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Thermoelastic Analysis of Functionally Graded Hollow Cylinder Subjected to Uniform Temperature Field

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Abstract

This paper deals with the determination of displacement function and thermal stresses of a finite length isotropic functionally graded hollow cylinder subjected to uniform temperature field. The solution of the governing thermoelastic equation is obtained, as suggested by Spencer et al. for anisotropic laminates. Numerical calculations are also carried out for FGM (Functionally graded material) system consisting of ceramic Alumina (Al_2O_3), along with Nickel (Ni) as the metallic component varying with distance in one direction and illustrated graphically.

Keywords: Uniformly heated, hollow cylinder, Thermoelastic stresses, Functionally graded material, Inverse problem.

1. Introduction

Functionally graded materials (FGMs) have been developed as a new material that is adaptable as heat-resistant material and have useful applications in furnace lines, space structures, fusion reactors and electronic component packaging. The mathematical modeling of FGMs is currently an active research area. Since the mathematical problems arising are complicated, much of the work on FGMs has been carried out numerically, e.g., using finite element method (FEM), perturbation method etc [1]. It is necessary to develop other approaches for such a problem particularly boundary value problems, which provide invaluable check on the accuracy. So, it is meaningful to investigate the thermoelastic behaviour with defined boundary conditions.

Noda et al. [1] used the Laplace transformation and perturbation method to obtain one dimensional transient thermal stress response of a functionally graded material. Horgan et al. [2] analyzed the classic problem of stress distribution in an inhomogeneous isotropic rotating solid disc and pressurized hollow cylinder. Zimmerman et al. [3] have presented solution for the problem of the uniform heating of a functionally graded cylinder with simple forms for the variation of the moduli with radius using the method of Frobenius series. Chen et al. [4] in his axisymmetric thermoelastic problem of a uniformly heated functionally graded isotropic hollow cylinder of finite length, described the stress-strain relation in terms of Lamé elastic constant.

The boundary value problem described by Noda et al. [1], Horgan et al. [2] and Chen et al. [4] occurs in design application, while interior value problem occurs in quenching studies and in the measurement of aerodynamic heating etc. For interior value problem, special methods are employed.

In a related theoretical study done by Horgan et al. [2] and Chen et al. [4] the variation of E is described by the nonlinear function $E = E^0 (r/r_a)^n$. As pointed out by Eraslan et al. [5], their model is not as flexible as the general parabolic model. With this general parabolic model, wide range of nonlinear and continuous profiles can be obtained to describe the reasonable variation of E in the material.

Shariyat [9] studied dynamic thermal buckling of suddenly heated temperature-dependent FGM cylindrical shells, under combined axial compression and external pressure. Na et al. [10] focused on the thermoelastic modeling and dynamic response of a rotating blade made of functionally graded ceramic-metal based materials. Ootao [11] done theoretical treatment of transient thermoelastic problems involving functionally graded thick plate, laminated composite strip with an interlayer of functionally graded material, and functionally graded hollow cylinder. Houari et al. [12] has determined thermoelastic bending of functionally graded sandwich plates using the two-variable refined plate theory. Marzocca et al. [13] reviews the state-of-the-art in linear and nonlinear aero-thermo-elasticity of FGM panels. Chang [14] studied an inverse algorithm to estimate the unknown time-dependent heat flux at the inner surface of a functionally graded hollow circular.

Fazelzadeh [15] studied thermal divergence of supersonic plates made of functionally graded materials. Sheng [16] analyzed Non-Linear Response of Functionally Graded Cylindrical Shells under Mechanical and Thermal Loads. The magneto-thermo-mechanical response of a functionally graded magnetoelastic material (FGMM) annular variable-thickness rotating disk is investigated by Bayat et al. [18]. Ashida [19] analyzed one-dimensional dynamic thermoelastic problem in a functionally graded material (FGM) thin film subjected to a thermal shock loading.

In the present paper, an attempt is made to study the analytical solution for an inverse thermoelastic problem to determine the unknown displacement and stress functions of an isotropic functionally graded finite-length hollow cylinder of thickness h occupying the space $D = \{(x, y, z) \in R^3 : \; r_a \le (x^2 + y^2)^{1/2} \le r_b, \; 0 \le z \le h\}$ with known interior displacement. Special solution suggested by Spencer et al. [7] for the thermoelastic distortions of anisotropic laminates are utilized to investigate the problem.

2. Gradation relations

The volume fraction distribution of metal obeying simple power law with exponent M is given as

$$V_m(r) = \left(\frac{r - r_c}{r_m - r_c} \right)^N = (\bar{r})^N \text{ for N≥0 and } r_c \le \bar{r} \le r_m \tag{1}$$

where V_m is the volume fraction of Nickel and the volume fraction parameter N measures the amount of metal and ceramic in the composition.

