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Bending Analysis of Thick Isotropic Plates by Using 5th Order Shear Deformation Theory

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Abstract

A 5th order shear deformation theory considering transverse shear deformation effect as well as transverse normal strain deformation effect is presented for static flexure analysis of simply supported isotropic plate. The assumed displacement field accounts for non-linear variation of in-plane displacements as well as transverse displacement through the plate thickness. The condition of zero transverse shear stresses on the upper and lower surface of plate is satisfied. Hence the present formulation does not require the shear correction factor generally associated with the first order shear deformable theory. Governing equations and boundary conditions of the theory are obtained using the principle of virtual work. Closed-form analytical solutions for simply supported square isotropic thick plates subjected to single sinusoidal distributed loads are obtained. Numerical results for static flexure analysis include the effects of side to thickness ratio and plate aspect ratio for simply supported isotropic plates. Numerical results are obtained using MATLAB programming. The results of present theory are in close agreement with those of higher order shear deformation theories and exact 3D elasticity solutions.

Keywords: Thick isotropic plate, 5th order shear deformation theory, static flexure, transverse shear stress, transverse normal stress, Navier solution.

1. Introduction

Plates are the basic structural components that are widely used in various engineering disciplines such as aerospace, civil, marine and mechanical engineering. The transverse shear and transverse normal deformation effects are more pronounced in shear flexible plates which may be made up of isotropic, orthotropic, anisotropic or laminated composite materials. To address the correct structural behavior of structural elements made up of these materials; development of refined theories, which consider refined effects in static and dynamic analysis of structural elements, becomes necessary.

The wide spread use of shear flexible materials has stimulated interest in the accurate prediction of structural behavior of thick plates. Thick beams and plates, either isotropic or anisotropic, basically form two and three dimensional problems of elasticity theory. Reduction of these problems to the corresponding one and two dimensional approximate problems for their analysis has always been the main objective of research workers. The shear deformation effects are more pronounced in the thick plates when subjected to transverse loads than in the thin plates under similar loading. These effects are neglected in classical plate theory. To describe the correct bending behavior of thick plates including shear deformation effects and the associated cross sectional warping, shear deformation theories are required. A comprehensive review of refined shear deformation theories for isotropic and anisotropic laminated plates is given by Ghugal and Shimpi [1] and further recently reviewed by Sayyad and Ghugal [2].

It is well-known that the classical plate theory based on Navier-Kirchhoff hypothesis underpredicts deflections and overpredicts natural frequencies and buckling loads due to the neglect of transverse shear deformation effects. This theory is known as classical plate theory. Timoshenko and Woinowsky-Krieger [3] presented the static flexure and buckling solutions various thin plates subjected various loading and boundary conditions.

The progress in the theory of plates formulation made in 1789-1988 has been carefully reviewed by Jemielita [4]. Refined theories, mainly due to Levy [5], Reissner [6], Hencky [7], Mindlin [8] and Kromm [9] are improvements over the classical plate theory in which the effect of transverse shear deformation is included. Reissner's theory is stress based and Hencky, Mindlin theories are displacement based in which the displacements are expanded in powers of the thickness of plate. These theories are well-known as first order shear deformation theories (FSDTs) in the literature and widely referred to as Reissner-Mindlin plate theory. These theories, however, do not satisfy the shear stress free boundary conditions on the surfaces of the plate and require shear correction factors to consider strain energy due shear deformation appropriately. The deficiencies in classical and first order shear deformation theories led to the development of higher order or equivalent shear deformation theories.

The first refined (higher order) theories are due to Levy's paper of 1877 [5]. Hundred years later in 1977 Lo, Christensen and Wu [10, 11] developed a consistent higher order theory, based on Levy's kinematic hypothesis, for homogeneous and laminated plates including effects of transverse shear deformation, transverse normal strain and a nonlinear distribution of the in-plane and transverse displacements with respect to the thickness coordinate. Lo et al. theory contains eleven unknown displacement variables. It is basically a third order unconstrained shear deformation theory which includes transverse shear and normal deformation effects.

Many higher order shear deformation theories are developed later based on Lo et al., theory which are either constrained or unconstrained theories. These theories are reviewed in great details in Ref. [1] and [2]. Kant [12] studied the bending behavior of a thick homogeneous and isotropic rectangular plate using higher order shear and normal deformation theory in conjunction with segmentation method. A linear elastic analysis is presented. Kant and Swaminathan [13] presented a review of various unconstrained models derived from Lo et al. third order theory for the static analysis of composite plates. The third-order parabolic shear deformation theories satisfying shear stress free boundary conditions on the bounding planes of plate are studied by many researchers and critically reviewed by Jemielita [14]. It is shown by the Jemielita that the kinematical hypotheses proposed by various authors are the sub set of kinematical hypothesis proposed by Vlasov in 1957 [15, 16]. Among the several, third order shear deformation theory of Reddy [17, 18] is very well known and well established in the static and dynamic analyses of isotropic and composite plates.

In 1877 Levy [5] also developed a refined theory for thick plate for the first time using sinusoidal functions in the displacement field in terms of thickness coordinate to include the effects of transverse shear deformation. This belongs to a non-polynomial class of theories and termed as trigonometric shear deformation theory. However, efficiency of this plate theory was not assessed for more than a century. The discussion on Levy's theory can be found in history of theory of elasticity by Todhunter and Pearson [19]. Jemielita in Ref. [4] remarked that the Levy's theory would be remembered by next generation of researchers and its results taken into account and referred to in further developments. Sine function for describing the warping through the thickness of plate has been implemented in shear deformation theory by Touratier [20]. Ghugal and Sayyad [21-23], Sayyad and Ghugal [24-28] presented further developments in this theory including transverse normal strain effect (a qusi-3D sinusoidal shear deformation theory) and applied to bidirectional and one dimensional static and dynamic analyses of isotropic and composite plates. It is shown that sinusoidal or trigonometric shear deformation theory based Levy's kinematical hypothesis is very simple, efficient and accurate for the analysis of thick plates. This theory satisfies shear stress free boundary conditions on bounding planes of the plate exactly and obviates the need of shear correction factors, and kinematic is independent of material behavior.

Third order shear deformation theories are extensively studied by many researchers for the analysis of homogeneous, isotropic, orthotropic, anisotropic and specially orthotropic composite plates to derive the correct structural characteristics of plates subjected to mechanical and thermal loads. Because of the isotropic nature of thermal loadings, plates theories that neglect transverse normal strain lead to very inaccurate results in both thick and thin plates analysis. At least a parabolic expansion of transverse displacement is required to capture linear thermal strains. In view of this, Carrara [29] and Rohwer et al. [30] recommended that the theories with fifth order expansion or higher than fifth order including transverse strain effect are required for acceptable description of thermal response. This has motivated the authors of present paper to investigate the bending response of thick isotropic plates and composite plates.

In this paper a fifth order shear deformation theory with transverse normal strain effect is developed for the first time and accurate description transverse shear stresses is presented. The displacements and stresses obtained are compared with those of exact elasticity solutions in case of isotropic plate. The generalized theory formulation suitable for homogeneous, linearly elastic, isotropic and orthotropic composite thick plate is presented.

2. Theoretical formulation

2.1. Isotropic Plate Under Consideration

Consider a plate made up of isotropic material as shown in Fig. 1. The plate occupies a region given by Eq. (1):

$$0 \le x \le a, \qquad 0 \le y \le b, \qquad -h/2 \le z \le h/2. \tag{1}$$

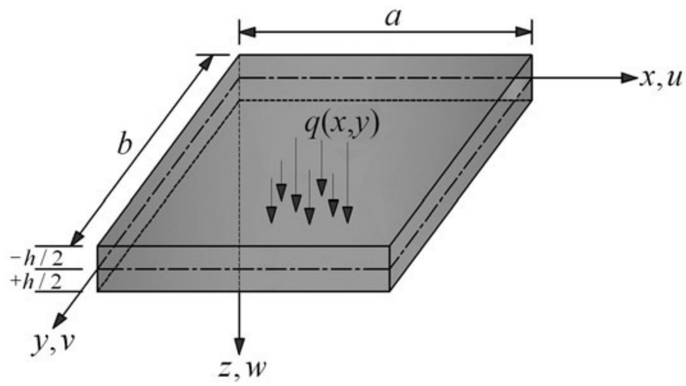

Fig. 1. Plate geometry and co-ordinate system

2.2. Kinematic Assumptions made in the proposed theory

The displacement components U and V are the displacements in x and y -directions and each consists of extension, bending and shear components as

$$U = u_0 + u_b + u_s \quad \text{and} \quad V = v_0 + v_b + v_s \tag{2}$$

The extension components u_0 and v_0 are the middle surface displacements in the x and y directions, respectively.

