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Springbackward Phenomenon of a Transversely Isotropic Functionally Graded Composite Cylindrical Shell

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Abstract

This study provides an approach to predict the springback phenomenon during post-solidification cooling in a functionally graded hybrid composite cylindrical shell with a transverse isotropic structure. Here, the material properties are given with a general parabolic power-law function. During the theoretical analysis, an appropriate transformation is introduced in the equilibrium equation, which is resulting in a hypergeometrical differential equation. Thermoelastic solutions are obtained and analyzed for a homogeneous, nonhomogeneous, and elastic-plastic state. The solution is validated by applying it to the multilayered functionally graded cylindrical shell using the transfer or propagator matrix method.

Keywords: Thermoelasticity; Functionally graded hybrid composites; Cylindrical shell; Springback effect.

1. Introduction

'Springback' refers to the effect that occurs in the processing of post-solidification cooling of functionally graded hybrid composite semi-cylindrical shells. The concept of hybrid composites has been discussed since the beginning of the 70's by several researchers describing various theoretical and practical aspects of these materials. After cooling, solidification and removal from the mould, the product is usually found to have distorted from the shape of the mould. This distortion is due to thermoelastic deformation that accompanies cooling to ambient temperature after solidification at a higher temperature. The amount of thermal contraction in the functionally graded composite materials that are microscopically inhomogeneous in unique characterization, both in its thermal and mechanical properties depends strongly on the direction of the material, which results in the change of shape as well as the uniform contraction found in an isotropic material.

The mathematical modeling of functionally graded materials (FGMs) is currently an active research area because of the increasing application of FGM in industrial engineering. Issues like the Spring-forward of channel sections, the buckling problem, and the three-dimensional thermal stress problem in the functionally graded materials (FGMs) have been studied by many researchers [1-5]. The Axisymmetric thermoelasticity, the weight reduction, the high thermal radiation, plane strain solutions, and stresses in the functionally graded infinite strip (FGIS) were analyzed in the previous research [6-10]. Huang et al. [11] suggested a reduced-basis method (RBM) to perform a real-time analysis of the transient response displacement in FGM. The Analytic solutions were developed in order to solve non-linear heterogeneous thermoelasticity equations in two-dimension and to determine deformations and stresses in the circular disks made of FGMs [12-14]. Ma et. al. [15] analyzed the heat conduction problem of non-homogeneous FGMs for a layer sandwiched between two half-planes and obtained the full-field solutions of temperature and heat flux by using the Fourier transform method.

Birman et al. [16] provided a brief review on the uses of FGM in Engineering. Chiba et al. [17] presented a method for the optimisation of the material composition of functionally graded materials (FGMs) in the case of thermal stress relaxation which consists of a multiscale thermoelastic analysis along with a genetic algorithm. Lamba et al. [18] studied the uncoupled thermoelastic response of a thick cylinder with the length of 2h in which heat sources are generated according to the linear function of the temperature, with boundary conditions of the radiation type. Gaikwad [19] determined temperature, displacement, and thermal stresses in a thin circular plate due to the uniform internal energy generation. Recently, Matveenko et al. [20] presented the results of analytical and numerical investigations on the singularity of stresses of the 2D elastic solids of FGMs. Williams et al. [21] studied the numerical correlation of a multiple concentric cylinder model for the thermoplastic response of metal matrix composites.

In the present investigation, the axisymmetric thermoelastic problem in determining the unknown displacement and the stress distribution of a functionally graded transversely isotropic hybrid composite cylindrical shell, which includes the springback effect along the radial direction in an interfacial zone during post-solidification cooling, is studied. A special solution suggested by Spencer et al. [22] used for the thermoelastic distortions of laminated anisotropic tubes and guideline which also adopted by Varghese [23] is selected to investigate the problem. In this general parabolic model, the numerical computation of the displacement and stress functions is demonstrated. Furthermore, the solution is validated by applying the springback effect during post-solidification cooling into a multilayered functionally graded cylindrical shell using the transfer or the propagator matrix method.

2. Formulation of the Problem

For the theoretical investigation, an interphase zone of metal-matrix fiber composites by which the behavior is greatly affected is considered. This interfacial region can arise naturally due to the chemical reaction between the fiber and the polymeric matrix or can be introduced deliberately [21]. In the composite system, the metal undergoes elastic-plastic deformations and the fiber deforms elastically. It is assumed that the cylindrical shell occupies the space $a \le r \le b$, $0 \le z \le h$, where a and b denote the inner and outer radii, respectively while θ is constant.

