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Springbackward Phenomenon of a Transversely Isotropic Functionally Graded Composite Cylindrical Shell

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Abstract

This article gives an approach to predict the springbackward phenomena during post solidification cooling in a functionally graded hybrid composite cylindrical shell with transverse isotropic structure. Here the material properties are considered to be given with a general parabolic power-law function. During theoretical analysis, appropriate transform is introduced in the equilibrium equation which is resulting into hyper geometrical differential equation. Thermoelastic solutions are obtained and are investigated for homogeneous, nonhomogenous and elastic-plastic state. The solution is validated by applying it to a multilayered functionally graded cylindrical shell using transfer or propagator matrix method.

Keywords: Thermoelasticity, Functionally Graded Hybrid Composites, Cylindrical Shell, Spring Backward Effect.

1. Introduction

'Springback' is an effect that occurs in the processing of post-solidification cooling of functionally graded hybrid composite semi-cylindrical shell. The concept of hybrid composites has been discussed since the beginning of the 70's by several researchers describing various theoretical and practical aspects of these materials. After cooling, solidification and removal from the mould, the product is usually found to have distorted from shape of the mould. This distortion is due to thermoelastic deformation that accompanies cooling to ambient temperature after solidification at a higher temperature. In the functionally graded composite materials that are microscopically inhomogeneous in unique characterization, both in its thermal and its mechanical properties, the amount of thermal contraction depends strongly on the direction in the material, which results in the change of shape as well as the uniform contraction found in an isotropic material.

The mathematical modeling of functionally graded materials is currently an active research area because of their increasing application in industrial engineering. Springforward of channel sections, buckling problem, three-dimensional thermal stress problem for functionally graded materials (FGMs) were studied by [1-5]. Axisymmetric thermoelasticity, weight reduction and high thermal radiation, plane strain solutions, stresses in functionally graded infinite strip (FGIS) were analyzed by [6-10]. Huang et. al. [11] suggested a reduced-basis method (RBM) to real-timely analyze the transient response displacement in FGM. Analytic solutions were developed by [12-14] to solve non-linear heterogeneous thermoelasticity equations in two-dimension and to determine deformations and stresses in circular disks made of FGMs. Ma et. al. [15] analyzed the heat conduction problem of nonhomogeneous functionally graded materials for a layer sandwiched between two half-planes and obtained the full-field solutions of temperature and heat flux using Fourier transform method.

Birman et. al. [16] gave a brief review on uses of FGM in Engineering. Chiba et. al. [17] presented a method for optimisation of the material composition of functionally graded materials (FGMs) for thermal stress relaxation which consists of a multiscale thermoelastic analysis and a genetic algorithm. Lamba et. al. [18] studied the uncoupled thermoelastic response of thick cylinder of length 2h in which heat sources are generated according to the linear function of the temperature, with boundary conditions of the radiation type. Gaikwad [19] determined temperature, displacement, and thermal stresses in a thin circular plate due to uniform internal energy generation. Recently Matveenko et. al. [20] presented the results of analytical and numerical investigations on estimating the character of the singularity of stresses of the 2D elastic solids made of FGMs. Williams et. al. [21] studied the numerical correlation of a multiple concentric cylinder model for the thermoplastic response of metal matrix composites.

In the present investigation, axisymmetric thermoelastic problem to determine the unknown displacement and stress distribution of a functionally graded transversely isotropic hybrid composite cylindrical shell having springback effect along radial direction in an interfacial zone during post-solidification cooling is studied. Special solution suggested by Spencer et. al. [22] for the thermoelastic distortions of laminated anisotropic tubes and guideline adopted by Varghese [23] are adopted to investigate the problem. The numerical computation for the displacement and stress functions with this general parabolic model is demonstrated. Further solution is validated by applying the concept of springback effect during post-solidification cooling to a multilayered functionally graded cylindrical shell using transfer or propagator matrix method.

2. Formulation of the Problem

For our theoretical study, we consider an interphase zone in a composite of fiber and the metal-matrix postulating that behavior is greatly affected at this phase. This interfacial region can arise naturally due to chemical reaction between fiber and the polymeric matrix, or can be introduced deliberately [21]. In the composite system, the metal undergoes elastic-plastic deformations and the fiber deforms elastically. It is assumed that the cylindrical shell is occupying the space $a \le r \le b,\quad 0 \le z \le h$, where a and b denote the inner and outer radii respectively and θ is constant.

The basic equations for the above problem are given by

a) Strain-displacement relationships [9]:

$$\varepsilon_{rr} = u_r' \ , \ \varepsilon_{\theta\theta} = r^{-1} u_r \ , \ \varepsilon_{zz} = \hat{u}_z \ 2\varepsilon_{\theta r} = 0 \ , \ 2\varepsilon_{rz} = 0 \ , 2\sigma_{\theta z} = 0 \tag{1}$$

where the prime (′) and (^) denotes differentiation with respect to r and z.