The bending components u_b and v_b are assumed to be same as those given by the CPT. Therefore, the expressions for bending components are given as:

$$u_b = -z\frac{\partial w(x,y)}{\partial x} \quad \text{and} \quad v_b = -z\frac{\partial w(x,y)}{\partial y} \tag{3}$$

The displacements due to shear deformation u_s and v_s are assumed to be cubic and fifth order in thickness coordinate z such that maximum shear stress occurs at neutral plane of the plate. These components are as follows:

$$\begin{aligned} u_s &= f_1(z)\,\phi_x(x,y) + f_2(z)\,\psi_x(x,y) \\ v_s &= f_1(z)\,\phi_y(x,y) + f_2(z)\,\psi_y(x,y) \end{aligned} \tag{4}$$

where the functions $f_1(z)$ and $f_2(z)$ are defined as follows:

$$f_1(z) = z\left(1 - \frac{4}{3}\frac{z^2}{h^2}\right) \text{ and } f_2(z) = z\left(1 - \frac{16}{5}\frac{z^4}{h^4}\right)$$

The transverse displacement w in z-direction is assumed to be a function of x, y, z Co-ordinates to include the effect of transverse normal strain /stress.

$$w = w(x,y) + g_1(z)\,\phi_z(x,y) + g_2(z)\,\psi_z(x,y) \tag{5}$$

where functions $g_1(z)$ and $g_2(z)$ are as given below:

$$g_1(z) = \left(1 - 4\frac{z^2}{h^2}\right),\ g_2(z) = \left(1 - 16\frac{z^4}{h^4}\right)$$

The body forces are ignored in the analysis. The plate is subjected to transverse load only.

2.3. Kinematics of the proposed plate theory

Based upon the before mentioned assumptions, the displacement field of the proposed plate theory is given as below:

$$U(x,y,z)=u_0-z\frac{\partial w}{\partial x}+f_1(z)\ \phi_x(x,y)+f_2(z)\ \psi_x(x,y)$$
$$V(x,y,z)=v_0-z\frac{\partial w}{\partial y}+f_1(z)\ \phi_y(x,y)+f_2(z)\ \psi_y(x,y) \tag{6}$$
$$W(x,y,z)=w(x,y)+g_1(z)\ \phi_z(x,y)+g_2(z)\ \psi_z(x,y)$$

2.4. About the Present Theory

The kinematics of the present theory is based on nine independent field variables. The inplane displacements are expressed in terms of thickness coordinate z of the plate and the highest power of it is five (quintic). Hence it is termed as fifth order shear deformation theory. The transverse displacement contains. The inplane displacements vary non-linearly through the thickness of the plate. The displacement field of the theory enforces the realistic variation of the transverse shear stresses (parabolic and quartic) across the thickness of the plate. These stresses satisfy the shear stress free boundary condition on the bounding planes of the plate. Thus, the theory obviates the need of shear correction factor. The effect of transverse normal strain is considered. Thus, the theory presents the quasi-3D description of bending response. The usage of theory could be very effective in the bending, buckling, vibration and thermal analysis of nonhomogeneous, anisotropic, composite or sandwich thick plates within the scope of linear elasticity with small deformations.

2.5. Strain-displacement relationships

Normal strains (ε_x, ε_y and ε_z) and shear strains (γ_{xy}, γ_{zx}, γ_{yz}) are obtained within the framework of linear theory of elasticity using the displacement field given by Eqns. (6).

$$\varepsilon_x=\frac{\partial U}{\partial x}=\frac{\partial u_0}{\partial x}-z\frac{\partial^2 w}{\partial x^2}+f_1(z)\ \frac{\partial^2\phi_x}{\partial x}+f_2(z)\ \frac{\partial^2\psi_x}{\partial x}$$

$$\varepsilon_y=\frac{\partial V}{\partial y}=\frac{\partial v_0}{\partial y}-z\frac{\partial^2 w}{\partial y^2}+f_1(z)\ \frac{\partial^2\phi_y}{\partial y}+f_2(z)\ \frac{\partial^2\psi_y}{\partial y}$$

$$\varepsilon_z=\frac{\partial W}{\partial z}=-8\frac{z}{h^2}\ \phi_z-64\frac{z^3}{h^4}\ \psi_z$$

$$\gamma_{xy}=\frac{\partial V}{\partial x}+\frac{\partial U}{\partial y}=\frac{\partial u_0}{\partial y}+\frac{\partial v_0}{\partial x}-2z\frac{\partial^2 w}{\partial x\,\partial y}+f_1(z)\frac{\partial\phi_x}{\partial y}+f_1(z)\frac{\partial\phi_y}{\partial x}+f_2(z)\frac{\partial\psi_x}{\partial y}+f_2(z)\frac{\partial\psi_y}{\partial x}$$

$$\gamma_{zx}=\frac{\partial U}{\partial z}+\frac{\partial W}{\partial x}=f_1'(z)\ \phi_x+f_2'(z)\ \psi_x+g_1(z)\frac{\partial\phi_z}{\partial x}+g_2(z)\frac{\partial\psi_z}{\partial x}$$

$$\gamma_{yz}=\frac{\partial V}{\partial z}+\frac{\partial W}{\partial y}=f_1'(z)\ \phi_y+f_2'(z)\ \psi_y+g_1(z)\frac{\partial\phi_z}{\partial y}+g_2(z)\frac{\partial\psi_z}{\partial y} \tag{7}$$

2.6. Constitutive Equations

Since the thick plate is made up of orthotropic layers, the stress-strain relations in the K^{th} orthotropic layer are given as:

$$\begin{Bmatrix}\sigma_x^k\\ \sigma_y^k\\ \sigma_z^k\end{Bmatrix}=\begin{bmatrix}\overline{Q}_{11}^k & \overline{Q}_{12}^k & \overline{Q}_{13}^k\\ \overline{Q}_{12}^k & \overline{Q}_{22}^k & \overline{Q}_{23}^k\\ \overline{Q}_{13}^k & \overline{Q}_{23}^k & \overline{Q}_{33}^k\end{bmatrix}\begin{Bmatrix}\varepsilon_x^k\\ \varepsilon_y^k\\ \varepsilon_z^k\end{Bmatrix}\quad\text{and}\quad\begin{Bmatrix}\tau_{xy}^k\\ \tau_{zx}^k\\ \tau_{yz}^k\end{Bmatrix}=\begin{bmatrix}\overline{Q}_{44}^k & 0 & 0\\ 0 & \overline{Q}_{55}^k & 0\\ 0 & 0 & \overline{Q}_{66}^k\end{bmatrix}\begin{Bmatrix}\gamma_{xy}^k\\ \gamma_{zx}^k\\ \gamma_{yz}^k\end{Bmatrix} \tag{8}$$

where $\overline{Q}_{ij}^k$ are the transformed material constants, are expressed as:

$$Q_{11}^k = Q_{11}\cos^4\theta_k + 2(Q_{12}+2Q_{44})\sin^2\theta_k\cos^2\theta_k + Q_{22}\sin^4\theta_k$$

$$Q_{12}^k = \left(Q_{11}+Q_{22}-4Q_{44}\right)\sin^2\theta_k\cos^2\theta_k + Q_{12}\left(\sin^4\theta_k+\cos^4\theta_k\right)$$

$$Q_{13}^k = Q_{13}\cos^2\theta_k + Q_{23}\sin^2\theta_k$$

$$Q_{22}^k = Q_{11}\sin^4\theta_k + 2(Q_{12}+2Q_{44})\sin^2\theta_k\cos^2\theta_k + Q_{22}\cos^4\theta_k$$

$$Q_{23}^k = Q_{13}\sin^2\theta_k + Q_{23}\cos^2\theta_k$$

$$Q_{33}^k = Q_{33},\ Q_{44}^k = (Q_{11}+Q_{22}-2Q_{12}-2Q_{44})\sin^2\theta_k\cos^2\theta_k + Q_{44}\left(\sin^4\theta_k+\cos^4\theta_k\right)$$

$$Q_{55}^k = Q_{55}\cos^2\theta_k + Q_{66}\sin^2\theta_k$$

$$Q_{66}^k = Q_{66}\cos^2\theta_k + Q_{55}\sin^2\theta_k \tag{9}$$

where θ_k is the angle of material axes with the reference coordinate axes of each layer and Q_{ij} are the plane stress-reduced stiffness and are known in terms of the engineering constants in the material axes of the layer:

$$Q_{11} = \frac{E_1\left(1-\mu_{23}\mu_{32}\right)}{\Delta};\ Q_{12} = \frac{E_1\left(\mu_{21}-\mu_{31}\mu_{23}\right)}{\Delta};\ Q_{13} = \frac{E_1\left(\mu_{31}-\mu_{21}\mu_{32}\right)}{\Delta}$$

$$Q_{22} = \frac{E_2\left(1-\mu_{13}\mu_{31}\right)}{\Delta};\ Q_{23} = \frac{E_2\left(\mu_{32}-\mu_{12}\mu_{31}\right)}{\Delta};\ Q_{33} = \frac{E_3\left(1-\mu_{12}\mu_{21}\right)}{\Delta}; \tag{10}$$

$$Q_{44} = G_{12};\ Q_{55} = G_{13};\ Q_{66} = G_{23}\ \text{and}\ \Delta = 1 - \mu_{12}\mu_{21} - \mu_{23}\mu_{32} - \mu_{31}\mu_{13} - 2\mu_{21}\mu_{32}\mu_{13}$$