The basic equations for the above-mentioned problem are given as follows:

a) Strain-displacement relationships [9]:

$$\varepsilon_{rr} = u'_r \;,\; \varepsilon_{\theta\theta} = r^{-1} u_r \;,\; \varepsilon_{zz} = \hat{u}_z \; 2\varepsilon_{\theta r} = 0 \;,\; 2\varepsilon_{rz} = 0 \;, 2\sigma_{\theta z} = 0 \tag{1}$$

where the prime (′) and (^) denote differentiation with respect to r and z.

b) Equilibrium equations with zero body force which are reduced to a single equation [22]:

$$\sigma'_{rr} + r^{-1}(\sigma_{rr} - \sigma_{\theta\theta}) = 0 \tag{2}$$

c) Non zero radial stress, σ_{rr}, tangential stress, $\sigma_{\theta\theta}$, and axial stress, σ_{zz}, are as follows:

$$\begin{aligned}
\sigma_{rr} &= c_{11}\varepsilon_{rr} + c_{12}\varepsilon_{\theta\theta} + c_{13}\varepsilon_{zz} - c_{11}(\varepsilon_{rr}^{p} + \varepsilon_{rr}^{per}) - c_{12}(\varepsilon_{\theta\theta}^{p} + \varepsilon_{\theta\theta}^{per}) \\
&\quad - c_{13}(\varepsilon_{zz}^{p} + \varepsilon_{zz}^{per}) + \beta_1 \Delta T \qquad \text{on the curved surfaces } r = a \text{ and } r = b \\
\sigma_{\theta\theta} &= c_{12}\varepsilon_{rr} + c_{11}\varepsilon_{\theta\theta} + c_{13}\varepsilon_{zz} - c_{12}(\varepsilon_{rr}^{p} + \varepsilon_{rr}^{per}) - c_{11}(\varepsilon_{\theta\theta}^{p} + \varepsilon_{\theta\theta}^{per}) \\
&\quad - c_{13}(\varepsilon_{zz}^{p} + \varepsilon_{zz}^{per}) + \beta_1 \Delta T \qquad \text{on the edges with } \theta = cons\tan t \\
\sigma_{zz} &= c_{13}\varepsilon_{rr} + c_{13}\varepsilon_{\theta\theta} + c_{33}\varepsilon_{zz} - c_{13}(\varepsilon_{rr}^{p} + \varepsilon_{rr}^{per}) - c_{13}(\varepsilon_{\theta\theta}^{p} + \varepsilon_{\theta\theta}^{per}) \\
&\quad - c_{33}(\varepsilon_{zz}^{p} + \varepsilon_{zz}^{per}) + \beta_3 \Delta T \qquad \text{on the ends } z = 0 \text{ and } z = h
\end{aligned} \tag{3}$$

where the stress σ is related to the infinitesimal strain tensor $e = \varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{zz}$ and $\Delta T = T_s - T$ is the change in temperature to a lower value than T_s at which the lay-up solidification process takes place. In the proceeding equations, ε_{ij} denotes the total strain, σ_{ij} denotes the total stress and superscripts p and per are used to designate plastic and permanent strains, respectively. In a plastically pre-deformed region ε_{ij}^{per} is nonzero and it is always zero otherwise. In a purely elastic deformation of the functionally graded hybrid composite cylindrical shell during post-solidification cooling we have $\varepsilon_{ij}^{p} = \varepsilon_{ij}^{per} = 0$. The stress-temperature coefficient β_i $(i = 1,2,3)$ is related to $(i = 1,2,3)$. α_i is given as

a) $\beta_1 = c_{11}\alpha_1 + c_{12}\alpha_1 + c_{13}\alpha_3$, b) $\beta_2 = c_{12}\alpha_1 + c_{11}\alpha_1 + c_{13}\alpha_3$, $\beta_3 = c_{13}\alpha_1 + c_{13}\alpha_1 + c_{33}\alpha_3$

According to the coordinates, displacement and stress components are denoted by $(u_r, 0, u_z)$ and $\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{zz}$, respectively. Therefore, we assume that the material constants c_{ij} and the coefficient of thermal expansion α_i are functions of r but not of θ and z, thereupon, the functionally graded hybrid composite cylindrical shell is a special case.

Substituting the Eqs. (1) and (3) in Eq. (2), the equilibrium equation of the functionally graded hybrid composite cylindrical shell can be obtained as

$$\begin{aligned} & c_{11} u_r'' + (c_{11}' + r^{-1} c_{11}) u_r' + (c_{12}' - r^{-1} c_{11})\, r^{-1} u_r \\ & = -\beta_1' \,\Delta T - c_{13}' \hat{u}_z + (\varepsilon_{rr}^{p} + \varepsilon_{rr}^{per})(c_{11}' + r^{-1}(c_{11} - c_{12})) \\ & \quad + (\varepsilon_{\theta\theta}^{p} + \varepsilon_{\theta\theta}^{per})(c_{12}' - r^{-1}(c_{11} - c_{12})) + c_{13}' (\varepsilon_{zz}^{p} + \varepsilon_{zz}^{per}) + c_{11} (\varepsilon_{rr}'^{p} + \varepsilon_{rr}'^{per}) \\ & \quad + c_{12} (\varepsilon_{\theta\theta}'^{p} + \varepsilon_{\theta\theta}'^{per}) \qquad for\ a \le r \le b \ \ and \ \ 0 \le z \le h \end{aligned} \tag{4}$$

