frac{\mu l_2^2}{2}\frac{d^2H}{dz^2}\left(\frac{dw}{dx}-\varphi\right) \tag{22}$$

The governing differential equations and corresponding boundary conditions are derived by implementing the minimum total potential energy principle. In this principle, the first variation of total potential energy should be equal to zero as following

$$\delta\prod=\delta U-\delta W=0 \tag{23}$$

where $\prod$ is the total potential energy, δU and δW are the first variations of strain energy and work done by external forces, respectively. After some mathematical manipulations and using the fundamental lemma of the calculus of variation, the following equilibrium equations for hyperbolic shear deformable microbeam model can be achieved as

$$\begin{aligned}\delta w:\quad &-\left(b_5+\frac{8}{15}b_{141}+\frac{1}{4}b_{142}\right)\frac{d^2w}{dx^2}\\
&+\Bigg(b_2-2b_3+b_4+2b_{60}-4b_{100}+2b_{110}-\frac{4}{5}(b_{121}-b_{131})+\frac{8}{15}b_{61}-\frac{32}{15}(b_{101}-b_{111})+b_{62}-b_{102}\\
&+\frac{1}{4}b_{112}\Bigg)\frac{d^4w}{dx^4}-\left(2(b_{70}-2b_{80}+b_{90})+\frac{4}{5}(b_{71}-2b_{81}+b_{91})\right)\frac{d^6w}{dx^6}\\
&+\left(b_5+\frac{8}{15}b_{141}+\frac{1}{4}b_{142}\right)\frac{d\varphi}{dx}-\Bigg(b_4-b_3+2(b_{110}-b_{100})\\
&+\frac{2}{5}(2b_{131}-b_{121})+\frac{16}{15}(2b_{111}-b_{101})+\frac{1}{4}b_{112}-\frac{1}{2}b_{102}\Bigg)\frac{d^3\varphi}{dx^3}\\
&+\left(2(b_{90}-b_{80})+\frac{4}{5}(b_{91}-b_{81})\right)\frac{d^5\varphi}{dx^5}-q=0\end{aligned} \tag{24}$$

$$\begin{aligned}\delta\varphi:\quad &-\left(b_5+\frac{8}{15}b_{141}+\frac{1}{4}b_{142}\right)\frac{dw}{dx}+\Bigg(b_4-b_3+2(b_{110}-b_{100})\\
&+\frac{2}{5}(2b_{131}-b_{121})+\frac{16}{15}(2b_{111}-b_{101})+\frac{1}{4}b_{112}-\frac{1}{2}b_{102}\Bigg)\frac{d^3w}{dx^3}\\
&-\left(2(b_{90}-b_{80})+\frac{4}{5}(b_{91}-b_{81})\right)\frac{d^5w}{dx^5}+\left(b_5+\frac{8}{15}b_{141}+\frac{1}{4}b_{142}\right)\varphi\\
&-\left(b_4+2b_{110}+\frac{4}{5}b_{131}+\frac{32}{15}b_{111}+\frac{1}{4}b_{112}\right)\frac{d^2\varphi}{dx^2}\\
&+\left(2b_{90}+\frac{4}{5}b_{91}\right)\frac{d^4\varphi}{dx^4}=0\end{aligned} \tag{25}$$

and corresponding boundary conditions are written at $x=0$ and $x=L$ as follows

$$\begin{aligned}&\left(b_5+\frac{8}{15}b_{141}+\frac{1}{4}b_{142}\right)\frac{dw}{dx}-\Bigg(b_2-2b_3+b_4+2b_{60}-4b_{100}+2b_{110}-\frac{4}{5}(b_{121}-b_{131})\\
&+\frac{8}{15}b_{61}-\frac{32}{15}(b_{101}-b_{111})+b_{62}-b_{102}+\frac{1}{4}b_{112}\Bigg)\frac{d^3w}{dx^3}+\left(2(b_{70}-2b_{80}+b_{90})+\frac{4}{5}(b_{71}-2b_{81}+b_{91})\right)\frac{d^5w}{dx^5}\\
&-\left(b_5+\frac{8}{15}b_{141}+\frac{1}{4}b_{142}\right)\varphi+\Bigg(b_4-b_3+2(b_{110}-b_{100})+\frac{2}{5}(2b_{131}-b_{121})\\
&+\frac{16}{15}(2b_{111}-b_{101})+\frac{1}{4}b_{112}-\frac{1}{2}b_{102}\Bigg)\frac{d^2\varphi}{dx^2}\\
&-\left(2(b_{90}-b_{80})+\frac{4}{5}(b_{91}-b_{81})\right)\frac{d^4\varphi}{dx^4}-\hat{Q}_1=0\quad\text{or}\qquad w=0\end{aligned} \tag{26}$$