b) Equilibrium equations, with zero body force, reduce to the single equation [22]:

$$\sigma_{rr}' + r^{-1}(\sigma_{rr} - \sigma_{\theta\theta}) = 0 \tag{2}$$

c) Non zero radial stress σ_{rr}, tangential stress $\sigma_{\theta\theta}$, and axial stress σ_{zz} are:

$$\begin{aligned}
\sigma_{rr} &= c_{11}\varepsilon_{rr} + c_{12}\varepsilon_{\theta\theta} + c_{13}\varepsilon_{zz} - c_{11}(\varepsilon_{rr}^{p} + \varepsilon_{rr}^{per}) - c_{12}(\varepsilon_{\theta\theta}^{p} + \varepsilon_{\theta\theta}^{per}) \\
&\quad - c_{13}(\varepsilon_{zz}^{p} + \varepsilon_{zz}^{per}) + \beta_1 \Delta T \qquad \textit{on the curved surfaces } r = a \textit{ and } r = b \\
\sigma_{\theta\theta} &= c_{12}\varepsilon_{rr} + c_{11}\varepsilon_{\theta\theta} + c_{13}\varepsilon_{zz} - c_{12}(\varepsilon_{rr}^{p} + \varepsilon_{rr}^{per}) - c_{11}(\varepsilon_{\theta\theta}^{p} + \varepsilon_{\theta\theta}^{per}) \\
&\quad - c_{13}(\varepsilon_{zz}^{p} + \varepsilon_{zz}^{per}) + \beta_1 \Delta T \qquad \textit{on the edges with } \theta = \textit{cons}\tan t \\
\sigma_{zz} &= c_{13}\varepsilon_{rr} + c_{13}\varepsilon_{\theta\theta} + c_{33}\varepsilon_{zz} - c_{13}(\varepsilon_{rr}^{p} + \varepsilon_{rr}^{per}) - c_{13}(\varepsilon_{\theta\theta}^{p} + \varepsilon_{\theta\theta}^{per}) \\
&\quad - c_{33}(\varepsilon_{zz}^{p} + \varepsilon_{zz}^{per}) + \beta_3 \Delta T \qquad \textit{on the ends } z = 0 \textit{ and } z = h
\end{aligned} \tag{3}$$

where the stress σ is related to the infinitesimal strain tensor $e = \varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{zz}$ and $\Delta T = T_s - T$ is the change in temperature below that temperature T_s at which the lay-up solidification process takes place. In the proceeding equations, ε_{ij} denotes the total strain, σ_{ij} denotes the total stress, superscripts p and per have been used to designate plastic and permanent strains, respectively. In a plastically pre-deformed region ε_{ij}^{per} is nonzero and it is always zero otherwise. In a purely elastic deformation of functionally graded hybrid composite cylindrical shell during post-solidification cooling we have $\varepsilon_{ij}^{p} = \varepsilon_{ij}^{per} = 0$. Stress-temperature coefficient β_i $(i = 1,2,3)$ are related to $(i = 1,2,3)$ α_i are given as:

a) $\beta_1 = c_{11}\alpha_1 + c_{12}\alpha_1 + c_{13}\alpha_3$, b) $\beta_2 = c_{12}\alpha_1 + c_{11}\alpha_1 + c_{13}\alpha_3$, $\beta_3 = c_{13}\alpha_1 + c_{13}\alpha_1 + c_{33}\alpha_3$

Referred to the coordinates, displacement components are denoted by $(u_r, 0, u_z)$ and stress components by $\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{zz}$ etc. Hence we assume that the material constants c_{ij} and the coefficient of thermal expansion α_i are functions of r but not of θ and z , so that the functionally graded hybrid composite cylindrical shell is a special case.

Substituting the equations (1) and (3) in equation (2), the equilibrium equation of the functionally graded hybrid composite cylindrical shell can be obtained as

$$\begin{aligned} & c_{11} u_r'' + (c_{11}' + r^{-1} c_{11}) u_r' + (c_{12}' - r^{-1} c_{11})\, r^{-1} u_r \\ & = -\beta_1' \,\Delta T - c_{13}' \hat{u}_z + (\varepsilon_{rr}^{p} + \varepsilon_{rr}^{per})(c_{11}' + r^{-1}(c_{11} - c_{12})) \\ & \quad + (\varepsilon_{\theta\theta}^{p} + \varepsilon_{\theta\theta}^{per})(c_{12}' - r^{-1}(c_{11} - c_{12})) + c_{13}' (\varepsilon_{zz}^{p} + \varepsilon_{zz}^{per}) + c_{11} (\varepsilon_{rr}'^{p} + \varepsilon_{rr}'^{per}) \\ & \quad + c_{12} (\varepsilon_{\theta\theta}'^{p} + \varepsilon_{\theta\theta}'^{per}) \qquad for\ a \le r \le b\ \ and\ \ 0 \le z \le h \end{aligned} \tag{4}$$

2.1 Boundary Conditions

As suggested by [22] for a complete solution of the thermoelastic problem, displacement field is to be determined such that, for $\Delta T \neq 0$, there is zero traction on all surfaces of the cylindrical shell during post-solidification cooling. Thus to seek we assume the following:

Traction vanishes at the inner and outer curved surfaces

$$\sigma_{rr} = 0,\ \sigma_{r\theta} = 0,\ \sigma_{rz} = 0 \quad at \quad r = a, b \tag{5}$$

Vanishing of the normal force on $z = 0, h$

$$2\pi \int_a^b \sigma_{zz}\, r\, dr = 0 \tag{6}$$

The cylindrical shell be simply supported at the two longitudinal edges,

$$u_r = 0,\ \sigma_{zz} = 0,\ \sigma_{\theta z} = 0,\ \sigma_{rz} = 0 \qquad at \quad z = 0, h \tag{7}$$

Thus in formulating these conditions we have taken into account from equations (5) to (7) that $\sigma_{rz} = 0,\ \sigma_{r\theta} = 0,\ \sigma_{\theta z} = 0$ and the other stress components depends only on r as a special case.