2.1 Boundary Conditions

As suggested by Spencer [22] for a complete solution of the thermoelastic problem, the displacement field has to be determined, and for $\Delta T \neq 0$, there would be zero traction on all surfaces of the cylindrical shell during post-solidification cooling. Thus we assume the following relations:

Traction vanishes at the inner and outer curved surfaces:

$$\sigma_{rr} = 0,\ \sigma_{r\theta} = 0,\ \sigma_{rz} = 0 \quad at \quad r = a, b \tag{5}$$

Vanishing of the normal force on $z = 0,\ h$:

$$2\pi \int_a^b \sigma_{zz}\, r\, dr = 0 \tag{6}$$

The cylindrical shell which is simply supported at the two longitudinal edges:

$$u_r = 0,\ \sigma_{zz} = 0,\ \sigma_{\theta z} = 0,\ \sigma_{rz} = 0 \qquad at \quad z = 0, h \tag{7}$$

Regarding Eqs. (5) to (7), in formulating these conditions we may assume that $\sigma_{rz} = 0,\ \sigma_{r\theta} = 0,\ \sigma_{\theta z} = 0$ and the other stress components depend only on r as a special case.

While the problem is only concerned with the radial direction, the zero resultant force and the bending moment on the edges $\theta = 0$, $\theta = \theta_0$ are not taken into consideration. In the earlier studies, it has been observed that the solution may be left out as an unequilibrated bending moment and a shear force on the ends of the finite-length functionally graded hybrid composite cylindrical shell. To neutralize this moment and force, an additional solution involving the stress that depends on the angle θ and r has been considered.

3. Reformulation of the Problem

Following the previous research [9], we consider the case where material constants c_{ij} and the thermal expansion coefficient α_i vary radially according to a general parabolic form given by

$$c_{ij} = c_{ij}^0 [1 - n (r/b)^m],\ \alpha_i = \alpha_i^0 [1 - n (r/b)^m] \tag{8}$$

where c_{ij}^0 and α_i^0 are arbitrary constants having the same dimension as c_{ij} and α_i , respectively. Moreover, n and m are the material parameters whose combination forms a wide range of nonlinear and continuous profiles to describe the reasonable variation of material constants and thermal expansion coefficients, when the thermal effect is neglected.

Eqs. (5), (6), (7) and (8) constitute the mathematical formulation of the problem under consideration.

4. Solution of the Problem

4.1 Analytical Stresses in the Elastic State

By substituting Eq. (8) in Eq. (4) and taking $\varepsilon_{ij}^{p} = \varepsilon_{ij}^{per} = 0$ into consideration, a standard form of differential equation is obtained as

$$\begin{aligned} & r^2 [1 - n (r/b)^m] u_r'' + r [1 - n (1 - m)(r/b)^m] u_r' - [1 - n (1 - m C_2^0)(r/b)^m] u_r \\ & = m\, r\, n (r/b)^m \{ -\beta_1^0 C_1^0 \,\Delta T - C_3^0 \hat{u}_z \} \\ & for\ a \le r \le b \ and \ 0 \le z \le h \end{aligned} \tag{9}$$

where $C_1^0 = 1/c_{11}^0,\ C_2^0 = c_{12}^0/c_{11}^0,\ C_3^0 = c_{13}^0/c_{11}^0$. Eq. (9) is a hypergeometric differential equation, which can be solved by introducing a new variable $x = x(r) = n (r/b)^m$ and applying the transformations as follows

$$u_r(r) = r\,u_r^*(x),\; u_\theta(\theta) = 0,\; u_z(z) = G\,z \tag{10}$$

where G is the unknown constant (i.e. independent of r) to be determined.

The term $u_r(r) = r\,u_r^*(x)$ represents a radial expansion or contraction in which the inner and outer radii change generally but the angle remains constant. The term $u_z(z) = G\,z$ is a uniform axial expansion or contraction. By substituting Eq. (10) in Eq. (9), and rewriting the governing equation, the following equation is obtained:

$$x\,[1-x]\,\widehat{\widehat{u}}_r^*(x) + [m + 2(1-x)]\,m^{-1}\,\widehat{u}_r^*(x) - [C_2^0 - 1]\,m^{-1}\,u_r^*(x) = m^{-1}\{-\beta_1^0 C_1^0\,\Delta T - C_3^0\,G\} \tag{11}$$

where the prime ($\frown$) denotes differentiation with respect to x.