$$\begin{aligned}&\Bigg(b_2-2b_3+b_4+2b_{60}-4b_{100}+2b_{110}-\frac{2}{5}(b_{121}-b_{131})\\
&+\frac{8}{15}b_{61}-\frac{32}{15}(b_{101}-b_{111})+b_{62}-b_{102}+\frac{1}{4}b_{112}\Bigg)\frac{d^2w}{dx^2}\\
&-\left(2(b_{70}-2b_{80}+b_{90})+\frac{4}{5}(b_{71}-2b_{81}+b_{91})\right)\frac{d^4w}{dx^4}\end{aligned}$$

$$-\left(b_4 - b_3 + 2(b_{110} - b_{100}) + \frac{2}{5}(b_{131} - b_{121}) + \frac{16}{15}(2b_{111} - b_{101}) + \frac{1}{4}b_{112} - \frac{1}{2}b_{102}\right)\frac{d\varphi}{dx} + \left(2(b_{90} - b_{80}) + \frac{4}{5}(b_{91} - b_{81})\right)\frac{d^3\varphi}{dx^3} - \hat{Q}_2 = 0 \quad \text{or} \quad \frac{dw}{dx} = 0 \tag{27}$$

$$\left(\frac{2}{5}(b_{121} - b_{131})\right)\frac{dw}{dx} + \left(2(b_{70} - 2b_{80} + b_{90}) + \frac{4}{5}(b_{71} - 2b_{81} + b_{91})\right)\frac{d^3w}{dx^3} - \left(\frac{2}{5}(b_{121} - b_{131})\right)\varphi - \left(2(b_{90} - b_{80}) + \frac{4}{5}(b_{91} - b_{81})\right)\frac{d^2\varphi}{dx^2} - \hat{Q}_3 = 0 \quad \text{or} \tag{28}$$

$$\frac{d^2w}{dx^2} = -\left(b_4 - b_3 + 2(b_{110} - b_{100}) + \frac{2}{5}b_{131} + \frac{16}{15}(2b_{111} - b_{101}) + \frac{1}{4}b_{112} - \frac{1}{2}b_{102}\right)\frac{d^2w}{dx^2} + \left(2(b_{90} - b_{80}) + \frac{4}{5}(b_{91} - b_{81})\right)\frac{d^4w}{dx^4} + \left(b_4 + \frac{2}{5}b_{131} + \frac{32}{15}b_{111} + \frac{1}{4}b_{112}\right)\frac{d\varphi}{dx} - \left(2b_{90} + \frac{4}{5}b_{91}\right)\frac{d^3\varphi}{dx^3} - \hat{Q}_4 = 0 \quad \text{or} \quad \varphi = 0 \tag{29}$$

$$\left(\frac{2}{5}b_{131}\right)\frac{dw}{dx} - \left(2(b_{90} - b_{80}) + \frac{4}{5}(b_{91} - b_{81})\right)\frac{d^3w}{dx^3} - \left(\frac{2}{5}b_{131}\right)\varphi + \left(2b_{90} + \frac{4}{5}b_{91}\right)\frac{d^2\varphi}{dx^2} - \hat{Q}_5 = 0 \quad \text{or} \quad \frac{d\varphi}{dx} = 0 \tag{30}$$

where $q(x)$ is the transverse distributed load. In addition, $\hat{Q}_j$ $(j = 1,2, \dots ,5)$ are the specified forces or moments of them at the end of the microbeam and

$$\{b_1, b_2, b_3, b_4\} = \int_A E\eta\{1, z^2, zH, H^2\}dA,$$
$$b_5 = \int_A \mu\left(\frac{dH}{dz}\right)^2 dA,$$
$$\{b_{6i}, b_{7i}, b_{8i}, b_{9i}, b_{10i}, b_{11i}\} = \int_A \mu l_i^2\left\{1, z^2, Hz, H^2, \left(\frac{dH}{dz}\right), \left(\frac{dH}{dz}\right)^2\right\}dA,$$
$$\{b_{12i}, b_{13}, b_{14i}\} = \int_A \mu l_i^2\left(\frac{d^2H}{dz^2}\right)\left\{z, H, \left(\frac{d^2H}{dz^2}\right)\right\}dA \tag{31}$$