As the problem is concerned with the radial direction only, we have not considered zero resultant force and bending moment on the edges $\theta = 0$, $\theta = \theta_0$. It has been observed from the earlier literature that the solution may leave unequilibrated bending moment and a shear force on the ends of the finite-length functionally graded hybrid composite cylindrical shell. To neutralize this moment and force, additional solution that involves stress that depends on the angle θ as well as *r* has been considered.

3. Reformulation of the Problem

Following [9], we consider the case where material constants c_{ij} and the thermal expansion coefficient α_i vary radially according to a general parabolic form given by

$$c_{ij} = c_{ij}^0 [1 - n (r/b)^m],\ \alpha_i = \alpha_i^0 [1 - n (r/b)^m] \tag{8}$$

where c_{ij}^0 and α_i^0 are arbitrary constants having the same dimension as c_{ij} and α_i respectively, n and m are the material parameters whose combination forms wide range of nonlinear and continuous profiles to describe reasonable variation of material constants and thermal expansion coefficients, when the thermal effect is neglected.

Equations (5), (6), (7) and (8) constitute the mathematical formulation of the problem under consideration.

4. Solution of the Problem

4.1 Analytical Stresses in the Elastic State

Using equation (8) in equation (4) and taking $\varepsilon_{ij}^{p} = \varepsilon_{ij}^{per} = 0$, we obtain a standard form of differential equation as

$$\begin{aligned} & r^2 [1 - n (r/b)^m] u_r'' + r [1 - n (1 - m)(r/b)^m] u_r' - [1 - n (1 - m C_2^0)(r/b)^m] u_r \\ & = m\, r\, n (r/b)^m \{-\beta_1^0 C_1^0 \Delta T - C_3^0 \hat{u}_z\} \\ & for\ a \le r \le b\ and\ 0 \le z \le h \end{aligned} \tag{9}$$

where $C_1^0 = 1/c_{11}^0,\ C_2^0 = c_{12}^0/c_{11}^0,\ C_3^0 = c_{13}^0/c_{11}^0$. Equation (9) is a hypergeometric differential equation, which can be solved by introducing a new variable $x = x(r) = n (r/b)^m$ and applying the transformations

$$u_r(r) = r u_r^*(x),\ u_\theta(\theta) = 0,\ u_z(z) = G z \tag{10}$$

where *G* is the unknown constant (i.e. independent of *r*) and to be determined.

The term $u_r(r) = r u_r^*(x)$ represents a radial expansion or contraction in which, in general, the inner and outer radii change but angle remains constant. The term $u_z(z) = G z$ is an uniform axial expansion or contraction.

Substituting equations (10) into equation (9), and rewriting the governing equation, we obtain

$$x\,[1-x\,]\widehat{u}_r^{*}(x\,)+[m+2(1-x\,)]\,m^{-1}\widehat{u}_r^{*}(x\,)-[C_2^0-1]\,m^{-1}u_r^{*}(x\,)=m^{-1}\{-\beta_1^0C_1^0\,\Delta T-C_3^0\,G\}\qquad(11)$$

where the prime ($\frown$) denotes differentiation with respect to *x*.

The general solution of equation (11) can be expressed as a combination of complimentary function and particular integral. The standard form of the homogenous hypergeometric differential equation can be obtained by setting right hand side of equation (11) to be zero and the solution can be cast in the form [24]

$$u_r^{*}(x\,)=\hat{C}_1\,F(\alpha,\beta,\delta;x\,)+\hat{C}_2\,(-1)^{-2/m}\,x^{-2/m}\,F(\alpha-\delta+1,\beta-\delta+1,2-\delta;x\,)\qquad(12)$$

where $\hat{C}_i$ $(i=1,2)$ is an arbitrary integration constant to be determined and $F(\alpha,\beta,\delta;x)$ is the hypergeometric function, which is defined as the analytic continuation of the so-called hypergeometric series,

$$\begin{aligned}F(\alpha,\beta,\delta;x\,)=1+\frac{\alpha\beta}{\delta}\frac{x}{1!}+\frac{\alpha\,(\alpha+1)\,\beta\,(\beta+1)}{\delta\,(\delta+1)}\frac{x^2}{2!}+\ldots\ldots\ldots\\+\frac{\alpha\,(\alpha+1)\ldots(\alpha+j-1)\,\beta\,(\beta+1)\ldots(\beta+j-1)}{\delta\,(\delta+1)\,(\delta+j-1)}\frac{x^j}{j\,!}+\ldots\ldots\end{aligned}\qquad(13)$$

As observed from [25], $F(\alpha,\beta,\delta;x)$ converges slowly within the unit circle $|x|\le 1$ provided that $\delta-(\alpha+\beta)>-1$. Since the problem under consideration is a realistic physical problem, these conditions are always satisfied and the series is always convergent. The arguments α , β and δ are real and dimensionless scalars of the hypergeometric function F in equation (12) have the following meanings:

$$\begin{aligned}&\alpha=-(1/2)+(1/m)-(1/2m)\sqrt{4+4m^2-4m\,C_2^0}\,,\ \beta=-(1/2)+(1/m)+(1/2m)\sqrt{4+m^2-4m\,C_2^0}\\&\delta=1+(2/m)\end{aligned}\qquad(14)$$