In which E_1, E_2, E_3 are the Young's moduli in the *x, y* and *z* directions respectively, G_{23}, G_{13}, G_{12} are the shear moduli and μ_{ij} are the Poisson's ratios for transverse strain in *j*-direction when stressed in the *i*-direction. Poisson's ratios and Young's moduli are related as:

$$\mu_{ij}\,E_j = \mu_{ji}\,E_i \qquad (i, j = 1,\ 2,\ 3)$$

3. Governing Equations and Boundary Conditions

Using the Eqs. (6) – (8) and the principle of virtual work, variationally consistent governing differential equations and associated boundary conditions for the plate under consideration can be obtained. The dynamic version of principle of virtual work when applied to the plate leads to

$$\int_{z=-h/2}^{z=+h/2}\int_{y=0}^{y=b}\int_{x=0}^{x=a}\begin{bmatrix}\sigma_x\ \delta\varepsilon_x + \sigma_y\ \delta\varepsilon_y + \sigma_z\ \delta\varepsilon_z \\ +\tau_{xy}\ \delta\gamma_{xy} + \tau_{zx}\ \delta\gamma_{zx} + \tau_{yz}\ \delta\gamma_{yz}\end{bmatrix} dx\ dy\ dz - \int_{y=0}^{y=b}\int_{x=0}^{x=a} q(x,y)\ \delta w\ dx\ dy = 0 \tag{11}$$

where symbol δ denotes the variational operator. Employing Green's theorem in Eq. (11) successively, we obtain the coupled Euler-Lagrange equations, which are the governing equations and the associated boundary conditions of the plate. The governing differential equations in terms unknown variables are obtained by collecting the coefficients of δu_0, δv_0, δw, $\delta\phi_x$, $\delta\psi_x$, $\delta\phi_y$, $\delta\psi_y$, $\delta\phi_z$, $\delta\psi_z$ and equating them to zero we get field equations and boundary conditions as follows:

$$\begin{aligned}\delta u_0:\ & A_{11}\frac{\partial^2 u_0}{\partial x^2} + A_{44}\frac{\partial^2 u_0}{\partial y^2} - A_{12}\frac{\partial^2 v_0}{\partial x\,\partial y} - A_{44}\frac{\partial^2 v_0}{\partial x\,\partial y} + \left(B_{11}\frac{\partial^3 w}{\partial x^3} + \left(B_{12}+2B_{44}\right)\frac{\partial^3 w}{\partial x\,\partial y^2}\right) \\ & -I_{11}\frac{\partial^2\phi_x}{\partial x^2} - I_{44}\frac{\partial^2\phi_x}{\partial y^2} - J_{11}\frac{\partial^2\psi_x}{\partial x^2} - J_{44}\frac{\partial^2\psi_x}{\partial y^2} - \left(I_{12}+I_{44}\right)\frac{\partial^2\phi_y}{\partial x\,\partial y} \\ & -\left(J_{12}+J_{44}\right)\frac{\partial^2\psi_y}{\partial x\,\partial y} + \frac{8}{h^2}B_{13}\frac{\partial\phi_z}{\partial x} + \frac{64}{h^4}E_{13}\frac{\partial\psi_z}{\partial x} = 0\end{aligned} \tag{12}$$

$$\delta v_0: \quad \begin{aligned} & -\left(A_{12}+A_{44}\right)\frac{\partial^2 u_0}{\partial x\,\partial y}-A_{44}\frac{\partial^2 v_0}{\partial x^2}-A_{22}\frac{\partial^2 v_0}{\partial y^2}+B_{22}\frac{\partial^3 w}{\partial y^3}+\left(B_{12}+2B_{44}\right)\frac{\partial^3 w}{\partial x^2\partial y} \\ & -\left(I_{12}+I_{44}\right)\frac{\partial^2 \phi_x}{\partial x\,\partial y}-\left(J_{12}+J_{44}\right)\frac{\partial^2 \psi_x}{\partial x\,\partial y}-I_{22}\frac{\partial^2 \phi_y}{\partial y^2}-I_{44}\frac{\partial^2 \phi_y}{\partial x^2}-J_{22}\frac{\partial^2 \psi_y}{\partial y^2} \\ & -J_{44}\frac{\partial^2 \psi_y}{\partial x^2}+\frac{8}{h^2}B_{23}\frac{\partial \phi_z}{\partial y}+\frac{64}{h^4}E_{23}\frac{\partial \psi_z}{\partial y}=0 \end{aligned} \tag{13}$$

$$\delta w: \quad \begin{aligned} & -B_{11}\frac{\partial^3 u_0}{\partial x^3}-\left(B_{12}+2B_{44}\right)\frac{\partial^3 u_0}{\partial x\,\partial y^2}-B_{22}\frac{\partial^3 v_0}{\partial y^3}-\left(B_{12}+2B_{44}\right)\frac{\partial^3 v_0}{\partial x^2\partial y}+D_{11}\frac{\partial^4 w}{\partial x^4} \\ & +\left(2D_{12}+4D_{44}\right)\frac{\partial^4 w}{\partial x^2\partial y^2}+D_{22}\frac{\partial^4 w}{\partial y^4}-L_{11}\frac{\partial^3 \phi_x}{\partial x^3}-\left(L_{12}+2L_{44}\right)\frac{\partial^3 \phi_x}{\partial x\,\partial y^2}-M_{11}\frac{\partial^3 \psi_x}{\partial x^3} \\ & -\left(M_{12}+2M_{44}\right)\frac{\partial^3 \psi_x}{\partial x\,\partial y^2}-L_{22}\frac{\partial^3 \phi_y}{\partial y^3}+\left(L_{12}+2L_{44}\right)\frac{\partial^3 \phi_y}{\partial x^2\partial y} \\ & -M_{22}\frac{\partial^3 \psi_y}{\partial y^3}-\left(M_{12}+2M_{44}\right)\frac{\partial^3 \psi_y}{\partial x^2\partial y}+D_{13}\frac{8}{h^2}\frac{\partial^2 \phi_z}{\partial x^2}+D_{23}\frac{8}{h^2}\frac{\partial^2 \phi_z}{\partial y^2}+F_{13}\frac{64}{h^4}\frac{\partial^2 \psi_z}{\partial x^2} \\ & +F_{23}\frac{64}{h^4}\frac{\partial^2 \psi_z}{\partial y^2}-q=0 \end{aligned} \tag{14}$$

$$\delta \phi_x: \quad \begin{aligned} & -I_{11}\frac{\partial^2 u_0}{\partial x^2}-I_{44}\frac{\partial^2 u_0}{\partial y^2}-\left(I_{12}+I_{44}\right)\frac{\partial^2 v_0}{\partial x\,\partial y}+L_{11}\frac{\partial^3 w}{\partial x^3}+\left(L_{12}+2L_{44}\right)\frac{\partial^3 w}{\partial x\,\partial y^2} \\ & -N_{11}\frac{\partial^2 \phi_x}{\partial x^2}-N_{44}\frac{\partial^2 \phi_x}{\partial y^2}+C_{55}\phi_x-P_{11}\frac{\partial^2 \psi_x}{\partial x^2}-P_{44}\frac{\partial^2 \psi_x}{\partial y^2}+H_{55}\psi_x \\ & -\left(N_{12}+N_{44}\right)\frac{\partial^2 \phi_y}{\partial x\,\partial y}-\left(P_{12}+P_{44}\right)\frac{\partial^2 \psi_y}{\partial x\,\partial y}+\left(\frac{8}{h^2}L_{13}+C_{55}\right)\frac{\partial \phi_z}{\partial x} \\ & +\left(\frac{64}{h^4}S_{13}+H_{55}\right)\frac{\partial \psi_z}{\partial x}=0 \end{aligned} \tag{15}$$