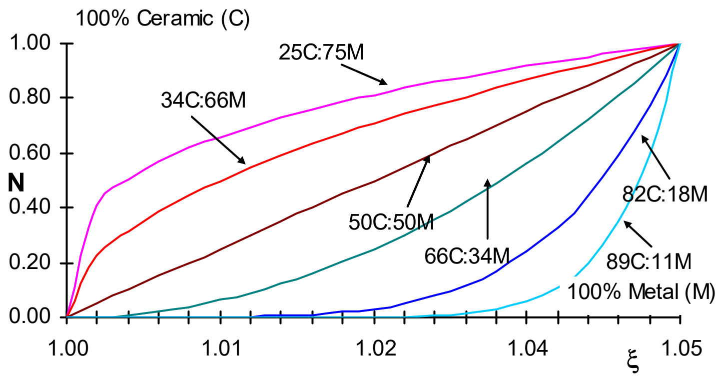

Fig. 1. Volume fraction distribution for various volume fraction parameter (N) along radius

It is observed from Figure 1 that when $N < 1$, the metallic volume fraction is greater than 52%, whereas with $N > 1$, the ceramic volume fraction increases with decrease in metallic volume fraction; and no significant change in maximum temperature was noticed with the increase in ceramic contents. Thus, we the assumption that at $N = 1$, one obtains the best composition of metal and ceramic which can withstand the maximum temperature, if power law distribution is considered for volume fraction distribution. The functionally graded property, at any point along the radius is given by: $M = M_m(V_m) + M_c(1 - V_m)$, where M_m & M_c are the material constants of two homogenous materials.

3. Notation and governing equations

Consider an isotropic functionally graded finite-length hollow cylinder of thickness h occupying the space D subjected to uniform heating. The hollow cylinder is bounded by the region $r_a \le (x^2 + y^2)^{1/2} \le r_b$, where r_a and r_b denote the inner and outer radii respectively; θ as constant. Referring to the coordinates, displacement components are denoted by $(u_r, 0, u_z)$ and stress components by $\sigma_{rr}, \sigma_{\theta\theta}$ etc. Hence we assume that,

E Young's modulus and α thermal expansion coefficients are functions of r but not of θ and z, so that the cylinder is a special case. The basic equations for the above problem can be summarized as follows:
In stress-strain relationships, it is assumed that the stress is related to the infinitesimal strain tensor $e = \varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{zz}$, and temperature ΔT as

$$\sigma_{rr} = \frac{\upsilon E}{(1+\upsilon)(1-2\upsilon)} e + \frac{E}{1+\upsilon}\varepsilon_{rr} - \frac{E}{1-2\upsilon}\alpha\ \Delta T$$

on the curved surfaces $r = r_a$ and $r = r_a$,

$$\sigma_{\theta\theta} = \frac{\upsilon E}{(1+\upsilon)(1-2\upsilon)} e + \frac{E}{1+\upsilon}\varepsilon_{\theta\theta} - \frac{E}{1-2\upsilon}\alpha\ \Delta T$$

on the edges with $\theta =$ constant, and

$$\sigma_{zz} = \frac{\upsilon E}{(1+\upsilon)(1-2\upsilon)} e + \frac{E}{1+\upsilon}\varepsilon_{zz} - \frac{E}{1-2\upsilon}\alpha\ \Delta T$$

on the ends $z = 0$ and $z = h$ (2)

where E is the Young's modulus, α is the thermal expansion coefficient and ΔT is the temperature increment relative to some reference temperature under consideration and where stresses are zero if the cylinder is unreformed.
Strain-displacement relationships:

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r},\ \varepsilon_{\theta\theta} = \frac{u_r}{r},\ \varepsilon_{zz} = \frac{\partial u_z}{\partial z}\ 2\varepsilon_{\theta r} = 0,\ 2\varepsilon_{rz} = 0,\ 2\sigma_{\theta z} = 0 \tag{3}$$

Equilibrium equations with zero body force is:

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0 \tag{4}$$

Substituting the equations (2) and (3) in equation (4), the thermoelastic equilibrium equation of the cylinder can be obtained as

$$\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r}\frac{\partial u_r}{\partial r} - \frac{u_r}{r^2} + \left(\frac{\upsilon}{1-\upsilon}\right)\frac{1}{E}\frac{\partial E}{\partial r}\left[\left(\frac{1-\upsilon}{\upsilon}\right)\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z}\right] + \left(\frac{\upsilon}{1-\upsilon}\right)\frac{\partial^2 u_z}{\partial r \partial z}$$
$$= \left(\frac{1+\upsilon}{1-\upsilon}\right)\frac{1}{E}\frac{\partial}{\partial r}[E\,\alpha\,\Delta T] \quad \text{for } r_a \le r \le r_b \text{ and } 0 \le z \le h \tag{5}$$

4. Boundary conditions

As suggested by Spencer et al. [7], a complete solution to the thermoelastic problem displacement field is to be determined such that, for $T \neq 0$, there is zero traction on all surfaces of the hollow cylinder. Thus, to seek the solution, we assume the following:
Zero traction conditions on the inner and outer curved surfaces

$$\sigma_{rr} = 0,\ \sigma_{r\theta} = 0,\ \sigma_{rz} = 0 \ \text{ at } \ r = r_a, r_b \tag{6}$$

Zero normal force on $z = 0, h$

$$2\pi \int_{r_a}^{r_b} \sigma_{zz}\ r\,dr = 0 \tag{7}$$

Boundary conditions of the finite-length cylinder be simply supported at the two longitudinal edges,

$$u_r = 0,\ \sigma_{zz} = 0,\ \sigma_{\theta z} = 0,\ \sigma_{rz} = 0 \ \text{ at } \ z = 0, h \tag{8}$$

Thus, in formulating these conditions, we have taken into account from equation (4), that $\sigma_{rz} = 0$, $\sigma_{r\theta} = 0$, $\sigma_{\theta z} = 0$ and the other stress components depend only on r as a special case.

Reformulation of the problem

Following the treatment of the gradient cylinder given in [5], we will now consider the case where Young's modulus E and thermal expansion coefficient α varies radially according to a general parabolic form given by

$$\frac{1}{E}\frac{dE}{dr} = \frac{k}{r} \tag{9}$$

and hence

$$E = E^0\{1 - n\,(r/r_a)^k\} \qquad \text{for } 0 \le n < 1 \text{ and } k > 0 \tag{10}$$

where E^0 is arbitrary constant having the same dimension as E, while n and k are the material parameters whose combination forms various concave, convex and linear E profiles.

Since the spatial variation in Poisson ratio is of much less significance than that in Young's modulus, we assume henceforth that υ is a constant while E and α are functions dependent on position. This assumption commonly made in the literature on FGM [2], leads to considerable mathematical simplification.

Substituting the equations (9) and (10) in equation (5), one obtains the thermoelastic equilibrium equation of the cylinder as

$$\frac{\partial^2 u_r}{\partial r^2}+\frac{(k+1)}{r}\frac{\partial u_r}{\partial r}+\left(\frac{k\,\upsilon}{1-\upsilon}-1\right)\frac{u_r}{r^2}+\frac{\upsilon}{1-\upsilon}\left[\frac{k}{r}\frac{\partial u_z}{\partial z}+\frac{\partial^2 u_z}{\partial r\,\partial z}\right]=2\left(\frac{1+\upsilon}{1-\upsilon}\right)\alpha^0\{1-n\,(r/r_a)^k\}\,\Delta T \tag{11}$$

where α^0 is an arbitrary constant having the same dimension as α. Now in order to simplify and obtain the result, we use a modified Poisson ratio $\upsilon'=\upsilon/(1-\upsilon)$ instead of υ in equation (11),

$$\frac{\partial^2 u_r}{\partial r^2}+\frac{(k+1)}{r}\frac{\partial u_r}{\partial r}+\left(k\,\upsilon'-1\right)\frac{u_r}{r^2}+\upsilon'\left[\frac{k}{r}\frac{\partial u_z}{\partial z}+\frac{\partial^2 u_z}{\partial r\,\partial z}\right]=2\left(1+2\upsilon'\right)\alpha^0\{1-n\,(r/r_a)^k\}\,\Delta T \tag{12}$$