4. Analytical solutions for static bending problem of microbeams

In this section, Navier's solution procedure is employed to solve the static bending problem of simply supported microbeams. The following expansions of generalized displacements which include undetermined Fourier coefficients and certain trigonometric functions can be successfully employed as

$$w(x) = \sum_{n=1}^{\infty} W_n \sin\beta\, x \tag{32}$$

$$\varphi(x) = \sum_{n=1}^{\infty} R_n \cos\beta x \tag{33}$$

where W_n and R_n are the undetermined Fourier coefficients and $\beta = n\pi/L$. This means that Eqs. (32) and (33) must satisfy the corresponding boundary conditions. It is seen from the previous works that these functions are valid for the simply supported microbeams [45-50]. On the other hand, the external applied force q can be expanded by Fourier series with Fourier coefficient Q_n as following

$$q(x) = \sum_{n=1}^{\infty} Q_n \sin\beta\, x \tag{34}$$

$$Q_n = \frac{2}{L}\int_0^L q(x)\sin\beta x\, dx \tag{35}$$

and Q_n can be expressed as in the case of point load at the middle of the microbeam as

$$q(x) = P\delta(x - L/2) \tag{36}$$
$$Q_n = \frac{2P}{L}\sin\frac{n|\pi}{2} \quad \text{for } n = 1,2,3, \dots \tag{37}$$

where δ is the Dirac delta function, P is the magnitude of the point load Substituting Eqs. (32)-(37) into Eqs. (24) and (25), the following relation is achieved as

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} W_n \\ R_n \end{Bmatrix} = \begin{Bmatrix} Q_n \\ 0 \end{Bmatrix} \tag{38}$$

where

$$\begin{aligned} K_{11} &= \beta^2\left(b_5 + \frac{8}{15}b_{141} + \frac{1}{4}b_{142}\right) + \beta^4\Big(b_2 - 2b_3 + b_4 + 2b_{60} - 4b_{100} + 2b_{110} \\ &\quad - \frac{4}{5}(b_{121} - b_{131}) + \frac{8}{15}b_{61} - \frac{32}{15}(b_{101} - b_{111}) + b_{62} - b_{102} + \frac{1}{4}b_{112}\Big) \\ &\quad + \beta^6\left(2(b_{70} - 2b_{80} + b_{90}) + \frac{4}{5}(b_{71} - 2b_{81} + b_{91})\right) \\ K_{12} = K_{21} &= -\beta\left(b_5 + \frac{8}{15}b_{141} + \frac{1}{4}b_{142}\right) - \beta^3\Big(b_4 - b_3 + 2(b_{110} - b_{100}) + \frac{2}{5}(2b_{131} - b_{121}) + \frac{16}{15}(2b_{111} - b_{101}) \\ &\quad + \frac{1}{4}b_{112} - \frac{1}{2}b_{102}\Big) - \beta^5\left(2(b_{90} - b_{80}) + \frac{4}{5}(b_{91} - b_{81})\right) \\ K_{22} &= \left(b_5 + \frac{8}{15}b_{141} + \frac{1}{4}b_{142}\right) + \beta^2\left(b_4 + 2b_{110} + \frac{4}{5}b_{131} + \frac{32}{15}b_{111} + \frac{1}{4}b_{112}\right) + \beta^4\left(2b_{90} + \frac{4}{5}b_{91}\right) \end{aligned} \tag{39}$$

Solving the above algebraic equations set in Eq. (38), the Fourier coefficients R_n and W_n can be determined. Analytical expressions of $w(x)$ and $\varphi(x)$ will be obtained for the static bending of simply supported microbeams under point load by substituting these coefficients into Eqs. (32) and (33).