$$\delta \psi_x: \quad \begin{aligned} & -J_{11}\frac{\partial^2 u_0}{\partial x^2}-J_{44}\frac{\partial^2 u_0}{\partial y^2}-\left(J_{12}+J_{44}\right)\frac{\partial^2 v_0}{\partial x\,\partial y}+M_{11}\frac{\partial^3 w}{\partial x^3}+\left(M_{12}+2M_{44}\right)\frac{\partial^3 w}{\partial x\,\partial y^2} \\ & -P_{11}\frac{\partial^2 \phi_x}{\partial x^2}-P_{44}\frac{\partial^2 \phi_x}{\partial y^2}+H_{55}\phi_x-R_{11}\frac{\partial^2 \psi_x}{\partial x^2}-R_{44}\frac{\partial^2 \psi_x}{\partial y^2}+G_{55}\psi_x \\ & -\left(P_{12}+P_{44}\right)\frac{\partial^2 \phi_y}{\partial x\,\partial y}-\left(R_{12}+R_{44}\right)\frac{\partial^2 \psi_y}{\partial x\,\partial y}+\left(\frac{8}{h^2}M_{13}+H_{55}\right)\frac{\partial \phi_z}{\partial x} \\ & +\left(\frac{64}{h^4}T_{13}+G_{55}\right)\frac{\partial \psi_z}{\partial x}=0 \end{aligned} \tag{16}$$

$$\delta \phi_y: \quad \begin{aligned} & -\left(I_{12}+I_{44}\right)\frac{\partial^2 u_0}{\partial x\,\partial y}-I_{22}\frac{\partial^2 v_0}{\partial y^2}-I_{44}\frac{\partial^2 v_0}{\partial x^2}+L_{22}\frac{\partial^3 w}{\partial y^3}+\left(L_{12}+2L_{44}\right)\frac{\partial^3 w}{\partial x^2\partial y} \\ & -\left(N_{12}+N_{44}\right)\frac{\partial^2 \phi_x}{\partial x\,\partial y}-\left(P_{12}+P_{44}\right)\frac{\partial^2 \psi_x}{\partial x\,\partial y}-N_{22}\frac{\partial^2 \phi_y}{\partial y^2}-N_{44}\frac{\partial^2 \phi_y}{\partial x^2}+C_{66}\phi_y \\ & -P_{22}\frac{\partial^2 \psi_y}{\partial y^2}-N_{44}\frac{\partial^2 \psi_y}{\partial x^2}+H_{66}\psi_y+\left(\frac{8}{h^2}L_{23}+C_{66}\right)\frac{\partial \phi_z}{\partial y} \\ & +\left(\frac{64}{h^4}S_{23}+H_{66}\right)\frac{\partial \psi_z}{\partial y}=0 \end{aligned} \tag{17}$$

$$\delta\psi_y: \begin{array}{l} -\left(J_{12}+J_{44}\right)\dfrac{\partial^2 u_0}{\partial x\,\partial y}-J_{22}\dfrac{\partial^2 v_0}{\partial y^2}-J_{44}\dfrac{\partial^2 v_0}{\partial x^2}+M_{22}\dfrac{\partial^3 w}{\partial y^3}+\left(M_{12}+2M_{44}\right)\dfrac{\partial^3 w}{\partial x^2\partial y} \\ -\left(P_{12}+P_{44}\right)\dfrac{\partial^2\phi_x}{\partial x\,\partial y}-\left(R_{12}+R_{44}\right)\dfrac{\partial^2\psi_x}{\partial x\,\partial y}-P_{22}\dfrac{\partial^2\phi_y}{\partial y^2}-P_{44}\dfrac{\partial^2\phi_y}{\partial x^2}+H_{66}\phi_y \\ -R_{22}\dfrac{\partial^2\psi_y}{\partial y^2}-R_{44}\dfrac{\partial^2\psi_y}{\partial x^2}+G_{66}\psi_y+\left(\dfrac{8}{h^2}M_{23}+H_{66}\right)\dfrac{\partial\phi_z}{\partial y} \\ +\left(\dfrac{64}{h^4}T_{23}+G_{66}\right)\dfrac{\partial\psi_z}{\partial y}=0 \end{array} \tag{18}$$

$$\delta\phi_z: \begin{array}{l} -\dfrac{8}{h^2}B_{13}\dfrac{\partial u_0}{\partial x}-\dfrac{8}{h^2}B_{23}\dfrac{\partial v_0}{\partial y}+\dfrac{8}{h^2}D_{13}\dfrac{\partial^2 w}{\partial x^2}+\dfrac{8}{h^2}D_{23}\dfrac{\partial^2 w}{\partial y^2}-\left(\dfrac{8}{h^2}L_{13}+C_{55}\right)\dfrac{\partial\phi_x}{\partial x} \\ -\left(\dfrac{8}{h^2}M_{13}+H_{55}\right)\dfrac{\partial\psi_x}{\partial x}-\left(\dfrac{8}{h^2}L_{23}+C_{66}\right)\dfrac{\partial\phi_y}{\partial y}-\left(\dfrac{8}{h^2}M_{23}+H_{66}\right)\dfrac{\partial\psi_y}{\partial y} \\ -C_{55}\dfrac{\partial^2\phi_z}{\partial x^2}-C_{66}\dfrac{\partial^2\phi_z}{\partial y^2}+D_{33}\dfrac{64}{h^4}\phi_z-H_{55}\dfrac{\partial^2\psi_z}{\partial x^2}-H_{66}\dfrac{\partial^2\psi_z}{\partial y^2}+F_{33}\dfrac{512}{h^6}\psi_z=0 \end{array} \tag{19}$$

$$\delta\psi_z: \begin{array}{l} -\dfrac{64}{h^4}E_{13}\dfrac{\partial u_0}{\partial x}-\dfrac{64}{h^4}E_{23}\dfrac{\partial v_0}{\partial y}+\dfrac{64}{h^4}F_{13}\dfrac{\partial^2 w}{\partial x^2}+\dfrac{64}{h^4}F_{23}\dfrac{\partial^2 w}{\partial y^2}-\left(\dfrac{64}{h^4}S_{13}+H_{55}\right)\dfrac{\partial\phi_x}{\partial x} \\ -\left(\dfrac{64}{h^4}T_{13}+G_{55}\right)\dfrac{\partial\psi_x}{\partial x}-\left(\dfrac{64}{h^4}S_{23}+H_{66}\right)\dfrac{\partial\phi_y}{\partial y}-\left(\dfrac{64}{h^4}T_{23}+G_{66}\right)\dfrac{\partial\psi_y}{\partial y} \\ -H_{55}\dfrac{\partial^2\phi_z}{\partial x^2}-H_{66}\dfrac{\partial^2\phi_z}{\partial y^2}+F_{33}\dfrac{512}{h^6}\phi_z-G_{55}\dfrac{\partial^2\psi_z}{\partial x^2}-G_{66}\dfrac{\partial^2\psi_z}{\partial y^2}+O_{33}\dfrac{4096}{h^8}\psi_z=0 \end{array} \tag{20}$$

The boundary conditions at $x = 0$ and $x = a$ obtained are of the following form:

$$\delta u_0: \begin{array}{l} A_{11}\dfrac{\partial u_0}{\partial x}-B_{11}\dfrac{\partial^2 w}{\partial x^2}+I_{11}\dfrac{\partial\phi_x}{\partial x}+J_{11}\dfrac{\partial\psi_x}{\partial x}+A_{12}\dfrac{\partial v_0}{\partial y}-B_{12}\dfrac{\partial^2 w}{\partial y^2}+I_{12}\dfrac{\partial\phi_y}{\partial y} \\ +J_{12}\dfrac{\partial\psi_y}{\partial y}-B_{13}\dfrac{8}{h^2}\phi_z-E_{13}\dfrac{64}{h^4}\psi_z=0 \text{ or } u_0 \text{ is prescribed.} \end{array} \tag{21}$$

$$\delta v_0: \begin{array}{l} A_{44}\dfrac{\partial u_0}{\partial y}+A_{44}\dfrac{\partial v_0}{\partial x}-2B_{44}\dfrac{\partial^2 w}{\partial x\,\partial y}+I_{44}\dfrac{\partial\phi_x}{\partial y}+I_{44}\dfrac{\partial\phi_y}{\partial x}+J_{44}\dfrac{\partial\psi_x}{\partial y} \\ +J_{44}\dfrac{\partial\psi_y}{\partial x}=0 \text{ or } v_0 \text{ is prescribed.} \end{array} \tag{22}$$