It is convenient to rewrite the radial, tangential and axial stresses in terms of the displacement functions

$$\sigma_{rr}=\frac{\upsilon'(1+\upsilon')E}{(1-\upsilon')(1+2\upsilon')}\left[\frac{1}{(1-\upsilon')}\frac{\partial u_r}{\partial r}+\frac{u_r}{r}+\frac{\partial u_z}{\partial z}\right]-\left(\frac{1+\upsilon'}{1-\upsilon'}\right)E\,\alpha\,\Delta T$$

$$\sigma_{\theta\theta}=\frac{\upsilon'(1+\upsilon')E}{(1-\upsilon')(1+2\upsilon')}\left[\frac{\partial u_r}{\partial r}+\frac{1}{(1-\upsilon')}\frac{u_r}{r}+\frac{\partial u_z}{\partial z}\right]-\left(\frac{1+\upsilon'}{1-\upsilon'}\right)E\,\alpha\,\Delta T$$

$$\sigma_{zz}=\frac{\upsilon'(1+\upsilon')E}{(1-\upsilon')(1+2\upsilon')}\left[\frac{\partial u_r}{\partial r}+\frac{u_r}{r}+\frac{1}{(1-\upsilon')}\frac{\partial u_z}{\partial z}\right]-\left(\frac{1+\upsilon'}{1-\upsilon'}\right)E\,\alpha\,\Delta T \tag{13}$$

Equations (6), (7), (12) and (13) constitute the mathematical formulation of the problem under consideration.

5. Solution of the problem

We can apply the transformation to equation (12) as

$$u_r(r)=r_a\,u_r^*(\xi),\quad u_\theta(\theta)=0\quad\text{and}\quad u_z(z)=G\,r_a\,\zeta \tag{14}$$

where G is the unknown constant (i.e. independent of variables) and is to be determined, $\xi=r/r_a$ and $\zeta=z/r_a$ are the dimensionless coordinates. The term $u_r(r)=r_a\,u_r^*(\xi)$ represent a radial expansion or contraction in which, in general, the inner and outer radii change but angle remains constant and $u_z(z)=G\,r_a\,\zeta$ is a uniform axial extension or contraction.

From equations (3) and (14) one obtains

$$\varepsilon_{rr}=\frac{\partial u_r^*}{\partial\xi},\ \ \varepsilon_{\theta\theta}=\frac{u_r^*}{\xi},\ \ \varepsilon_{zz}=\frac{\partial u_z^*}{\partial\zeta},\ \ 2\varepsilon_{\theta r}=0,\ \ 2\varepsilon_{rz}=0,\ \ 2\sigma_{\theta z}=0 \tag{15}$$

Obviously all the strains and stresses depend only on the variable *r*. Substituting the equations (14) into the equation (12), and rewriting the governing equation in a dimensionless form as

$$\frac{\partial^2 u_r^*}{\partial\xi^2}+\frac{(k+1)}{\xi}\frac{\partial u_r^*}{\partial\xi}+\left(k\,\upsilon'-1\right)\frac{u_r^*}{\xi^2}=2\left(1+2\upsilon'\right)\alpha^0\left[1-n\,\xi^k\right]\Delta T-k\,\upsilon'\frac{G}{\xi} \tag{16}$$

the general solution of equation (16) can be expressed as combination of complimentary function plus particular solution. The homogenous solution can be obtained by assuming [6]

$$u_r^*(\xi)=C\,\xi^m \tag{17}$$

where C is an arbitrary constant and m is an undetermined power.

Substituting equation (17) into the equation (16) and omitting the right hand side to get a quadratic equation for *m* as

$$m^2+k\,m+(k\,\upsilon'-1)=0 \tag{18}$$

we get two roots as

$$m_1=\frac{-k-\sqrt{k^2-4(k\,\upsilon'-1)}}{2},\qquad m_2=\frac{-k+\sqrt{k^2-4(k\,\upsilon'-1)}}{2} \tag{19}$$

Now, to find the nature of the roots, we obtain the discriminate of equation (18) as $D=k^2-4(k\,\upsilon'-1)$, which is greater than zero for a given Poisson's ratio $0\le\upsilon'\le 1/2$. Thus, the two roots given in equation (19) are real and unequal.

The particular solution can be easily obtained by assuming

$$u_r^*(\xi)=C_1\,\xi^{k+1}+C_2\,\xi \tag{20}$$

and substituting into the equation (16) with the unknown parameters C_1 and C_2. The general required solution of equation (16) is

$$u_r^*(\xi)=A\,\xi^{-(k+l)/2}+B\,\xi^{(-k+l)/2}+C_1\,\xi^{k+1}+C_2\,\xi \tag{21}$$

where $l>0$ is given by $l=(k^2+4-4k\upsilon')^{1/2}$, A and B are two arbitrary constants, and

$$C_1=\frac{2\,(1+2\upsilon')}{2k^2+(\upsilon'-3)k}\,\alpha^0\,\Delta T \quad\text{and}\quad C_2=-\frac{\upsilon'}{1+\upsilon'}G \tag{22}$$

It is assumed that $2k^2+(\upsilon'-3)k\neq 0$. The boundary conditions (6) to (7) can be rewritten using equations (14) and (13) as:

The initial condition

$$u_r^*(\xi)=0\,,\ \ \frac{\partial u_r^*(\xi)}{\partial \xi}=0 \qquad \text{for } \xi=\xi_0 \text{ and all } n \text{ and } k \tag{23}$$

The radial component of stress is given by σ_{rr} in such a way that the modified third kind boundary conditions are

$$\left[\frac{\partial u_r^*}{\partial \xi}+(1-\upsilon')\frac{u_r^*}{\xi}+G(1-\upsilon')\right]_{\xi=\xi_1}=f\,(\xi_1,n,k)\,, \qquad \text{for all } n \text{ and } k \tag{24}$$

$$[u^*]_{\xi=\xi_R}=g\,(\xi_R\,,n,k)\,, \qquad \text{for all } n \text{ and } k \tag{25}$$

The interior condition is

$$\left[\frac{\partial u_r^*}{\partial \xi}+(1-\upsilon')\frac{u_r^*}{\xi}+G(1-\upsilon')\right]_{\xi=\xi_i}=h\,(\xi_i\,,n,k)\,, \qquad \text{for all } n \text{ and } k \tag{26}$$

where $f\,(\xi_1,\,n,\,k)$, $g\,(\xi_R\,,\,n,\,k)$ and $h(\xi_i\,,\,n,\,k)$ are functions having fixed radius with the inhomogeneity parameter n and k. Apart from the boundary conditions $\sigma_{rr}=0$, $\sigma_{r\theta}=0$, $\sigma_{rz}=0$ at $r=r_a$ and $r=r_b$, we should also consider the conditions $\sigma_{zz}=0$, $\sigma_{\theta z}=0$, $\sigma_{rz}=0$ at the two ends $z=0$ and $z=h$, where the last two boundary conditions of both the case have been automatically satisfied at both surfaces as shown earlier in the paper. The boundary conditions (8) can be rewritten for homogenous case using equations (14) and (13) as

$$G=\left[\frac{(1-\upsilon')(1+2\upsilon')}{\upsilon'}\right]\alpha\,\Delta T \tag{27}$$

6. Further investigation

6.1 The homogenous case

Solution for k = 0 and n = 0

For k = 0 and n = 0 one obtain all material constants of the governing equation (13) that are independent of radial coordinates, then $E=E^0=$ constant and the right-hand side of equation (13) vanishes and reduces to the Euler differential equation. The general solution takes the form

$$u_r^*(\xi)=A\,\xi^{-1}+B\,\xi \tag{28}$$

From the third kind boundary conditions (24), (26) and (27), the values of A and B are

$$A=0 \text{ and } B=-\frac{1-\upsilon'}{2-\upsilon'}G \tag{29}$$

It can be concluded that the stress component vanishes everywhere in a homogenous hollow cylinder when it is uniformly heated with k = 0 and n = 0 as the inhomogeneity parameter.

Solution for k = 0 and n $\neq$ 0

Consider the homogeneous case in which $k\to 0$, then $E=E^0(1-n)=$ constant, irrespective of n. One discovers that all material constants of the governing equation (13) are independent of the radial coordinate and the right-hand side of equation (13) vanishes and equation (13) reduces to the Euler differential equation. The general solution takes the form similar to equation (28) and from the boundary conditions (24), (26) and (27), the values of A and B are

$$A=\frac{f\,(\xi_1,n,0)-h(\xi_i\,,n,0)}{\upsilon'(\xi_i^{-2}-\xi_1^{-2})}$$

$$B = \frac{1}{2-\upsilon'}\left[f(\xi_1,n,0) - G(1-\upsilon') + \frac{f(\xi_1,n,0)-h(\xi_1,n,0)}{(\xi_i^{-2}-\xi_1^{-2})}\xi_1^{-2} \right] \tag{30}$$

Again, it can be concluded that the stress component vanishes everywhere in a homogenous hollow cylinder when it is uniformly heated with k = 0 and n ≠ 0 as the inhomogeneity parameter.