The general solution of Eq. (11) can be expressed as a combination of a complimentary function and a particular integral. The standard form of the homogeneous hypergeometric differential equation can be obtained by setting the right-hand side of Eq. (11) to be zero and the solution can be cast in the following form [24]:

$$u_r^*(x) = \hat{C}_1\,F(\alpha,\beta,\delta;x) + \hat{C}_2\,(-1)^{-2/m}\,x^{-2/m}\,F(\alpha-\delta+1,\beta-\delta+1,2-\delta;x) \tag{12}$$

where $\hat{C}_i$ $(i = 1,2)$ is an arbitrary integration constant to be determined and $F(\alpha,\beta,\delta;x)$ is the hypergeometric function which is defined as the analytic continuation of the so-called hypergeometric series,

$$F(\alpha,\beta,\delta;x) = 1 + \frac{\alpha\beta}{\delta}\frac{x}{1!} + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\delta(\delta+1)}\frac{x^2}{2!} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \frac{\alpha(\alpha+1)\ldots(\alpha+j-1)\beta(\beta+1)\ldots(\beta+j-1)}{\delta(\delta+1)(\delta+j-1)}\frac{x^j}{j!} + \ldots\ldots\ldots\ldots\ldots \tag{13}$$

As observed in the previous research [25], $F(\alpha,\beta,\delta;x)$ converges slowly within the unit circle $|x| \le 1$ provided that $\delta - (\alpha+\beta) > -1$. Since the problem under consideration is a real physical problem, these conditions are always satisfied and the series is always convergent. The arguments α , β and δ are the real and dimensionless scalars of the hypergeometric function F in Eq. (12) and have the following meanings:

$$\alpha = -(1/2) + (1/m) - (1/2m)\sqrt{4 + 4m^2 - 4m\,C_2^0},\;\; \beta = -(1/2) + (1/m) + (1/2m)\sqrt{4 + m^2 - 4m\,C_2^0}$$
$$\delta = 1 + (2/m) \tag{14}$$

The complimentary function given by Eq. (12) can be re-written as (see Appendix B)

$$u_r(r) = C_1\,P(r) + C_2\,Q(r) \tag{15}$$

where P and Q are the fundamental solutions of the reduced differential equation as

$$P(r) = r\,F(\alpha,\beta,\delta;n(r/b)^m)$$
$$Q(r) = r^{-1}\,F(\alpha-\delta+1,\beta-\delta+1,2-\delta;n(r/b)^m) \tag{16}$$

The particular integral solution $R(r)$ for Eq. (9) is determined by the parameters variation method which is assumed to be in the following form: [2]

$$R(r) = \hat{U}_1(r)P(r) + \hat{U}_2(r)Q(r) \tag{17}$$

where the following equations represent the nonhomogeneous term of the differential equation.

$$\hat{U}_1(r) = -\int_a^r \frac{Q(\xi)f(\xi)}{W(\xi)}d\xi$$
$$\hat{U}_2(r) = \int_a^r \frac{P(\xi)f(\xi)}{W(\xi)}d\xi$$
$$W(r) = \det\begin{pmatrix} P & Q \\ P' & Q' \end{pmatrix} = P(r)Q'(r) - Q(r)P'(r) \tag{18}$$
$$f(r) = (1/(n\,m\,(r/b)^m\,[1 - n\,(r/b)^m]))\{-\beta_1^0 C_1^0\,\Delta T - C_3^0\,G\}$$

The essential solution of Eq. (10) can be expressed as a combination of the complimentary function and the particular integral as follows:

$$u_r(r) = C_1 P(r) + C_2 Q(r) + R(r) \tag{19}$$

The derivatives $P'(r)$ and $Q'(r)$ in Eq. (18) are evaluated using the differentiation rule as follows [25]:

$$\begin{aligned} &F'(\alpha,\beta,\delta;x(r)) = (\alpha\beta/\delta)x'(r)F(\alpha+1,\beta+1,\delta+1;x(r)), \\ &R'(r) = \hat{U}_1(r)P'(r) + \hat{U}_2(r)Q'(r) \end{aligned} \tag{20}$$

Since $P(r)$, $Q(r)$ and $W(r)$ are polynomials, the integrals (18) may be evaluated analytically by expanding the integrands into Taylor series. If these expansions are not possible because of the product $f(r)$, accurate evaluations may nevertheless be accomplished by the application of the Gaussian Quadrature rule of integration. Regarding the form of the radial displacement (19), the stresses become

$$\begin{aligned} \sigma_{rr} &= C_1(c_{11}P'(r) + r^{-1}c_{12}P(r)) + C_2(c_{11}Q'(r) + r^{-1}c_{12}Q(r)) \\ &\quad + (c_{11}R'(r) + r^{-1}c_{12}R(r)) + c_{13}G + \beta_1\,\Delta T \\ \sigma_{\theta\theta} &= C_1(c_{12}P'(r) + r^{-1}c_{11}P(r)) + C_2(c_{12}Q'(r) + r^{-1}c_{11}Q(r)) \\ &\quad + (c_{12}R'(r) + r^{-1}c_{11}R(r)) + c_{13}G + \beta_1\,\Delta T \\ \sigma_{zz} &= C_1(c_{13}P'(r) + r^{-1}c_{13}P(r)) + C_2(c_{13}Q'(r) + r^{-1}c_{13}Q(r)) \\ &\quad + (c_{13}R'(r) + r^{-1}c_{13}R(r)) + c_{33}G + \beta_3\,\Delta T \end{aligned} \tag{21}$$

Thermoelastic solution is completed by the application of boundary conditions.