5. Numerical results and discussion

Static bending problem of a simply supported microbeam subjected to a concentrated load at the mid-span of the microbeam is analytically solved with Navier's solution procedure based on hyperbolic shear deformable microbeam model. For illustration purpose, the microbeam is taken to be made of epoxy with the following material properties: the Young's modulus E$= 1.44\ GPa$, the Poisson's ratio $v = 0.38$, $\eta = 1$, $b = 2h$ and the material length scale parameter $(l_0 = l_1 = l_2 = l)$ $l = 11.01\mu m$ [33,54].

Variations of dimensionless maximum deflections of the simply supported microbeam under point load are depicted in Figs. 1 and 2 based on different beam theories for various values of slenderness ratio of microbeam, respectively. It is shown that an increase in the slenderness ratio leads to a decrease in the dimensionless deflections. Moreover, it can be observed that the dimensionless maximum deflections evaluated by HBT are higher than those obtained by BET due to effects of shear deformation and it is more considerable for lower slenderness ratios.

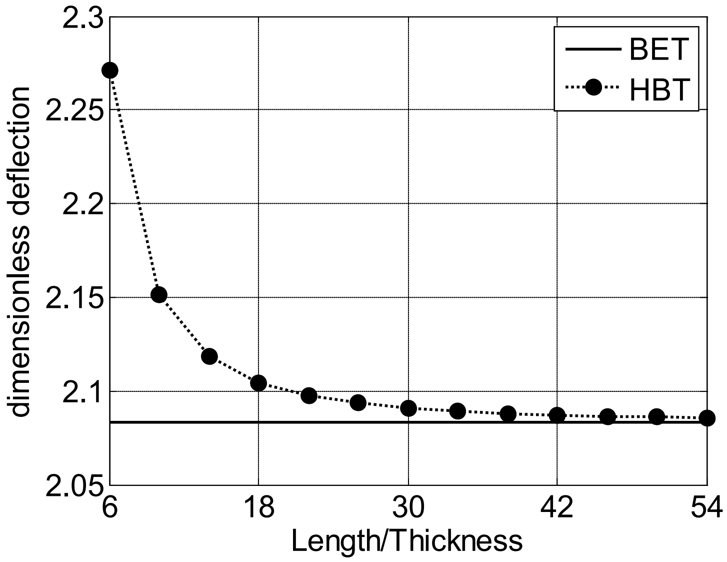

Fig. 1 Variation of the dimensionless center deflection with respect to slenderness ratio corresponding to different beam theories based on CT

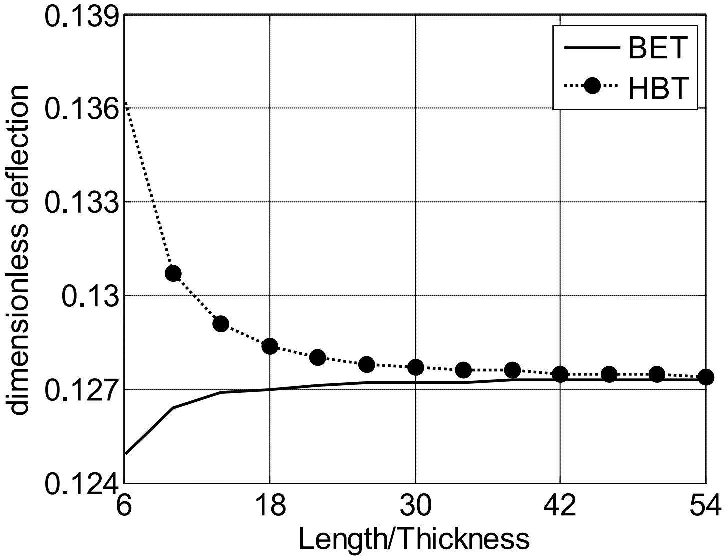

Fig. 2 Variation of the dimensionless center deflection with respect to slenderness ratio corresponding to different beam theories based on MSGT

6. Conclusions

Size-dependent static bending analysis of a microbeam subjected to a point load is investigated based on hyperbolic shear deformable beam theory in conjunctions with modified strain gradient theory. The equilibrium equations and corresponding boundary conditions in bending are derived by implementing minimum total potential energy principle. Analytical solutions for deflections under sinusoidal load for simply supported homogeneous microbeams are presented by Navier solution procedure. The influences of shear deformation, material length scale parameter and slenderness ratio on the bending response of microbeams are investigated in detail. It is observed from the results that effects of shear deformation lead to an increment in deflections and these effects may become more considerable for lower slenderness ratios. Moreover, it can be interpreted that the microbeam model based on MSGT is stiffer than those based on CT.