Solution for n = 0 and k ≠ 0

For $n=0$ from equation (10) we obtain $E = E^0$, irrespective of k. The general solution takes the form of equation (28) and from the boundary conditions (24), (26) and (27), the values of A and B takes the form as

$$A = \frac{f(\xi_1,0,k)-h(\xi_i,0,k)}{\upsilon'(\xi_i^{-2}-\xi_1^{-2})},\quad B = \frac{1}{2-\upsilon'}\left[f(\xi_1,0,k) - G(1-\upsilon') + \frac{f(\xi_1,0,k)-h(\xi_1,0,k)}{(\xi_i^{-2}-\xi_1^{-2})}\xi_1^{-2} \right] \tag{31}$$

Therefore it can be concluded that the stress component vanishes everywhere in a homogenous hollow cylinder when it is uniformly heated with k ≠ 0 and n = 0 as the inhomogeneity parameter. With this form of the cylindrical profile function, a uniform thickness of the cylinder is obtained by setting $n=0$ and a linearly decreasing thickness is obtained by setting $k=1$. Furthermore, if $k<1$, the profile is concave; and if $k>0$, it is convex. It can be shown that when the material is isotropic, the model represented by equation (2) is the same as employed by Eraslan [5] when the thermal effect is neglected.

6.2 The inhomogeneous case

Solution for n ≠ 0 and k ≠ 0

As pointed out by Spencer et al. [7], the form of solution considered, does not permit the point-by-point specification of traction at the two ends $z=0$ and $z=h$. Only resultant forces and moments can be specified on the basis of Saint Venant's principle. From the problem that we have considered, we get the boundary condition (7), which gives rise to a dimensionless form as

$$\int_{\xi_1}^{\xi_i} E\left[\frac{\partial u_r^*}{\partial \xi} + \frac{u_r^*}{\xi} + \frac{G}{(1-\upsilon')}\right]\xi\, d\xi = \frac{1+2\upsilon'}{\upsilon'}\Delta T \int_{\xi_1}^{\xi_i} E\,\alpha\,\xi\, d\xi \tag{32}$$

Thus we have

$$\begin{aligned}
&A\left\{\xi_i^{-(k+l-2)/2}\left[1+\frac{n(k+l-2)}{(k-l+2)}\xi_i^{k}\right] - \xi_1^{-(k+l-2)/2}\left[1+\frac{n(k+l-2)}{(k-l+2)}\xi_1^{k}\right]\right\}\\
&+B\left\{\xi_i^{(-k+l+2)/2}\left[1+\frac{n(k-l-2)}{(k+l+2)}\xi_i^{k}\right] - \xi_1^{(-k+l+2)/2}\left[1+\frac{n(k-l-2)}{(k+l+2)}\xi_1^{k}\right]\right\}\\
&+C_1\left\{\xi_i^{k+2}\left[1-\frac{n(k+2)}{2(k+1)}\xi_i^{k}\right] - \xi_1^{k+2}\left[1-\frac{n(k+2)}{2(k+1)}\xi_1^{k}\right]\right\}\\
&+\left(C_2+\frac{G}{2(1-\upsilon)}\right)\left\{\xi_i^{2}\left[1-\frac{n}{(k+2)}\xi_i^{k}\right] - \xi_1^{2}\left[1-\frac{n}{(k+2)}\xi_1^{k}\right]\right\}\\
&=\alpha^0\left(\frac{1+2\upsilon'}{\upsilon'}\right)\left\{\frac{1}{2}(\xi_i^{2}-\xi_1^{2}) - \frac{2n}{(k+2)}(\xi_i^{k+2}-\xi_1^{k+2}) + \frac{n^2}{2(k+1)}(\xi_i^{2k+2}-\xi_1^{2k+2})\right\}\Delta T
\end{aligned} \tag{33}$$

Using equation (21) in the equations (24), (26), (27) and (33), the values of *A, B* and *G* for $n \neq 0$ and $k \neq 0$ can be obtained.

7. Numerical results and discussion

Consider a hollow cylinder under thermoelastic third kind boundary value problem with an inner radius r_a = 0.0126 m and an outer radius r_b =0.0189 m. It is assumed that the hot gases which are flowing within the hollow cylinder are at uniform temperature of ΔT = 1423 K. The initial condition is chosen to be uniform at the room temperature of 296.23 K. The radial displacement at an inner surface of the cylinder is assumed to be known; while at an outer surface, third kind boundary conditions will be imposed. The ceramic chosen is Alumina (Al_2O_3) along with metal Nickel (Ni) as the metallic component of the FGM having the following values of material properties [1]:

CTE: α_c =8.60 X 10-6 [/K], α_m = 18.30 X 10-6 [/K]

Poisson's ratio: υ_c = 0.22, υ_m = 0.30

Elasticity: E_c = 390 [GPa], E_m = 203 [GPa]

7.1. Thermal stresses in hollow cylinder

It is observed that for inhomogeneous thin hollow cylinder, the radial stress distribution, the tangential stress distribution and an axial stress distribution all with respect to $E^0\alpha^0\Delta T$ versus dimensionless radius ξ having the inner radius, the outer radius, and the interior point for the inhomogeneity parameter k = 0 irrespective of n, occur highest at the inner edge and decay with the power function of radius when heated uniformly.