4.2 Further Investigation

The Homogenous Case:

Regarding $n=0$ and $m>0$, all material constants of the equations (8) are obtained which are independent of radial coordinates, then, irrespective of m, $c_{ij} = c_{ij}^0$, $\alpha_i = \alpha_i^0$, and $F(\alpha,\beta,\delta;0)=1$. From Eq. (16) we obtain $P(r)=r$ and $Q(r)=r^{-1}$, thus the right-hand side of the general solution (15) reduces to

$$u_r(r) = C_1 r + r^{-1}C_2 \tag{22}$$

The boundary conditions to evaluate integration constants C_1 and C_2 are $\sigma_{rr} = 0$, $\sigma_{r\theta} = 0$, $\sigma_{rz} = 0$ at $r = a$ and $r = b$, where the last two boundary conditions of both cases have been automatically satisfied at both surfaces as shown earlier in the paper. Then, by the virtue of Eqs. (22) and (5), Eq. (21) can be rewritten as

$$\begin{aligned} &C_1(c_{11}+c_{12}) - a^{-2}C_2(c_{11}-c_{12}) = -\beta_1\,\Delta T - c_{13}G \\ &C_1(c_{11}+c_{12}) - (a\eta)^{-2}C_2(c_{11}-c_{12}) = -\beta_1\,\Delta T - c_{13}G \end{aligned} \tag{23}$$

where $\eta = b/a$ is the outer radius-to-inner radius ratio. Apart from the boundary conditions given in Eq. (7) along the radial direction, the conditions at $z=0$ and $z=h$ should also be taken into consideration.

The last two conditions $\sigma_{\theta z} = 0$ and $\sigma_{rz} = 0$ at $z = 0, h$ automatically are met and the first condition $\sigma_{zz} = 0$ at $z = 0, h$ gives

$$2C_1 c_{13} = -\beta_3\,\Delta T - c_{33}G \tag{24}$$

From Eqs. (23) and (24), the following relations are obtained:

$$C_1 = -\alpha_1^0\,\Delta T \quad , \; C_2 = 0 \;\text{ and }\; G = -\alpha_3^0\,\Delta T \tag{25}$$

It can be concluded that the stress component vanishes everywhere in a homogenous transversely isotropic functionally graded hybrid composite cylindrical shell during post-solidification cooling with $n=0$ and $m>0$ as the inhomogeneity parameters. It is also observed that as $m \to 0$, $c_{ij} = c_{ij}^0(1-n)$ as well as $\alpha_i = \alpha_i^0(1-n)$ $\to$ constant, irrespective of n. Therefore, all material constants are independent of the radial coordinate and the governing equation reduces to the Euler's differential equation.

The Inhomogeneous Case:

For the inhomogeneity parameter $n \neq 0$ and $m \neq 0$, the radial stress expression (21) can be rewritten utilizing boundary conditions (5) as

$$\begin{aligned} &C_1(c_{11}P'(a) + a^{-1}c_{12}P(a)) + C_2(c_{11}Q'(a) + a^{-1}c_{12}Q(a)) \\ &+(c_{11}R'(a) + a^{-1}c_{12}R(a)) = -\beta_1\,\Delta T - c_{13}G \\ &C_1(c_{11}P'(a\eta) + (a\eta)^{-1}c_{12}P(a\eta)) + C_2(c_{11}Q'(a\eta) + (a\eta)^{-1}c_{12}Q(a\eta)) \\ &+(c_{11}R'(a\eta) + (a\eta)^{-1}c_{12}R(a\eta)) = -\beta_1\,\Delta T - c_{13}G \end{aligned} \tag{26}$$

As pointed out by Spencer [22], the form of the considered solution does not permit the point-by-point specification of the traction at the two ends; $z = 0$ and $z = h$. The resultant forces and moments can only be specified on the basis of Saint-Venant's principle. Regarding the problem that have been considered, the boundary conditions (6) are obtained as follows:

$$C_1 \int_a^{a\eta} (P'(r) + r^{-1} P(r))\, r\, dr + C_2 \int_a^{a\eta} (Q'(r) + r^{-1} Q(r))\, r\, dr + \int_a^{a\eta} (R'(r) + r^{-1} R(r))\, r\, dr = (-\beta_3 \Delta T - c_{33} G) \int_a^{a\eta} r\, dr \tag{27}$$

As explained earlier, to carry out integrals involving Hypergeometric functions, we can expand the function into Taylor series and integrate the series. However, *n* and *m* should be assigned numerical values e.g. $n = 0.5$, $m = 1.5$, or the Gaussian Quadrature rule of integration can also be used. As the Hypergeometric functions are infinite series, the result will be exact. The general expressions for the stress and displacement contain unknown integration constants C_1, C_2 and the unknown coefficient constant G. For the determination of all three unknown constants, the three non-redundant conditions (26) and (27) are available.