The complimentary function given by equation (12) can be re-written as (For details see Appendix B)

$$u_r(r)=C_1\,P(r)+C_2\,Q(r)\qquad(15)$$

where P and Q are the fundamental solutions of the reduced differential equation as

$$\begin{aligned}&P(r)=r\,F(\alpha,\beta,\delta;n(r/b)^m)\\&Q(r)=r^{-1}\,F(\alpha-\delta+1,\beta-\delta+1,2-\delta;n(r/b)^m)\end{aligned}\qquad(16)$$

The particular integral solution $R(r)$ of equation (9) is determined by the method of variation of parameters. It is assumed to be of the form [2]

$$R(r)=\hat{U}_1(r)P(r)+\hat{U}_2(r)Q(r)\qquad(17)$$

where

$$\begin{aligned}&\hat{U}_1(r)=-\int_a^r\frac{Q(\xi)f(\xi)}{W(\xi)}d\xi\\&\hat{U}_2(r)=\int_a^r\frac{P(\xi)f(\xi)}{W(\xi)}d\xi\\&W(r)=\det\begin{pmatrix}P&Q\\P'&Q'\end{pmatrix}=P(r)Q'(r)-Q(r)P'(r)\\&f(r)=(1/(n\,m\,(r/b)^m\,[1-n\,(r/b)^m\,]))\{-\beta_1^0C_1^0\,\Delta T-C_3^0\,G\}\end{aligned}\qquad(18)$$

represents the nonhomogeneous term of the differential equation.

The general required solution of equation (10) can be expressed as a combination of complimentary function and particular integral as

$$u_r(r)=C_1\,P(r)+C_2\,Q(r)+R(r)\qquad(19)$$

The derivatives $P'(r)$ and $Q'(r)$ in equation (18) are evaluated using the differentiation rule given by [25] as

$$\begin{aligned}&F'(\alpha,\beta,\delta;x\,(r))=(\alpha\beta/\delta)\,x\,'(r)F(\alpha+1,\beta+1,\delta+1;x\,(r)),\\&R'(r)=\hat{U}_1(r)P'(r)+\hat{U}_2(r)Q'(r)\end{aligned}\qquad(20)$$

Since $P(r)$, $Q(r)$ and $W(r)$ are polynomials, the integrals (18) may be evaluated analytically by expanding the integrands into Taylor series. If these expansions are not possible because of the product $f(r)$, accurate evaluations may nevertheless be accomplished by the application of the Gaussian Quadrature rule of integration. With the form (19) of the radial displacement, the stresses become

$$\begin{aligned}\sigma_{rr} &= C_1(c_{11}P'(r)+r^{-1}c_{12}P(r))+C_2(c_{11}Q'(r)+r^{-1}c_{12}Q(r)) \\ &\quad +(c_{11}R'(r)+r^{-1}c_{12}R(r))+c_{13}G+\beta_1\,\Delta T \\ \sigma_{\theta\theta} &= C_1(c_{12}P'(r)+r^{-1}c_{11}P(r))+C_2(c_{12}Q'(r)+r^{-1}c_{11}Q(r)) \\ &\quad +(c_{12}R'(r)+r^{-1}c_{11}R(r))+c_{13}G+\beta_1\,\Delta T \\ \sigma_{zz} &= C_1(c_{13}P'(r)+r^{-1}c_{13}P(r))+C_2(c_{13}Q'(r)+r^{-1}c_{13}Q(r)) \\ &\quad +(c_{13}R'(r)+r^{-1}c_{13}R(r))+c_{33}G+\beta_3\,\Delta T\end{aligned} \tag{21}$$

Thermoelastic solution is completed by the application of boundary conditions.

4.2 Further Investigation

The Homogenous Case:

For $n=0$ and $m>0$, one obtains all material constants of the equations (8) that are independent of radial coordinates, then $c_{ij}=c_{ij}^0$, $\alpha_i=\alpha_i^0$, and $F(\alpha,\beta,\delta;0)=1$ irrespective of m. From equation (16) we obtain $P(r)=r$ and $Q(r)=r^{-1}$, thus the right-hand side of the general solution (15) reduces to

$$u_r(r)=C_1\,r+r^{-1}C_2 \tag{22}$$

The boundary conditions to evaluate integration constants C_1 and C_2 are $\sigma_{rr}=0$, $\sigma_{r\theta}=0$, $\sigma_{rz}=0$ at $r=a$ and $r=b$, where the last two boundary conditions of both the cases have been automatically satisfied at both surfaces as shown earlier in the paper. Then by the virtue of equation (22) and equation (5), equation (21) can be rewritten as

$$\begin{aligned}C_1(c_{11}+c_{12})-a^{-2}C_2(c_{11}-c_{12}) &= -\beta_1\,\Delta T-c_{13}G \\ C_1(c_{11}+c_{12})-(a\eta)^{-2}C_2(c_{11}-c_{12}) &= -\beta_1\,\Delta T-c_{13}G\end{aligned} \tag{23}$$

where $\eta=b/a$ is the outer radius-to-inner radius ratio. Apart from the boundary conditions given in equation (7) along radial direction, we should also consider the conditions at $z=0$ and $z=h$.