$$\delta w: \begin{array}{l} B_{11}\dfrac{\partial^2 u_0}{\partial x^2}+2B_{44}\dfrac{\partial^2 u_0}{\partial y^2}+\left[B_{12}+2B_{44}\right]\dfrac{\partial^2 v_0}{\partial x\,\partial y}-D_{11}\dfrac{\partial^3 w}{\partial x^3}-\left[D_{12}+4D_{44}\right]\dfrac{\partial^3 w}{\partial x\,\partial y^2} \\ +L_{11}\dfrac{\partial^2\phi_x}{\partial x^2}+2L_{44}\dfrac{\partial^2\phi_x}{\partial y^2}+M_{11}\dfrac{\partial^2\psi_x}{\partial x^2}+2M_{44}\dfrac{\partial^2\psi_x}{\partial y^2}+\left[L_{12}+2L_{44}\right]\dfrac{\partial^2\phi_y}{\partial x\,\partial y} \\ +\left[M_{12}+2M_{44}\right]\dfrac{\partial^2\psi_y}{\partial x\,\partial y}-D_{13}\dfrac{8}{h^2}\dfrac{\partial\phi_z}{\partial x}-F_{13}\dfrac{64}{h^4}\dfrac{\partial\psi_z}{\partial x}=0 \text{ or } w \text{ is prescribed.} \end{array} \tag{23}$$

$$\frac{\partial\delta w}{\partial x}: \begin{array}{l} -B_{11}\dfrac{\partial u_0}{\partial x}+D_{11}\dfrac{\partial^2 w}{\partial x^2}-L_{11}\dfrac{\partial\phi_x}{\partial x}-M_{11}\dfrac{\partial\psi_x}{\partial x}-B_{12}\dfrac{\partial v_0}{\partial y}+D_{12}\dfrac{\partial^2 w}{\partial y^2} \\ -L_{12}\dfrac{\partial\phi_y}{\partial y}-M_{12}\dfrac{\partial\psi_y}{\partial y}+D_{13}\dfrac{8}{h^2}\phi_z+F_{13}\dfrac{64}{h^4}\psi_z=0 \text{ or } \dfrac{\partial w}{\partial x} \text{ is prescribed.} \end{array} \tag{24}$$

$$\delta\phi_x \; : I_{11}\frac{\partial u_0}{\partial x} - L_{11}\frac{\partial^2 w}{\partial x^2} + N_{11}\frac{\partial \phi_x}{\partial x} + P_{11}\frac{\partial \psi_x}{\partial x} + I_{12}\frac{\partial v_0}{\partial y} - L_{12}\frac{\partial^2 w}{\partial y^2} + N_{12}\frac{\partial \phi_y}{\partial y} + P_{12}\frac{\partial \psi_y}{\partial y} - L_{13}\frac{8}{h^2}\phi_z - S_{13}\frac{64}{h^4}\psi_z = 0 \text{ or } \phi_x \text{ is prescribed.} \tag{25}$$

$$\delta\psi_x \; : J_{11}\frac{\partial u_0}{\partial x} - M_{11}\frac{\partial^2 w}{\partial x^2} + P_{11}\frac{\partial \phi_x}{\partial x} + R_{11}\frac{\partial \psi_x}{\partial x} + J_{12}\frac{\partial v_0}{\partial y} - M_{12}\frac{\partial^2 w}{\partial y^2} + P_{12}\frac{\partial \phi_y}{\partial y} + R_{12}\frac{\partial \psi_y}{\partial y} - M_{13}\frac{8}{h^2}\phi_z - T_{13}\frac{64}{h^4}\psi_z = 0 \text{ or } \psi_x \text{ is prescribed.} \tag{26}$$

$$\delta\phi_y \; : I_{44}\left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\right) - 2L_{44}\frac{\partial^2 w}{\partial x\,\partial y} + N_{44}\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right) + P_{44}\left(\frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x}\right) = 0 \text{ or } \phi_y \text{ is prescribed.} \tag{27}$$

$$\delta\psi_y \; : J_{44}\left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\right) - 2M_{44}\frac{\partial^2 w}{\partial x\,\partial y} + P_{44}\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right) + R_{44}\left(\frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x}\right) = 0 \text{ or } \psi_y \text{ is prescribed.} \tag{28}$$

$$\delta\phi_z \; : \; C_{55}\phi_y + H_{55}\psi_y + C_{55}\frac{\partial \phi_z}{\partial y} + H_{55}\frac{\partial \psi_z}{\partial y} = 0 \text{ or } \phi_z \text{ is prescribed.} \tag{29}$$

$$\delta\psi_z \; : \; H_{55}\phi_y + G_{55}\psi_y + H_{55}\frac{\partial \phi_z}{\partial y} + G_{55}\frac{\partial \psi_z}{\partial y} = 0 \text{ or } \psi_z \text{ is prescribed.} \tag{30}$$

The boundary conditions at $y = 0$ and $y = b$ obtained is of the following form:

$$\delta u_0 \; : A_{44}\frac{\partial u_0}{\partial y} + A_{44}\frac{\partial v_0}{\partial x} - 2B_{44}\frac{\partial^2 w}{\partial x\,\partial y} + I_{44}\frac{\partial \phi_x}{\partial y} + I_{44}\frac{\partial \phi_y}{\partial x} + J_{44}\frac{\partial \psi_x}{\partial y} + J_{44}\frac{\partial \psi_y}{\partial x} = 0 \text{ or } u_0 \text{ is prescribed.} \tag{31}$$

$$\delta v_0 \; : \; A_{12}\frac{\partial u_0}{\partial x} - B_{12}\frac{\partial^2 w}{\partial x^2} + I_{12}\frac{\partial \phi_x}{\partial x} + J_{12}\frac{\partial \psi_x}{\partial x} + A_{22}\frac{\partial v_0}{\partial y} - B_{22}\frac{\partial^2 w}{\partial y^2} + I_{22}\frac{\partial \phi_y}{\partial y} + J_{22}\frac{\partial \psi_y}{\partial y} - B_{23}\frac{8}{h^2}\phi_z - E_{23}\frac{64}{h^4}\psi_z = 0 \text{ or } v_0 \text{ is prescribed.} \tag{32}$$

$$\delta w \; : \left(B_{12} + 2B_{44}\right)\frac{\partial^2 u_0}{\partial x\,\partial y} + B_{22}\frac{\partial^2 v_0}{\partial y^2} + 2B_{44}\frac{\partial^2 v_0}{\partial x^2} - D_{22}\frac{\partial^3 w}{\partial y^3} - \left(D_{12} + 4D_{44}\right)\frac{\partial^3 w}{\partial x^2\partial y} + \left(L_{12} + 2L_{44}\right)\frac{\partial^2 \phi_x}{\partial x\,\partial y} + \left(M_{12} + 2M_{44}\right)\frac{\partial^2 \psi_x}{\partial x\,\partial y} + L_{22}\frac{\partial^2 \phi_y}{\partial y^2} + 2L_{44}\frac{\partial^2 \phi_y}{\partial x^2} + M_{22}\frac{\partial^2 \psi_y}{\partial y^2} + 2M_{44}\frac{\partial^2 \psi_y}{\partial x^2} - D_{23}\frac{8}{h^2}\frac{\partial \phi_z}{\partial y} - F_{23}\frac{64}{h^4}\frac{\partial \psi_z}{\partial y} = 0 \text{ or } w \text{ is prescribed.} \tag{33}$$

$$\frac{\partial \delta w}{\partial y} : -B_{12}\frac{\partial u_0}{\partial x} + D_{12}\frac{\partial^2 w}{\partial x^2} - L_{12}\frac{\partial \phi_x}{\partial x} - M_{12}\frac{\partial \psi_x}{\partial x} - B_{22}\frac{\partial v_0}{\partial y} + D_{22}\frac{\partial^2 w}{\partial y^2} - L_{22}\frac{\partial \phi_y}{\partial y} - M_{22}\frac{\partial \psi_y}{\partial y} + D_{23}\frac{8}{h^2}\phi_z + F_{23}\frac{64}{h^4}\psi_z = 0 \text{ or } \frac{\partial w}{\partial y} \text{ is prescribed.} \tag{34}$$

$$\delta\phi_x : I_{44}\left(\frac{\partial u_0}{\partial y}+\frac{\partial v_0}{\partial x}\right)-2L_{44}\frac{\partial^2 w}{\partial x\,\partial y}+N_{44}\left(\frac{\partial \phi_x}{\partial y}+\frac{\partial \phi_y}{\partial x}\right)+P_{44}\left(\frac{\partial \psi_x}{\partial y}+\frac{\partial \psi_y}{\partial x}\right)=0 \text{ or } \phi_x \text{ is prescribed.} \tag{35}$$

$$\delta\psi_x : J_{44}\left(\frac{\partial u_0}{\partial y}+\frac{\partial v_0}{\partial x}\right)-2M_{44}\frac{\partial^2 w}{\partial x\,\partial y}+P_{44}\left(\frac{\partial \phi_x}{\partial y}+\frac{\partial \phi_y}{\partial x}\right)+R_{44}\left(\frac{\partial \psi_x}{\partial y}+\frac{\partial \psi_y}{\partial x}\right)=0 \text{ or } \psi_x \text{ is prescribed.} \tag{36}$$