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References

[1] Younis, M.I., Abdel-Rahman, E.M., Nayfeh, A.H., "A reduced-order model for electrically actuated microbeam-based MEMS", Journal of Microelectromechanical Systems, Vol. 12, pp. 672–680, 2003.

[2] Li, P., Fang, Y., "A molecular dynamics simulation approach for the squeeze-film damping of MEMS devices in the free molecular regime", Journal of Micromechanics and Microengineering, Vol. 20, 035005, 2010.

[3] Wu, Z.Y., Yang, H., Li, X.X., Wang, Y.L., "Self-assembly and transfer of photoresist suspended over trenches for microbeam fabrication in MEMS", Journal of Micromechanics and Microengineering, Vol. 20, 115014, 2010.

[4] Zook, J.D., Burns, D.W., Guckel, H., Sniegowski, J.J., Engelstad, R.L., Feng, Z., "Characteristics of polysilicon resonant microbeams", Sensors and Actuators A: Physics, Vol. 35, pp. 51–59, 1992.

[5] Torii, A., Sasaki, M., Hane, K., Okuma, S., "Adhesive force distribution on microstructures investigated by an atomic force microscope", Sensors and Actuators A: Physics, Vol. 44, pp. 153–158, 1994.

[6] Hung, E.S., Senturia, S.D., "Extending the travel range of analog-tuned electrostatic actuators", Journal of Microelectromechanical Systems, Vol. 8, pp. 497–505, 1999.

[7] Acquaviva, D., Arun, A., Smajda, R., Grogg, D., Magrez, A., Skotnicki, T., Ionescu, A.M., "Micro-Electro-Mechanical Switch Based on Suspended Horizontal Dense Mat of CNTs by FIB Nanomanipulation", Procedia Chemistry, Vol. 1, pp. 1411–1414, 2009.

[8] Poole, W.J., Ashby, M.F., Fleck, N.A., "Micro-hardness of annealed and work- hardened copper polycrystals", Scripta Materialia, Vol. 34, pp. 559–564, 1996.

[9] Stölken, J.S., Evans, A.G., "A microbend test method for measuring the plasticity length scale", Acta Materialia, Vol. 46, pp. 5109–5115, 1998.

[10] Lam, D.C.C., Yang, F., Chong, A.C.M., Wang, J., Tong, P., "Experiments and theory in strain gradient elasticity", Journal of the Mechanics and Physisc of Solids, Vol. 51, pp. 1477–1508, 2003.

[11] McFarland, A.W., Colton, J.S., "Role of material microstructure in plate stiffness with relevance to microcantilever sensors", Journal of Micromechanics and Microengineering, Vol. 15, pp. 1060–1067, 2005.

[12] Mindlin, R.D., Tiersten, H.F., "Effects of couple-stresses in linear elasticity", Archive for Rational Mechanics and Analysis, Vol. 11, pp. 415–448, 1962.