4.3 Analytical Stresses in the Elastic-Plastic State

Total strains are expressed as the superposition of elastic and plastic parts in the form of $\varepsilon_{ij} = \varepsilon_{ij}^{e} + \varepsilon_{ij}^{p} + \varepsilon_{ij}^{per}$, while the superscripts e and $(p + per)$ denote the elastic and the plastic, respectively. The total axial strain along radial symmetric in the interfacial zone during post-solidification cooling of the functionally graded hybrid composite cylindrical shell can be assumed as [9]

$$\sigma_{zz} - \upsilon(\sigma_{rr} + \sigma_{\theta\theta}) = 0 \tag{28}$$

Since $\sigma_{\theta\theta} > \sigma_{zz} > \sigma_{rr}$ throughout the functionally graded hybrid composite cylindrical shell, Tresca's yield criterion is as follows:

$$\sigma_{\theta\theta} - \sigma_{rr} = \sigma_0 \tag{29}$$

While the corresponding flow rule with this yielding is $\varepsilon_{\theta\theta}^{p} = -\varepsilon_{rr}^{p}$ and $\varepsilon_{zz}^{p} = 0$.

Substituting Eqs. (28) and (29) in Eq. (2) leads to the following equation:

$$c_{11} u_r'' + (c_{11}' + c_{12} r^{-1} - c_{12} + c_{11}) u_r' + (-c_{12} r^{-4} + r^{-1} c_{12}' + c_{12} r^{-1} - c_{11} r^{-1}) u_r + 2c_{12} \varepsilon_{rr}^{p} - 2c_{11} \varepsilon_{rr}^{p} + (c_{12} - c_{11}) \varepsilon_{rr}'^{p} + (c_{12}' - c_{11}') \varepsilon_{rr}^{p} + \beta_1' \Delta T \tag{30}$$

5. Numerical Results and Discussion

To present some results corresponded to the thermoelastic solution, the nondimensional variables as listed in nomenclature are used. A plot which shows the distribution of stresses against the nondimensional radial coordinate for the homogenous case, while $m = 1$ and $n = 0$, is presented in Fig.1; whereas for the inhomogeneous case $m = 1$and $n = 1$ in Fig.2 , $m = 1$ and $n = 2$ in Fig.3 along with $m = 1$and $n = -1$ in Fig.4 are taken into consideration.