$$\delta\phi_y : I_{12}\frac{\partial u_0}{\partial x}-L_{12}\frac{\partial^2 w}{\partial x^2}+N_{12}\frac{\partial \phi_x}{\partial x}+P_{12}\frac{\partial \psi_x}{\partial x}+I_{22}\frac{\partial v_0}{\partial y}-L_{22}\frac{\partial^2 w}{\partial y^2}+N_{22}\frac{\partial \phi_y}{\partial y}+P_{22}\frac{\partial \psi_y}{\partial y}-L_{23}\frac{8}{h^2}\phi_z-S_{23}\frac{64}{h^4}\psi_z=0 \text{ or } \phi_y \text{ is prescribed.} \tag{37}$$

$$\delta\psi_y : J_{12}\frac{\partial u_0}{\partial x}-M_{12}\frac{\partial^2 w}{\partial x^2}+P_{12}\frac{\partial \phi_x}{\partial x}+R_{12}\frac{\partial \psi_x}{\partial x}+J_{22}\frac{\partial v_0}{\partial y}-M_{22}\frac{\partial^2 w}{\partial y^2}+P_{22}\frac{\partial \phi_y}{\partial y}+R_{22}\frac{\partial \psi_y}{\partial y}-M_{23}\frac{8}{h^2}\phi_z-T_{23}\frac{64}{h^4}\psi_z=0 \text{ or } \psi_y \text{ is prescribed.} \tag{38}$$

$$\delta\phi_z : C_{66}\phi_y+H_{66}\psi_y+C_{66}\frac{\partial \phi_z}{\partial y}+H_{66}\frac{\partial \psi_z}{\partial y}=0 \text{ or } \phi_z \text{ is prescribed.} \tag{39}$$

$$\delta\psi_z : H_{66}\phi_y+G_{66}\psi_y+H_{66}\frac{\partial \phi_z}{\partial y}+G_{66}\frac{\partial \psi_z}{\partial y}=0 \text{ or } \psi_z \text{ is prescribed.} \tag{40}$$

The boundary condition at corners ($x = 0, y = 0$), ($x = a, y = 0$), ($x = 0, y = b$) and ($x = a, y = b$) obtained in terms of displacements is of the following form:

$$A_{44}\left(\frac{\partial u_0}{\partial y}+\frac{\partial v_0}{\partial x}\right)-2B_{44}\frac{\partial^2 w}{\partial x\,\partial y}+I_{66}\left(\frac{\partial \phi_x}{\partial y}+\frac{\partial \phi_y}{\partial x}\right)+J_{66}\left(\frac{\partial \psi_x}{\partial y}+\frac{\partial \psi_y}{\partial x}\right)=0 \tag{41}$$

where A_{ij}, B_{ij}, etc., are the plate stiffnesses, as defined below:

$$\left(A_{ij},I_{ij},J_{ij},N_{ij},P_{ij},R_{ij}\right)=\int_{-h/2}^{+h/2}\bar{Q}_{ij}\left(1,f_1(z),f_2(z),f_1^2(z),f_1(z)f_2(z),f_2^2(z)\right)dz, \quad (i,j = 1,2,4)$$

$$\left(B_{ij},D_{ij},L_{ij},M_{ij}\right)=\int_{-h/2}^{+h/2}\bar{Q}_{ij}\left(z,z^2,zf_1(z),zf_2(z)\right)dz, \quad (i,j = 1,2,3,4)$$

$$\left(E_{ij},F_{ij},S_{ij},T_{ij}\right)=\int_{-h/2}^{+h/2}\bar{Q}_{ij}\left(z^3,z^4,z^3f_1(z),z^3f_2(z)\right)dz, \quad (i,j = 1,2,3)$$

$$\left(C_{ij},G_{ij},H_{ij}\right)=\int_{-h/2}^{+h/2}\bar{Q}_{ij}\left(g_1^2(z),g_2^2(z),g_1(z)g_2(z)\right)dz, \quad (i,j = 5,6)$$

$$\left(O_{ij}\right)=\int_{-h/2}^{+h/2}\bar{Q}_{ij}\left(z^6\right)dz, \quad (i,j = 3)$$

$$f_1(z)=\left(z-\frac{4}{3}\frac{z^3}{h^2}\right);\; f_2(z)=\left(z-\frac{16}{5}\frac{z^5}{h^4}\right);\; g_1(z)=\left(1-4\frac{z^2}{h^2}\right)\; ;\text{ and } g_2(z)=\left(1-16\frac{z^4}{h^4}\right) \tag{42}$$

4. Closed-Form Solutions for Simply Supported Plate

Example 1: Bending analysis of isotropic plates subjected to sinusoidal load.

The rectangular thick isotropic plate with all edges simply supported is subjected to transverse sinusoidal loading $q(x,y)=q_0\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right)$ on the top surface of the plate where q_0 is the magnitude of the sinusoidal distributed load at the center of the plate. The following material properties are used.

E = 210 GPa, μ = 0.3 and $G = E/2(1+\mu)$

4.1 Navier Solution

The governing differential equations and the associated boundary conditions for static flexure of rectangular plate under consideration can be obtained directly from Eqs. (10) through (41). The following are the boundary conditions of the simply supported isotropic plate.

$$v_0 = w = \phi_x = \psi_x = \phi_z = \psi_z = 0 \text{ at } x=0,\; x=a \tag{43}$$

$$u_0 = w = \phi_y = \psi_y = \phi_z = \psi_z = 0 \text{ at } y=0,\; y=b \tag{44}$$

The transverse load acting on the top surface of the plate is presented in the double trigonometric series,

$$q(x,y)=\sum_{m=1,3,5,\ldots}^{\infty}\sum_{n=1,3,5,\ldots}^{\infty} q_{mn}\sin\alpha x\sin\beta y$$

where $\alpha = m\pi/a$ and $\beta = n\pi/b$. The coefficient q_{mn} are the coefficient of double Fourier expansion. For single sine load in both the directions m =1 and n =1 and $q_{mn} = q_0$. The unknown variables $u_0(x,y)$, $v_0(x,y)$, $w(x,y)$, $\phi_x(x,y)$, $\psi_x(x,y)$, $\phi_y(x,y)$, $\psi_y(x,y)$, $\phi_z(x,y)$, $\psi_z(x,y)$ are represented in the following trigonometric form, which satisfy governing equations and boundary conditions exactly.

$$\begin{Bmatrix} u_0 \\ v_0 \\ w \\ \phi_x \\ \psi_x \\ \phi_y \\ \psi_y \\ \phi_z \\ \psi_z \end{Bmatrix} = \sum_{m=1,3,5,\ldots}^{\infty}\sum_{n=1,3,5,\ldots}^{\infty} \begin{Bmatrix} u_{0mn}\sin\alpha x\cos\beta y \\ v_{0mn}\cos\alpha x\sin\beta y \\ w_{mn}\sin\alpha x\sin\beta y \\ \phi_{xmn}\sin\alpha x\cos\beta y \\ \psi_{xmn}\cos\alpha x\sin\beta y \\ \phi_{ymn}\sin\alpha x\cos\beta y \\ \psi_{ymn}\cos\alpha x\sin\beta y \\ \phi_{zmn}\sin\alpha x\sin\beta y \\ \psi_{zmn}\sin\alpha x\sin\beta y \end{Bmatrix} \tag{45}$$

where u_{0mn}, v_{0mn}, w_{mn}, ϕ_{xmn}, ψ_{xmn}, ϕ_{ymn}, ψ_{ymn}, ϕ_{zmn}, ψ_{zmn} are the unknown coefficients of the respective Fourier expansions are governed by,

$$\left[K_{ij}\right]\{\Delta\}=\{F\} \qquad (i,j=1,\;9) \tag{46}$$

where

$\{\Delta\}=\{u_{0mn}\ v_{0mn} w_{mn}\ \phi_{xmn}\ \psi_{xmn}\ \phi_{ymn}\ \psi_{ymn}\ \phi_{zmn}\ \psi_{zmn}\}^T$, $\{F\}=\{0\,0\,q_{mn}\ 0\,0\,0\,0\,0\,0\}^T$ and the elements of stiffness matrix [K] are given as follows:

$$K_{11}=\left[A_{11}\alpha^2+A_{44}\beta^2\right];K_{12}=\left(A_{12}+A_{44}\right)\alpha\beta;K_{13}=-\left[B_{11}\ \alpha^3+\ (B_{12}+2B_{44})\ \alpha\ \beta^2\right];$$