[13] Koiter, W.T., "Couple stresses in the theory of elasticity: I and II", Proceedings of the Koninklijke Nederlandse Akademie van Wetenschappen (B), Vol. 67, pp. 17–44, 1964.
[14] Toupin, R.A., "Theory of elasticity with couple stresses", Archive for Rational Mechanics and Analysis, Vol. 17, pp. 85–112, 1964.
[15] Eringen, A.C., "Theory of micropolar plates". Zeitschrift für angewandte Mathematik und Physik, Vol. 18, pp. 12–30, 1967.
[16] Eringen, A.C., "Nonlocal polar elastic continua", International Journal of Engineering Science, Vol. 10, pp. 1–16, 1972.
[17] Eringen, A.C., "On differential equations of nonlocal elasticity and solutions of screw dislocation and surface waves", Journal of Applied Physics, Vol. 54, pp. 4703–4710, 1983.
[18] Fleck, N.A., Hutchinson, J.W., "A phenomenological theory for strain gradient effects in plasticity", Journal of the Mechanics and Physics of Solids, Vol. 41, pp. 1825–1857, 1993.
[19] Vardoulakis, I., Sulem, J., "*Bifurcation Analysis in Geomechanics*". Blackie/Chapman and Hall, London, 1995.
[20] Aifantis, E.C., "Gradient deformation models at nano, micro, and macro scales", Journal of Engineering Materials and Technology, Vol. 121, pp. 189–202, 1999.
[21] Fleck, N.A., Hutchinson, J.W., "A reformulation of strain gradient plasticity", Journal of the Mechanics and Physics of Solids, Vol. 49, pp. 2245–2271, 2001.
[22] Akgöz, B., Civalek, Ö., "Longitudinal vibration analysis for microbars based on strain gradient elasticity theory", Journal of Vibration and Control, Vol. 20, pp. 606–616, 2001.
[23] Akgöz, B., Civalek, Ö., "Longitudinal vibration analysis of strain gradient bars made of functionally graded materials (FGM)", Composites Part B, Vol. 55, pp. 263–268, 2013.
[24] Kahrobaiyan, M.H., Asghari, M., Ahmadian, M.T., "Longitudinal behavior of strain gradient bars", International Journal of Engineering Science, Vol. 66–67, pp. 44–59, 2013.
[25] Kahrobaiyan, M.H., Tajalli, S.A., Movahhedy, M.R., Akbari, J., Ahmadian, M.T., "Torsion of strain gradient bars", International Journal of Engineering Science, Vol. 49, pp. 856–866, 2011.
[26] Kong, S., Zhou, S., Nie, Z., Wang, K., "Static and dynamic analysis of micro beams based on strain gradient elasticity theory", International Journal of Engineering Science, Vol. 47, pp. 487–498, 2009.
[27] Wang, B., Zhao, J., Zhou, S., "A micro scale Timoshenko beam model based on strain gradient elasticity theory", European Journal of Mechanics A/Solids, Vol. 29, pp. 591–599, 2010.
[28] Akgöz, B., Civalek, Ö., "Strain gradient elasticity and modified couple stress models for buckling analysis of axially loaded micro-scaled beams", International Journal of Engineering Science, Vol. 49, pp. 1268–1280, 2011.
[29] Akgöz, B., Civalek, Ö., "Analysis of micro-sized beams for various boundary conditions based on the strain gradient elasticity theory", Archive of Applied Mechanics, Vol. 82, pp. 423–443, 2012.
[30] Akgöz, B., Civalek, Ö., "Buckling analysis of linearly tapered micro-columns based on strain gradient elasticity", Structural Engineering and Mechanics, Vol. 48, pp. 195–205, 2013.
[31] Asghari, M., Kahrobaiyan, M.H., Nikfar, M., Ahmadian, M.T., "A size-dependent nonlinear Timoshenko microbeam model based on the strain gradient theory", Acta Mechanica, Vol. 223, 1233–1249, 2012.
[32] Ghayesh, M.H., Amabili, M., Farokhi, H., "Nonlinear forced vibrations of a microbeam based on the strain gradient elasticity theory", International Journal of Engineering Science, Vol. 63, pp. 52–60, 2013.
[33] Kahrobaiyan, M.H., Asghari, M., Ahmadian, M.T., "Strain gradient beam element", Finite Element Analysis in Design, Vol. 68, pp. 63–75, 2013.
[34] Zhang, B., He, Y., Liu, D., Gan, Z., Shen, L., "Non-classical Timoshenko beam element based on the strain gradient elasticity theory", Finite Element Analysis in Design, Vol. 79, pp. 22–39, 2014.
[35] Mercan, K., Civalek, Ö., "DSC method for buckling analysis of boron nitride nanotube (BNNT) surrounded by an elastic matrix", Composite Structures, Vol. 143, pp. 300–309, 2016.
[36] Demir, Ç., Civalek, Ö., "Torsional and longitudinal frequency and wave response of microtubules based on the nonlocal continuum and nonlocal discrete models", Applied Mathematical Modelling, Vol. 37, pp. 9355–9367, 2013.
[37] Levinson, M., "A new rectangular beam theory", Journal of Sound and Vibration, Vol. 74, pp. 81–87, 1981.
[38] Reddy, J.N., "A simple higher-order theory for laminated composite plates", Journal of Applied Mechanics, Vol. 51, pp. 745–752, 1984.
[39] Touratier, M., "An efficient standard plate theory", International Journal of Engineering Science, Vol. 29, pp. 901–916, 1991.
[40] Soldatos, K.P., "A transverse shear deformation theory for homogeneous monoclinic plates", Acta Mechanica, Vol. 94, pp. 195–220, 1992.
[41] Karama, M., Afaq, K.S., Mistou, S., "Mechanical behaviour of laminated composite beam by the new multi-layered laminated composite structures model with transverse shear stress continuity", International Journal of Solids and Structures, Vol. 40, pp. 1525–1546, 2003.
[42] Aydogdu, M., "A new shear deformation theory for laminated composite plates", Composite Structures, Vol. 89, pp. 94–101, 2009.
[43] Nateghi, A., Salamat-talab, M., Rezapour, J., Daneshian, B., "Size dependent buckling analysis of functionally graded micro beams based on modified couple stress theory", Applied Mathematical Modelling, Vol. 36, pp. 4971–4987, 2012.
[44] Salamat-talab, M., Nateghi, A., Torabi, J., "Static and dynamic analysis of third-order shear deformation FG micro beam based on modified couple stress theory", International Journal of Mechanical Sciences, Vol. 57, pp. 63–73, 2012.
[45] Akgöz, B., Civalek, Ö., "A size-dependent shear deformation beam model based on the strain gradient elasticity theory", International Journal of Engineering Science, Vol. 70, pp. 1–14, 2013.
[46] Lei, J., He, Y., Zhang, B., Gan, Z., Zeng, P., "Bending and vibration of functionally graded sinusoidal microbeams based on the strain gradient elasticity theory", International Journal of Engineering Science, Vol. 72, pp. 36–52, 2013.
[47] Şimşek, M., Reddy, J.N., "Bending and vibration of functionally graded microbeams using a new higher order beam theory and the modified couple stress theory", International Journal of Engineering Science, Vol. 64, pp. 37–53, 2013.
[48] Şimşek, M., Reddy, J.N., "A unified higher order beam theory for buckling of a functionally graded microbeam embedded in elastic medium using modified couple stress theory", Composite Structures, 101, pp. 47–58, 2013.