The last two conditions $\sigma_{\theta z}=0$, $\sigma_{rz}=0$ at $z=0,h$ automatically gets satisfied and the first condition $\sigma_{zz}=0$ at $z=0,h$ gives

$$2C_1c_{13}=-\beta_3\,\Delta T-c_{33}G \tag{24}$$

From equations (23) and (24), we have

$$C_1=-\alpha_1^0\,\Delta T\quad,\; C_2=0\;\text{ and }\; G=-\alpha_3^0\,\Delta T \tag{25}$$

It can be concluded that the stress component vanishes everywhere in a homogenous transversely isotropic functionally graded hybrid composite cylindrical shell during post-solidification cooling with $n=0$ and $m>0$ as the inhomogeneity parameter. It is also observed that as $m\to 0$, $c_{ij}=c_{ij}^0(1-n)$ as well as $\alpha_i=\alpha_i^0(1-n)$ $\to$ constant, irrespective of n. We find that all material constants are independent of the radial coordinate and governing equation reduces to the Euler's differential equation.

The Inhomogeneous Case:

For the inhomogeneity parameter $n\neq 0$ and $m\neq 0$, the radial stress expression (21) can be rewritten utilizing boundary conditions (5) as

$$\begin{aligned}&C_1(c_{11}P'(a)+a^{-1}c_{12}P(a))+C_2(c_{11}Q'(a)+a^{-1}c_{12}Q(a)) \\ &+(c_{11}R'(a)+a^{-1}c_{12}R(a)) = -\beta_1\,\Delta T-c_{13}G \\ &C_1(c_{11}P'(a\eta)+(a\eta)^{-1}c_{12}P(a\eta))+C_2(c_{11}Q'(a\eta)+(a\eta)^{-1}c_{12}Q(a\eta)) \\ &+(c_{11}R'(a\eta)+(a\eta)^{-1}c_{12}R(a\eta)) = -\beta_1\,\Delta T-c_{13}G\end{aligned} \tag{26}$$

As pointed out by [22], the form of solution considered does not permit the point-by-point specification of traction at the two ends $z=0$ and $z=h$. Only resultant forces and moments can be specified on the basis of Saint Venant's principle. From the problem that we have considered, we get the boundary condition (6) as

$$\begin{aligned}&C_1\int_a^{a\eta}(P'(r)+r^{-1}P(r))\,r\,dr+C_2\int_a^{a\eta}(Q'(r)+r^{-1}Q(r))\,r\,dr \\ &+\int_a^{a\eta}(R'(r)+r^{-1}R(r))\,r\,dr = (-\beta_3\,\Delta T-c_{33}G)\int_a^{a\eta}r\,dr\end{aligned} \tag{27}$$

As explained earlier to carry out integrals involving Hypergeometric functions we can expand the function into Taylor series and integrate the series. However, for this, n and m should be assigned numerical value e.g. $n=0.5$, m

= 1.5, or Gaussian Quadrature rule of integration can also be used. As the Hypergeometric functions are infinite series, the result will be exact. The general expressions for stress and displacement contains unknown integration constants C_1, C_2 and unknown coefficient constant G. For the determination of all three unknown constants three non-redundant conditions (26) and (27) are available.

4.3 Analytical Stresses in the Elastic-Plastic State

Total strain are expressed as the superposition of elastic and plastic parts in the form $\varepsilon_{ij} = \varepsilon_{ij}^{e} + \varepsilon_{ij}^{p} + \varepsilon_{ij}^{per}$, while the superscripts e and $(p+per)$ denote elastic and plastic, respectively. The total axial strain along radial symmetric in the interfacial zone during post-solidification cooling of functionally graded hybrid composite cylindrical shell can be assumed as [9]

$$\sigma_{zz} - \upsilon(\sigma_{rr} + \sigma_{\theta\theta}) = 0 \tag{28}$$

Since $\sigma_{\theta\theta} > \sigma_{zz} > \sigma_{rr}$ throughout the functionally graded hybrid composite cylindrical shell, Tresca's yield criterion reads

$$\sigma_{\theta\theta} - \sigma_{rr} = \sigma_0 \tag{29}$$

and the flow rule associated with this yielding is $\varepsilon_{\theta\theta}^{p} = -\varepsilon_{rr}^{p}$ and $\varepsilon_{zz}^{p} = 0$.

Substituting equations (28) and (29) in equation (2)

$$\begin{aligned} &c_{11}u_r'' + (c_{11}' + c_{12}r^{-1} - c_{12} + c_{11})u_r' + (-c_{12}r^{-4} + r^{-1}c_{12}' + c_{12}r^{-1} - c_{11}r^{-1})u_r \\ &+ 2c_{12}\varepsilon_{rr}^{p} - 2c_{11}\varepsilon_{rr}^{p} + (c_{12} - c_{11})\varepsilon_{rr}'^{p} + (c_{12}' - c_{11}')\varepsilon_{rr}^{p} + \beta_1'\Delta T \end{aligned} \tag{30}$$

5. Numerical Results and Discussion

To present some results pertaining to the thermoelastic solution, the nondimensional variables listed in nomenclature are used. A plot of distribution of stresses against the nondimensional radial coordinate for homogenous case by taking m = 1, n = 0 is shown in Fig.1, whereas for inhomogeneous case we have considered m = 1, n = 1 in Fig.2 and m = 1, n = 2 in Fig.3 and m = 1, n = -1 in Fig.4.

From fig.1 it is observed that the absolute values of radial, tangential and axial stresses are high in the steady state at the outer radius and are slowly decreasing towards the inner radius exponentially. During analysis it was observed that the stresses change with the increase or decrease in the material parameter m.

From fig.2 and fig.3 it is seen that the variation of radial, tangential and axial stresses is similar to that of homogeneous case as seen in fig.1, but the magnitude is increasing with increase in the material parameter n.