$$K_{14}=\left(I_{11}\alpha^2+I_{44}\beta^2\right);K_{15}=\left(J_{11}\alpha^2+J_{44}\beta^2\right);K_{16}=\left(I_{12}+I_{44}\right)\alpha\beta;$$

$$K_{17}=\left(J_{12}+J_{44}\right)\alpha\beta;K_{18}=\frac{8}{h^2}B_{13}\alpha;K_{19}=\frac{64}{h^4}E_{13}\alpha;$$

$$K_{21}=\left(A_{12}+A_{44}\right)\alpha\beta;K_{22}=\left(A_{22}\alpha^2+A_{44}\beta^2\right);K_{23}=-\left[B_{22}\ \beta^3+\ (B_{12}+2B_{44})\ \alpha^2\beta\right];$$

$$K_{24}=\left(I_{12}+I_{44}\right)\alpha\beta;K_{25}=\left(J_{12}+J_{44}\right)\alpha\beta;K_{26}=\left(I_{22}\beta^2+I_{44}\alpha^2\right);$$

$$K_{27}=\left(J_{22}\beta^2+J_{44}\alpha^2\right);K_{28}=\frac{8}{h^2}B_{23}\beta;K_{29}=\frac{64}{h^4}E_{23}\beta;$$

$$K_{31}=-\left[B_{11}\ \alpha^3+\ (B_{12}+2B_{44})\alpha\beta^2\right];K_{32}=-\left[B_{22}\ \beta^3+\ (B_{12}+2B_{44})\ \alpha^2\beta\right];$$

$$K_{33}=\left[D_{11}\ \alpha^4+\left(2D_{12}+4D_{44}\right)\ \alpha^2\beta^2+D_{22}\ \beta^2\right];$$

$$K_{34}=-\left[L_{11}\ \alpha^3+\ (L_{12}+2L_{44})\ \alpha\beta^2\right];K_{35}=-\left[M_{11}\ \alpha^3+\ (M_{12}+2M_{44})\ \alpha\beta^2\right];$$

$$K_{36}=-\left[L_{22}\ \beta^3+\ (L_{12}+2L_{44})\ \alpha^2\beta\right];K_{37}=-\left[M_{22}\ \beta^3+\ (M_{12}+2M_{44})\ \alpha^2\beta\right];$$

$$K_{38}=-\frac{8}{h^2}\left[D_{13}\alpha^2+D_{23}\beta^2\right];K_{39}=-\frac{64}{h^4}\left[F_{13}\alpha^2+F_{23}\beta^2\right];$$

$$K_{41}=\left(I_{11}\alpha^2+I_{44}\beta^2\right);K_{42}=\left(I_{12}+I_{44}\right)\alpha\beta;$$

$$K_{43}=-\left[L_{11}\ \alpha^3+\ (L_{12}+2L_{44})\ \alpha\beta^2\right];K_{44}=\left(N_{11}\alpha^2+N_{44}\beta^2\right)+C_{55};$$

$$K_{45}=\left(P_{11}\alpha^2+P_{44}\beta^2\right)+H_{55};K_{46}=\left(N_{12}+N_{44}\right)\alpha\beta;$$

$$K_{47}=\left(P_{12}+P_{44}\right)\alpha\beta;K_{48}=\left(\frac{8}{h^2}L_{13}+C_{55}\right)\alpha;K_{49}=\left(\frac{64}{h^4}S_{13}+H_{55}\right)\alpha;$$

$$K_{51}=J_{11}\alpha^2+J_{44}\beta^2;K_{52}=(J_{12}+J_{44})\ \alpha\beta;K_{53}=-\left[M_{11}\ \alpha^3+\ (M_{12}+2M_{44})\ \alpha\ \beta^2\right];$$

$$K_{54}=P_{11}\alpha^2+P_{44}\beta^2+H_{55};K_{55}=R_{11}\alpha^2+R_{44}\beta^2+G_{55};$$

$$K_{56}=(P_{12}+P_{44})\ \alpha\beta;K_{57}=\left(R_{12}+R_{44}\right)\alpha\beta;$$

$$K_{58}=\left(\frac{8}{h^2}M_{13}+H_{55}\right)\alpha;K_{59}=\left(\frac{64}{h^4}T_{13}+G_{55}\right)\alpha;$$

$$K_{61}=\left(I_{12}+I_{44}\right)\alpha\beta;K_{62}=\left(I_{22}\beta^2+I_{44}\alpha^2\right);$$

$$K_{63}=-\left[L_{22}\ \beta^3+\ \left(L_{12}+2L_{44}\right)\alpha^2\beta\right];K_{64}=\left[\left(N_{12}+N_{44}\right)\alpha\beta\right];$$

$$K_{65}=\left(P_{12}+P_{44}\right)\alpha\beta;K_{66}=\left(N_{22}\beta^2+N_{44}\alpha^2\right)+C_{66};$$

$$K_{67}=\left(P_{22}\beta^2+P_{44}\alpha^2\right)+H_{66};K_{68}=\left(\frac{8}{h^2}L_{23}+C_{66}\right)\beta;$$

$$K_{69}=\left(\frac{64}{h^4}S_{23}+H_{66}\right)\beta;$$

$$K_{71} = (J_{12} + J_{44})\alpha\beta; K_{72} = \left(J_{22}\beta^2 + J_{44}\alpha^2\right);$$
$$K_{73} = -\left[M_{22}\,\beta^3 + \left(M_{12} + 2M_{44}\right)\alpha^2\beta\right];$$
$$K_{74} = \left(P_{12} + P_{44}\right)\alpha\beta; K_{75} = (R_{12} + R_{44})\alpha\beta;$$
$$K_{76} = \left[(P_{22}\beta^2 + P_{44}\alpha^2) + H_{66}\right]; K_{77} = \left[(R_{22}\beta^2 + R_{44}\alpha^2) + G_{66}\right];$$
$$K_{78} = \left[\left(\frac{8}{h^2}M_{23} + H_{66}\right)\beta\right]; K_{79} = \left[\left(\frac{64}{h^4}T_{23} + G_{66}\right)\beta\right];$$

$$K_{81} = \frac{8}{h^2}B_{13}\alpha; K_{82} = \frac{8}{h^2}B_{23}\beta; K_{83} = -\frac{8}{h^2}\left(D_{13}\alpha^2 + D_{23}\beta^2\right); K_{84} = \left(\frac{8}{h^2}L_{13} + C_{55}\right)\alpha;$$
$$K_{85} = \left(\frac{8}{h^2}M_{13} + H_{55}\right)\alpha; K_{86} = \left(\frac{8}{h^2}L_{23} + C_{66}\right)\beta; K_{87} = \left(\frac{8}{h^2}M_{23} + H_{66}\right)\beta;$$
$$K_{88} = C_{55}\alpha^2 + C_{66}\beta^2 + D_{33}\frac{64}{h^4}; K_{89} = H_{55}\alpha^2 + H_{66}\beta^2 + F_{33}\frac{512}{h^6};$$

$$K_{91} = \frac{64}{h^4}E_{13}\alpha; K_{92} = \frac{64}{h^4}E_{23}\beta; K_{93} = -\frac{64}{h^4}\left(F_{13}\alpha^2 + F_{23}\beta^2\right);$$
$$K_{94} = \left(\frac{64}{h^4}S_{13} + H_{55}\right)\alpha; K_{95} = \left(\frac{64}{h^4}T_{13} + G_{55}\right)\alpha; K_{96} = \left(\frac{64}{h^4}S_{23} + H_{66}\right)\beta;$$
$$K_{97} = \left(\frac{64}{h^4}T_{23} + G_{66}\right)\beta; K_{98} = H_{55}\alpha^2 + H_{66}\beta^2 + F_{33}\frac{512}{h^6};$$
$$K_{99} = G_{55}\alpha^2 + G_{66}\beta^2 + O_{33}\frac{4096}{h^8};$$

6. Numerical results and discussion

6.1. Numerical Results

The results for examples solved on flexure analysis are obtained for different side to thickness ratios (a / *h* = 4, 10, 20, 50, 100) and aspect ratios (a/ b = 1) are presented in Table 1. The results of present theory are compared with those of first order shear deformation theory Mindlin [8], higher order shear deformation of Reddy [17, 18], trigonometric shear deformation theory of Sayyad and Ghugal [21, 25], exponential shear deformation theory of Sayyad and Ghugal [31], hyperbolic shear deformation theory of Ghugal and Pawar [32] and classical plate theory of Kirchhoff. Exact elasticity results are generated from the theory provided by Pagano [33] for comparison of results. The results obtained for in-plane displacements, transverse displacement, in-plane normal and shear stresses; and transverse shear stress using constitutive relation are presented in the following non-dimensional forms commonly used in the literature.

$$\bar{u}\left(0,\frac{b}{2},\frac{z}{h}\right) = \frac{uE_3}{qhS^3};\ \bar{w}\left(\frac{a}{2},\frac{b}{2},\frac{z}{h}\right) = \frac{100wE_3}{qhS^4};\ (\bar{\sigma}_x,\ \bar{\sigma}_y)\left(\frac{a}{2},\frac{b}{2},\frac{z}{h}\right) = \frac{(\sigma_x,\ \sigma_y)}{qS^2};$$
$$\bar{\tau}_{xy}\left(\frac{a}{2},\frac{b}{2},\frac{z}{h}\right) = \frac{(\tau_{xy})}{qS^2};\ \bar{\tau}_{zx}\left(0,\frac{b}{2},\frac{z}{h}\right) = \frac{\tau_{zx}}{qS};\ \bar{\tau}_{yz}\left(\frac{a}{2},0,\frac{z}{h}\right) = \frac{\bar{\tau}_{yz}}{qS};$$

where $S = a/h$ and $E_3 = E$ is elastic modulus.