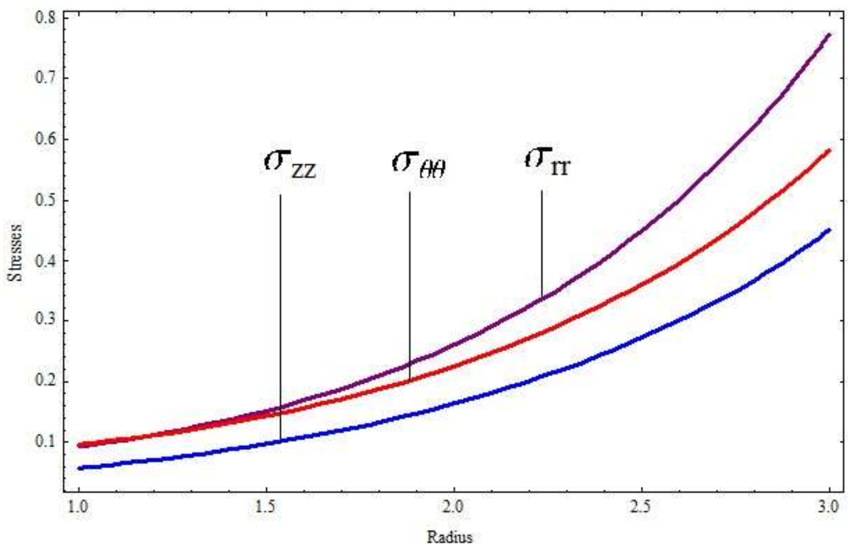

Fig. 1. Thermal Stresses in the homogeneous case with $m = 1$, $n = 0$

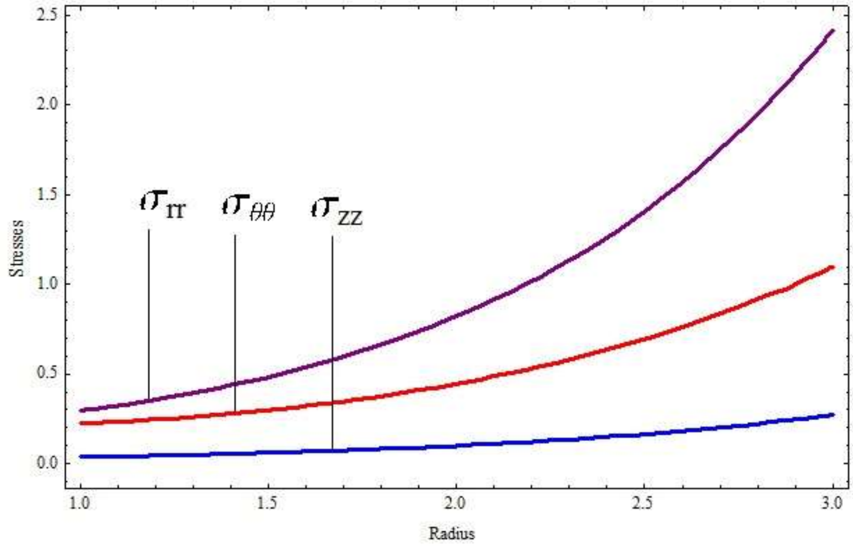

Fig. 2. Thermal Stresses in the inhomogeneous case with $m = 1$, $n = 1$

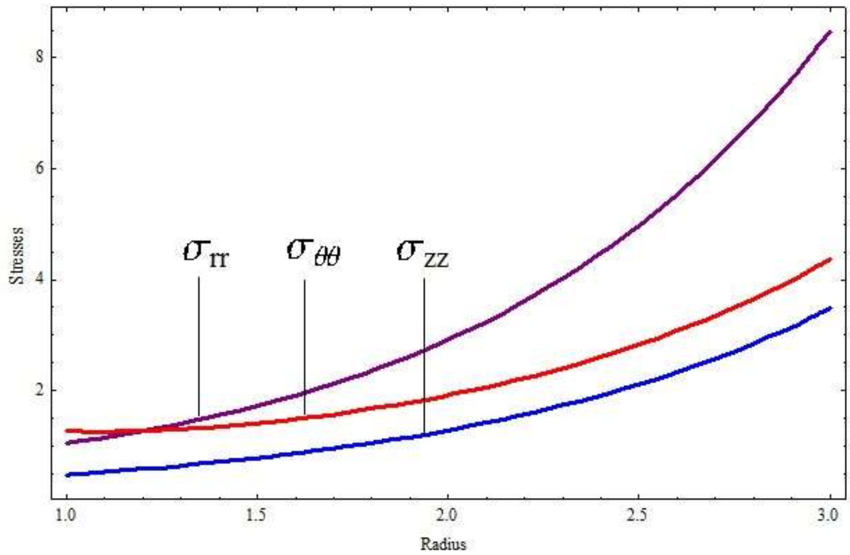

Fig. 3. Thermal Stresses in the inhomogeneous case with $m = 1$, $n = 2$

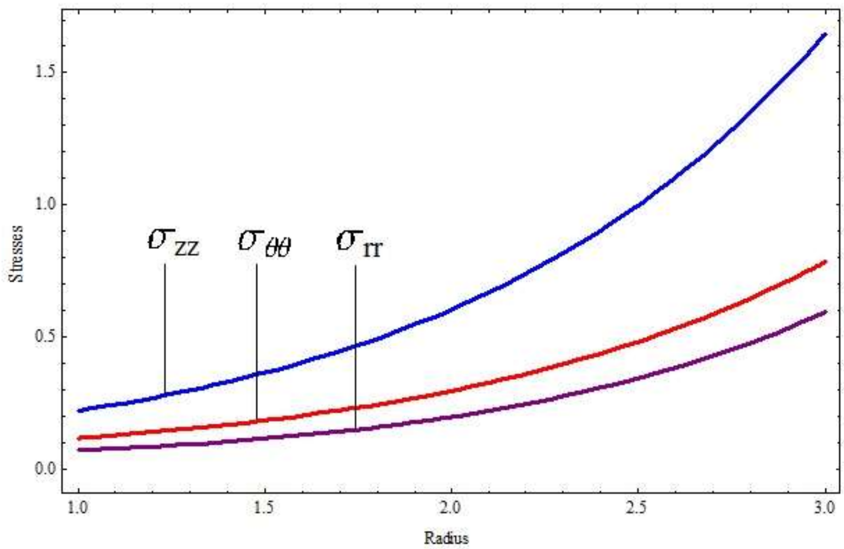

Fig. 4. Thermal Stresses in the inhomogeneous case with $m = 1$, $n = -1$

It can be observed in Fig.1 that the absolute values of radial, tangential and axial stresses are high in the steady state at the outer radius and are slowly decreasing towards the inner radius exponentially. During the analysis it was observed that the stresses change with the increase or decrease in the material parameter m.

As shown in Figs. 2 and 3, the variation of radial, tangential, and axial stresses are similar to that of the homogeneous case as shown in Fig.1, but the magnitude increases by an increase in the material parameter n.

Fig.4 shows that the absolute values of radial, tangential, and axial stresses are high at the outer radius and are slowly decreasing towards the inner radius exponentially. Moreover, the magnitude of the axial stress is more than that of radial and tangential stresses.