From fig.4 it is seen that the absolute values of radial, tangential and axial stresses are high at the outer radius and are slowly decreasing towards the inner radius exponentially. Also the magnitude of axial stress is more as compared to that of radial and tangential stresses.

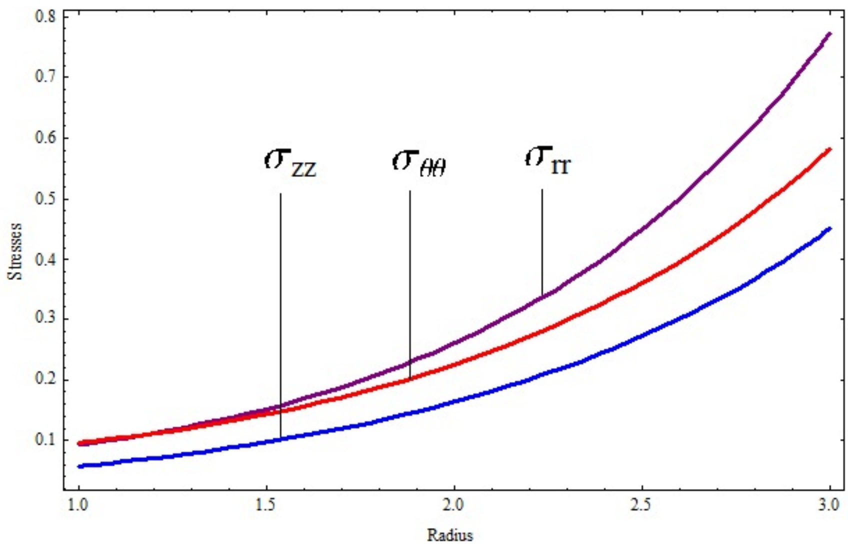

Fig.1. Thermal Stresses in Homogeneous Case with m = 1, n = 0

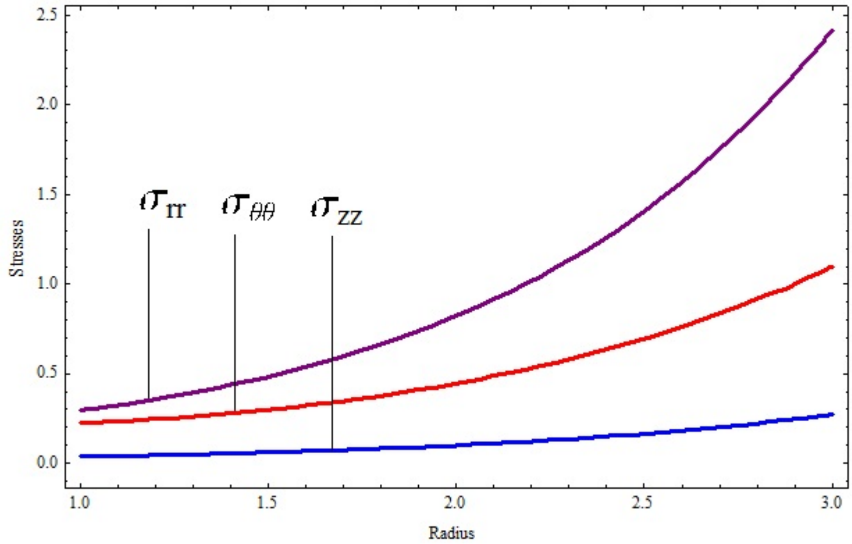

Fig.2. Thermal Stresses in Inhomogeneous Case with m = 1, n = 1

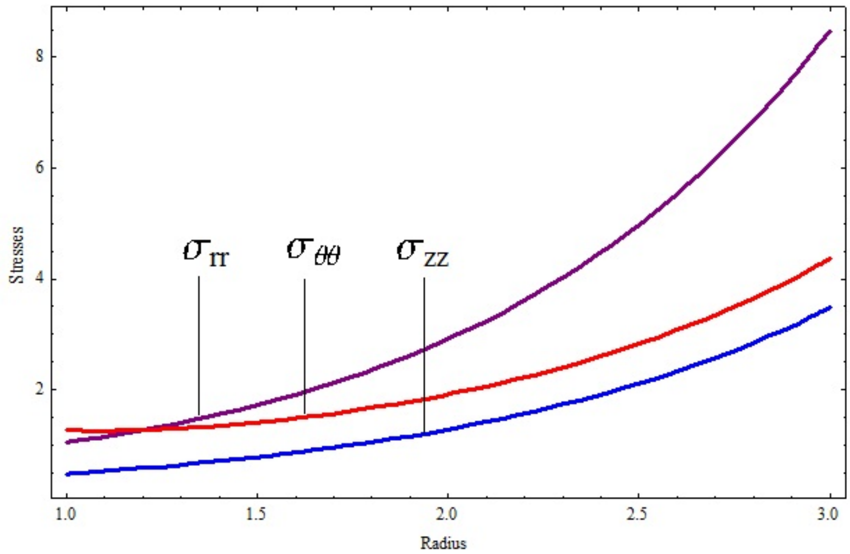

Fig.3. Thermal Stresses in Inhomogeneous Case with m = 1, n = 2

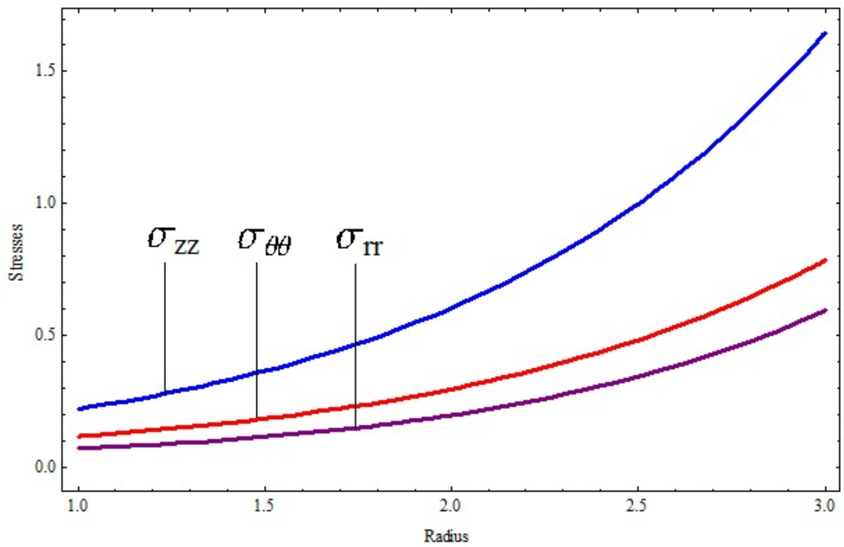

Fig.4. Thermal Stresses in Inhomogeneous Case with m = 1, n = -1

6. Multilayered Solution

The above analysis can be further verified by extending the concept for multilayered functionally graded hybrid composite cylindrical shell using transfer or propagator matrix method. The propagator matrix method has been applied extensively to the problems of anisotropic media [22]. Here we formulate the theory of propagator matrix method for the multilayered functionally graded hybrid composites. For simplicity we consider functionally graded hybrid composite cylindrical shell with transverse isotropy comprised of *m* layers in the interfacial zone. The layers are labeled from 1 to *m* so that layer 1 is bounded by $r = a$ and $r = r_1$, layer *h* by $r = r_{h-1}$ and $r = r_h$ and layer *m* by $r = r_{m-1}$ and $r = b$. The h^{th} layer is characterized by moduli $c_{ij}^{(h)}$ and expansion coefficients $\alpha_i^{(h)}$. We seek solution of the form given in equation (9), with *G* the same in each layer. For layer *h* after some manipulation, equations (1) to (3) can be written as:

$$\frac{d}{dr}\begin{bmatrix} u \\ \sigma_{rr} \end{bmatrix} = A_h \begin{bmatrix} u \\ \sigma_{rr} \end{bmatrix} + C_h d \quad \text{and} \quad \begin{bmatrix} \sigma_{\theta\theta} \\ \sigma_{zz} \end{bmatrix} = B_h \begin{bmatrix} u \\ \sigma_{rr} \end{bmatrix} + H_h d \tag{31}$$