[49] Akgöz, B., Civalek, Ö., “A new trigonometric beam model for buckling of strain gradient microbeams”, International Journal of Mechanical Sciences, Vol. 57, pp. 88–94, 2014.

[50] Akgöz, B., Civalek, Ö., “Shear deformation beam models for functionally graded microbeams with new shear correction factors”, Composite Structures, Vol. 112, pp. 214–225, 2014.

[51] Akgöz, B., Civalek, Ö., “Thermo-mechanical buckling behavior of functionally graded microbeams embedded in elastic medium”. International Journal of Engineering Science, Vol. 85, pp. 90–104, 2014.

[52] Darijani, H., Mohammadabadi, H., “A new deformation beam theory for static and dynamic analysis of microbeams”, International Journal of Mechanical Sciences, Vol. 89, pp. 31–39, 2014.

[53] Akgöz, B., Civalek, Ö., “A microstructure-dependent sinusoidal plate model based on the strain gradient elasticity theory”, Acta Mechanica, Vol. 226, pp. 2277–2294, 2015.

[54] Akgöz, B., Civalek, Ö., “A novel microstructure-dependent shear deformable beam model”, International Journal of Mechanical Sciences, Vol. 99, pp. 10–20, 2015.

[55] Akgöz, B., Civalek, Ö., “Bending analysis of FG microbeams resting on Winkler elastic foundation via strain gradient elasticity”, Composite Structures, Vol. 134, pp. 294–301, 2015.

[56] Zhang, B., He, Y., Liu, D., Gan, Z., Shen, L., “Size-dependent functionally graded beam model based on an improved third-order shear deformation theory”, European Journal of Mechanics-A/Solids, Vol. 47, pp. 211–230, 2014.