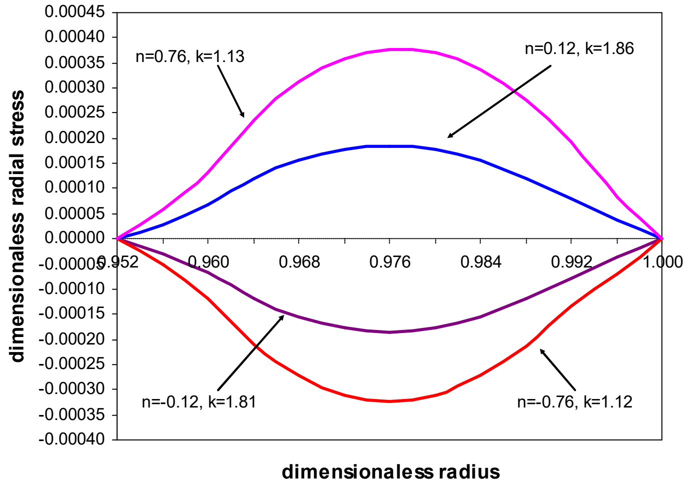

Fig. 2. Distribution of the radial stress versus radius for different values of material parameters for thin hollow cylinder

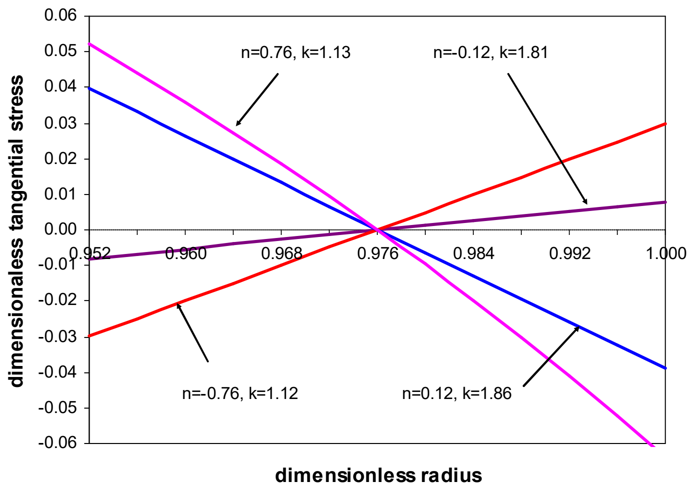

Fig. 3. Distribution of the tangential stress versus radius for different values of material parameters for thin hollow cylinder

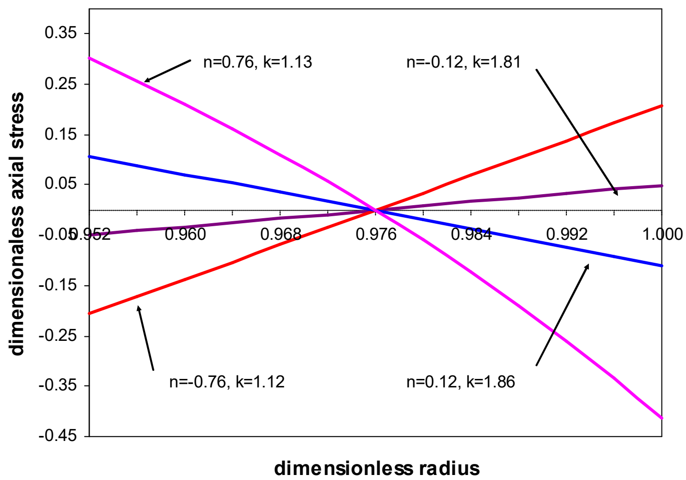

Fig. 4. Distribution of the axial stress versus radius for different values of material parameters for thin hollow cylinder

Figures 2 to 4 are typical dimensionless plots of the radial stress distribution, the tangential stress distribution and an axial stress distribution versus dimensionless radius ξ in a thin circular hollow cylinder with $\xi_b = r_b / r_a$ = 1.05, where ξ_b is an outer radius–to–inner radius ratio. In these figures we take $\upsilon' = 0.26$ as the constant factor, and non-dimensional stresses with respect to $E^0 \alpha^0 \Delta T$ would occur during uniform heating.
When the inhomogeneity parameter $n \neq 0$ and $k > 0$, non dimensional radial stress is maximum at the interior and so the outer edges of the cylinder tend to expand more than the inner core, leading to the inner part being under tensile stress, whereas with the decrease of inhomogeneity parameter k, the non-dimensional radial stress also decreases at fixed dimensionless radius ξ leading to compressive radial stress. The tangential stress and an axial stress will also have maximum at the interior and gradually decreases.