where

$$\begin{aligned} A_h(r) &= \frac{1}{c_{11}^{(h)}}\begin{bmatrix} c_{12}^{(h)} r^{-1} & 1 \\ (r^2 c_{11}^{(h)})^{-1} Q_{11} & -(c_{11}^{(h)} - c_{12}^{(h)}) r^{-1} \end{bmatrix}, \\ B_h(r) &= \frac{1}{c_{11}^{(h)}}\begin{bmatrix} (c_{11}^{(h)})^{-1} Q_{11} & c_{12}^{(h)} \\ (c_{11}^{(h)})^{-1} Q_{13} & c_{13}^{(h)} \end{bmatrix}, \\ C_h(r) &= \frac{1}{c_{11}^{(h)}}\begin{bmatrix} -c_{12}^{(h)} & -c_{13}^{(h)} & \beta_1 \\ (r c_{11}^{(h)})^{-1} Q_{11} & (r c_{11}^{(h)})^{-1} Q_{13} & (1 - (r c_{11}^{(h)})^{-1})\beta_1 \end{bmatrix}, \\ d &= \begin{bmatrix} G & \Delta T \end{bmatrix}^T, \quad Q_{ij} = c_{ij} - \frac{c_{1i} c_{1j}}{c_{11}} \end{aligned} \tag{32}$$

Q_{ij} is the reduced material constants and β_i are Stress-temperature coefficient related to α_i given in equation (1). The solution for *u* and σ_{rr} in layer *h* is

$$\begin{bmatrix} u \\ \sigma_{rr} \end{bmatrix} = P_h(r)\begin{bmatrix} u_{h-1} \\ \sigma_{h-1} \end{bmatrix} + K_h(r) d \quad , \quad r_{h-1} < r < r_h \tag{33}$$

where the propagator matrix $P_h(r)$ for homogenous functionally graded cylindrical shell is defined as

$$\begin{aligned} P_h(r) &= \frac{1}{2c_{11}^{0}}\begin{bmatrix} (c_{11}^{(h)} - c_{12}^{(h)})Y + (c_{11}^{(h)} + c_{12}^{(h)})Y^{-1} & r_{h-1}\{Y - Y^{-1}\} \\ c_{11}^{(h)} Q_{11}^{(h)}\{Y^{-1} - Y^{-2}\} & (c_{11}^{(h)} + c_{12}^{(h)})Y^{-1} + (c_{11}^{(h)} - c_{12}^{(h)})Y^{-2} \end{bmatrix} \\ Y(r) &= Y = r/r_{h-1} \end{aligned} \tag{34}$$

and for inhomogeneous functionally graded cylindrical shell as

$$\begin{aligned} P_h(r) = I &+ \int_{r_{h-1}}^{r} A(\xi) d\xi + \int_{r_{h-1}}^{r}\int_{r_{h-1}}^{\xi} A(\xi) A(\eta) d\eta d\xi \\ &+ \int_{r_{h-1}}^{r}\int_{r_{h-1}}^{\xi}\int_{r_{h-1}}^{\eta} A(\xi) A(\eta) A(\zeta) d\zeta d\eta d\xi + \ldots\ldots \end{aligned} \tag{35}$$