Table.1 Comparison of non-dimensional in-plane displacement $\bar{u}$, transverse displacement $\bar{w}$, in-plane normal stress $(\bar{\sigma}_x)$, in-plane shear stress $(\bar{\tau}_{xy})$ and transverse shear stress $(\bar{\tau}_{zx})$ of simply supported square isotropic plate (*a* =*b*) subjected to sinusoidal distributed load.

S	Theory	Model	$\bar{u}$	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\tau}_{zx}$	$\bar{\tau}_{xy}$
			(-*h*/2)	(0)	(-*h*/2)	(0)	(-*h*/2)
4	Present	5th OSDT	0.0440	3.5350	0.2108	0.2362	0.1342

	Sayyad and Ghugal [25]	SSNDT	0.0440	3.6534	0.2267	0.2444	0.1063
	Pagano [33]	Exact	0.0454	3.6630	0.2040	0.2361	–
	Sayyad and Ghugal [31]	ESDT	0.0460	3.7480	0.2130	0.2380	0.1140
	Reddy [17, 18]	TSDT	0.0460	3.7870	0.2090	0.2370	0.1120
	Sayyad and Ghugal [21]	HSDT	0.0440	3.6530	0.2260	0.2440	0.1330
	Ghugal and Pawar [32]	HPSDT	0.0470	3.7790	0.2090	0.2360	0.1120
	Mindlin [8]	FSDT	0.0440	3.6260	0.1970	0.1590	0.1060
	Kirchhoff	CPT	0.0440	2.8030	0.1970	----	0.1060
10	Present	5th OSDT	0.0440	2.9143	0.1996	0.2383	0.1106
	Sayyad and Ghugal [25]	SSNDT	0.0439	2.9333	0.2125	0.2454	0.1060
	Pagano [33]	Exact	0.0443	2.9425	0.1988	0.2383	–
	Sayyad and Ghugal [31]	ESDT	0.0440	2.9540	0.2000	0.2390	0.1080
	Reddy [17, 18]	TSDT	0.0440	2.9610	0.1990	0.2380	0.1070
	Sayyad and Ghugal [21]	HSDT	0.0440	2.9330	0.2120	0.2450	0.1100
	Ghugal and Pawar [32]	HPSDT	0.0440	2.9590	0.1990	0.2370	0.1070
	Mindlin [8]	FSDT	0.0440	2.9340	0.1970	0.1690	0.1060
	Kirchhoff	CPT	0.0440	2.8020	0.1970	----	0.1060
20	Present	5th OSDT	0.0440	2.8303	0.1981	0.2386	0.1074
	Sayyad and Ghugal [25]	SSNDT	0.0439	2.8286	0.2105	0.2455	0.1060
	Pagano [33]	Exact	0.0440	2.8377	0.1979	0.2386	–
50	Present	5th OSDT	0.0440	2.8070	0.1977	0.2387	0.1066
	Sayyad and Ghugal [25]	SSNDT	0.0439	2.7991	0.2100	0.2456	0.1060
	Pagano [33]	Exact	0.0440	2.8082	0.1976	0.2386	–
100	Present	5th OSDT	0.0440	2.8037	0.1976	0.2387	0.1064
	Sayyad and Ghugal [25]	SSNDT	0.0439	2.7949	0.2099	0.2456	0.1060
	Pagano [33]	Exact	0.0440	2.8040	0.1976	0.2387	–

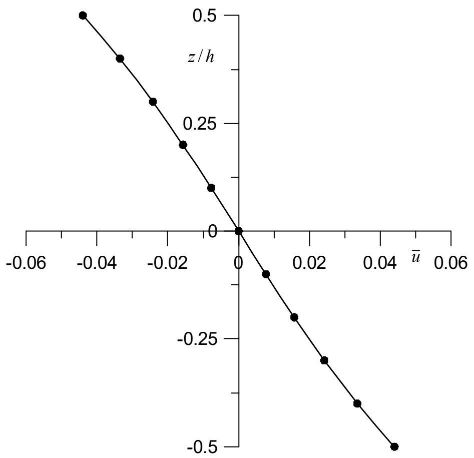

Fig. 2. Variation of in-plane displacement $\overline{u}$ through the thickness of the simply supported thick isotropic square plate subjected to sinusoidal distributed load for aspect ratio (a/h = 4).

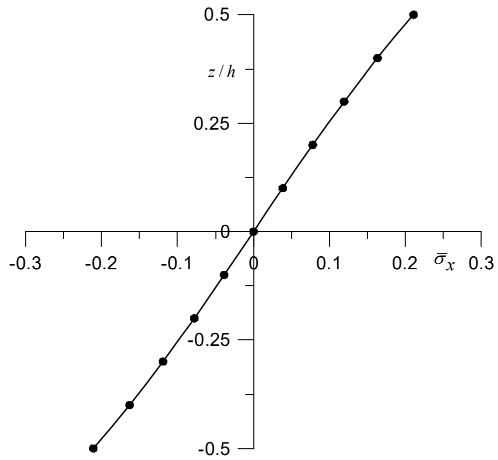

Fig. 3. Variation of in-plane normal stress $\bar{\sigma}_x$ through the thickness of the simply supported thick isotropic square plate subjected to sinusoidal distributed load for aspect ratio (a/h = 4)

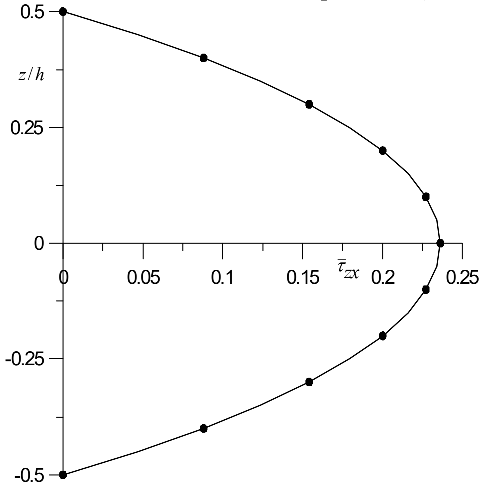

Fig. 4. Variation of transverse shear stress $\bar{\tau}_{zx}$ via constitutive equation through the thickness of the simply supported thick isotropic square plate with sinusoidal distributed load for aspect ratio (a/h = 4).

6.2. Discussion of Results

Table1 shows comparison of displacements and stresses for simply supported square isotropic thick plates subjected to sinusoidal distributed load. The in-plane displacements obtained by present theory are very close to those of refined theories and exact solution. Present theory underpredicts the transverse displacement slightly for aspect ratio 4 compared to other refined theories and the exact solution. For moderately thick plate, all the theories yield identical results. The results of inplane normal stress are in close agreement with those of refined theories and the exact theory. The transverse shear stresses predicted by present theory using constitutive relations are exactly matching with those of elasticity solutions. The variations of inplane displacement, inplane normal stress and transverse shear stress are shown in Figs. 2 through 4. The results of displacement and stresses match with those of classical plate theory in thin plate limits.

7. Conclusions

From the study of bending analysis of thick isotropic plates by using 5th order shear deformation theory, following conclusions are drawn:

1. The governing differential equations and the associated boundary conditions are variationally consistent.
2. Present theory obviates the need of shear correction factor due to the realistic variation of transverse shear stress.
3. The in-plane displacements and in plane stresses obtained by present theory are identical with those other refined theories and the exact solutions.
4. Transverse shear stresses obtained by present theory using constitutive relations satisfy shear stress free boundary conditions on the top and bottom surface of the plate. The present theory predicts exact transverse shear stresses as compared to exact solution.

5. The present quasi three-dimensional theory could be used effectively in the static thermal, bending, buckling and vibration analyses of nonhomogeneous, anisotropic, composite or sandwich thick plates.

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