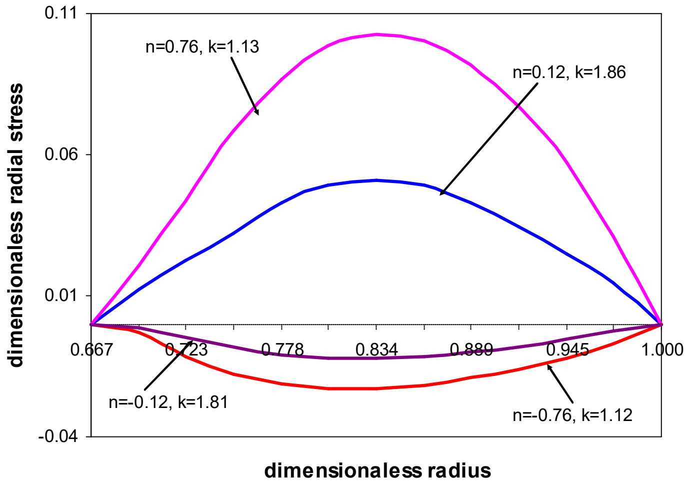

Fig. 5. Distribution of the radial stress versus radius for different values of material parameters for thick hollow cylinder

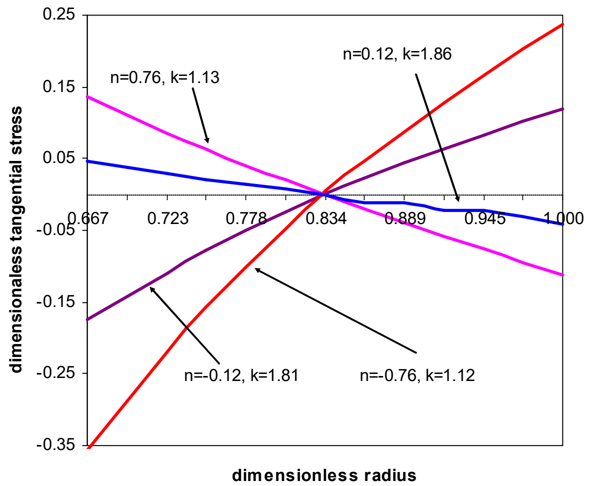

Fig. 6. Distribution of the tangential stress versus radius for different values of material parameters for thick hollow cylinder

Figures 5 to 7 are typical dimensionless plots for thick hollow cylinder with $\xi_b = r_b / r_a$ = 1.50, where ξ_b is an outer radius–to–inner radius ratio. It is observed that the inhomogeous parameters and the inner radius to outer radius ratio have significant influence on the distributions of thermoelastic stresses.

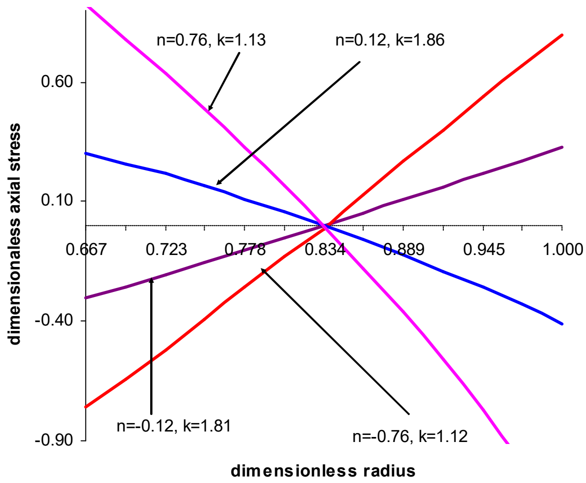

Fig. 7. Distribution of the axial stress versus radius for different values of material parameters for thick hollow cylinder

8. Conclusion

The main objective of this work is to obtain tractable analytical solutions for further parametric studies. Stress and displacement solutions for a functionally graded isotropic functionally graded hollow cylinder with parabolically–varying elastic modulus in the radial direction. Although the case of FGM cylinder with variation of elastic properties obeying a simple power law is extensively studied, the results for parabolically–varying properties are scarcely available in the literature. In this theoretical study, it was noticed that the positive inhomogeneity constant refers to increasing stiffness in the radial direction and provides a stress shielding effect, while negative inhomogeneity would create a stress amplification effect. The inhomogeneity constants, which includes continuously varying volume fraction of the constituents, can be empirically determined and may be useful parameter for optimizing the design in terms of material usage and performance.

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