It is observed that the successive terms in equation (35) are of the order of increasing powers of the shell thickness, and so for a thin inhomogeneous cylindrical shell it is justifiable to truncate to a low order and hence

$$\begin{aligned} K_h(r) &= P_h(r)\{L(r) - L(r_{h-1})\} \\ L(r) &= \int P_h(r)^{-1} C(r) dr \end{aligned} \tag{36}$$

It is observered from equation (33) that solution has effectively reduced to constants u_h and σ_h for $h = 0, 1, 2, \ldots m-1$ and thus it is imply that the remaining other stress components for layer *h* also reduces to constants u_h and σ_h. Thus it is sufficient to satisfy the following boundary and continuity conditions:

$\sigma_{rr} = 0$ at r = a, b

σ_{rr} and u are continuous at $r = r_h$ for h = 0, 1, 2,…m-1

The algebraic relation which results from applying these conditions are straightforward but too lengthy to be meaningful to the reader to be given here.

7. Conclusion

An analytical solution was obtained for two-dimensional axisymmetric thermoelastic problem of a transversely isotropic functionally graded hybrid composite cylindrical shell during post-solidification cooling. The solution was further verified by applying it to a multilayered functionally graded cylindrical shell. During our analysis it was observed that the nature of all stresses was exponential and magnitude was increased with increase in the material parameter *m*. The method of solution presented in this paper are useful in the analysis of functionally graded hybrid composite cylindrical shell for optimizing the design in terms of material usage and performance. For isotropic material the required formulation can be employed as $c_{11} = c_{33} = \lambda + 2\mu$, $c_{12} = c_{13} = \lambda$, $c_{44} = \mu$, where λ and μ are Lame constants.

Nomenclature

r	radial coordinate (dimensionless form $\bar{r} = r/b$)
z	axial coordinate (dimensionless form $\bar{z} = z/b$)
a , b	inner and outer radii of cylindrical shell
n , m	material parameters
u_r	radial displacement (dimensionless form $\bar{u}_r = u_r/(\bar{b}\,\alpha_1^0 T)$)
σ_{rr}	radial stress component (dimensionless form $\sigma_{rr}^* = \sigma_{rr}/(c_{11}^0 \alpha_1^0 T)$)
$\sigma_{\theta\theta}$	tangential stress component (dimensionless form $\sigma_{\theta\theta}^* = \sigma_{\theta\theta}/(c_{11}^0 \alpha_1^0 T)$)
σ_{zz}	axial stress component (dimensionless form $\sigma_{zz}^* = \sigma_{zz}/(c_{11}^0 \alpha_1^0 T)$)
C_i $(i = 1, 2)$	integration constants
$F(\alpha, \beta, \delta; x)$	hypergeometric function, where α, β and δ are independent of x

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Appendix A. A Short Note on Hypergeometric Functions

For $\delta \neq 0, -1, -2, -3, ..$ the hypergeometric function $F(\alpha, \beta, \delta; x)$ described in equation (13), which, *a fortiori*, is convergent for $|x| \leq 1$ and it is one of the two linearly independent solutions of the standard form of homogenous differential equation [24]

$$x(1-x)y'' + \{\delta - (\alpha + \beta + 1)x\}y' - \alpha\beta y = 0 \tag{A1}$$

If δ is not an integer, the general solution of the hypergeometric equation has the form:

$$y = C_1 F(\alpha, \beta, \delta; x) + C_2 x^{1-\delta} F(\alpha - \delta + 1, \beta - \delta + 1, 2 - \delta; x) \tag{A2}$$

Appendix B. Solution of Equation (11)

Introducing a new variable $x = n(r/b)^m$ and applying the transformations $u_r(r) = r u_r^*(x)$, $u_\theta(\theta) = 0$ and $u_z(z) = G z$, the homogenous equation is transformed into equation (11). Equation (11) is the standard form of the homogenous hypergeometric differential equation with the solution shown in equation (12). Back transforming using $r u_r^*(x) = u_r(r)$ we obtain

$$\begin{aligned} u_r(r) &= \hat{C}_1 r F(\alpha, \beta, \delta; n(r/b)^m) + \hat{C}_2 r (n(r/b)^m)^{-2/m} (-1)^{-2/m} F(\alpha - \delta + 1, \beta - \delta + 1, 2 - \delta; n(r/b)^m) \\ &= C_1 r F(\alpha, \beta, \delta; n(r/b)^m) + C_2 r^{-1} F(\alpha - \delta + 1, \beta - \delta + 1, 2 - \delta; n(r/b)^m) \end{aligned} \tag{B1}$$

Hence,

$$\begin{aligned} P(r) &= r F(\alpha, \beta, \delta; n(r/b)^m) \\ Q(r) &= r^{-1} F(\alpha - \delta + 1, \beta - \delta + 1, 2 - \delta; n(r/b)^m) \end{aligned} \tag{B2}$$