6. Multilayered Solution

The above-mentioned analysis can be further verified by extending the concept of the multilayered functionally graded hybrid composite cylindrical shell using the transfer or propagator matrix method. The propagator matrix method has been applied extensively for the problems of the anisotropic media [22]. Here, the theory of propagator matrix method for the multilayered functionally graded hybrid composites was formulated. For simplicity, the functionally graded hybrid composite cylindrical shell with transverse isotropy comprised of m layers in the interfacial zone was taken into the consideration. The layers are labeled from 1 to m, so that the layer 1 is bounded by $r = a$ and $r = r_1$, the layer h by $r = r_{h-1}$ and $r = r_h$ and the layer m by $r = r_{m-1}$ and $r = b$. The h^{th} layer is characterized by moduli $c_{ij}^{(h)}$ and expansion coefficients $\alpha_i^{(h)}$. We seek the solution of the form given in Eq. (9) with the same G in each layer. Regarding the layer h, after some manipulation Eqs. (1) to (3) can be written as:

$$\frac{d}{dr}\begin{bmatrix} u \\ \sigma_{rr} \end{bmatrix} = A_h \begin{bmatrix} u \\ \sigma_{rr} \end{bmatrix} + C_h d \quad \text{and} \quad \begin{bmatrix} \sigma_{\theta\theta} \\ \sigma_{zz} \end{bmatrix} = B_h \begin{bmatrix} u \\ \sigma_{rr} \end{bmatrix} + H_h d \tag{31}$$

where

$$\begin{aligned} A_h(r) &= \frac{1}{c_{11}^{(h)}}\begin{bmatrix} c_{12}^{(h)} r^{-1} & 1 \\ (r^2 c_{11}^{(h)})^{-1} Q_{11} & -(c_{11}^{(h)} - c_{12}^{(h)}) r^{-1} \end{bmatrix}, \\ B_h(r) &= \frac{1}{c_{11}^{(h)}}\begin{bmatrix} (c_{11}^{(h)})^{-1} Q_{11} & c_{12}^{(h)} \\ (c_{11}^{(h)})^{-1} Q_{13} & c_{13}^{(h)} \end{bmatrix}, \\ C_h(r) &= \frac{1}{c_{11}^{(h)}}\begin{bmatrix} -c_{12}^{(h)} & -c_{13}^{(h)} & \beta_1 \\ (r c_{11}^{(h)})^{-1} Q_{11} & (r c_{11}^{(h)})^{-1} Q_{13} & (1-(r c_{11}^{(h)})^{-1})\beta_1 \end{bmatrix}, \\ d &= \begin{bmatrix} G & \Delta T \end{bmatrix}^T, \quad Q_{ij} = c_{ij} - \frac{c_{1i} c_{1j}}{c_{11}} \end{aligned} \tag{32}$$

Q_{ij} is the reduced material constants and β_i is the stress-temperature coefficient related to α_i as given in Eq. (1). The solution for u and σ_{rr} in the layer h is

$$\begin{bmatrix} u \\ \sigma_{rr} \end{bmatrix} = P_h(r)\begin{bmatrix} u_{h-1} \\ \sigma_{h-1} \end{bmatrix} + K_h(r) d \quad , \quad r_{h-1} < r < r_h \tag{33}$$

where the propagator matrix $P_h(r)$ for the homogeneous functionally graded cylindrical shell is defined as

$$\begin{aligned} P_h(r) &= \frac{1}{2c_{11}^{0}}\begin{bmatrix} (c_{11}^{(h)} - c_{12}^{(h)})Y + (c_{11}^{(h)} + c_{12}^{(h)})Y^{-1} & r_{h-1}\{Y - Y^{-1}\} \\ c_{11}^{(h)} Q_{11}^{(h)}\{Y^{-1} - Y^{-2}\} & (c_{11}^{(h)} + c_{12}^{(h)})Y^{-1} + (c_{11}^{(h)} - c_{12}^{(h)})Y^{-2} \end{bmatrix} \\ Y(r) &= Y = r/r_{h-1} \end{aligned} \tag{34}$$

and for the inhomogeneous functionally graded cylindrical shell, it is defined as

$$\begin{aligned} P_h(r) = I &+ \int_{r_{h-1}}^{r} A(\xi) d\xi + \int_{r_{h-1}}^{r}\int_{r_{h-1}}^{\xi} A(\xi) A(\eta) d\eta d\xi \\ &+ \int_{r_{h-1}}^{r}\int_{r_{h-1}}^{\xi}\int_{r_{h-1}}^{\eta} A(\xi) A(\eta) A(\zeta) d\zeta d\eta d\xi + \dots\dots \end{aligned} \tag{35}$$

It is observed that the successive terms in Eq. (35) are of the order of increasing powers of the shell thickness, and thus, for a thin inhomogeneous cylindrical shell, it is justifiable to truncate to a low order and therefore,

$$\begin{aligned} K_h(r) &= P_h(r)\{L(r) - L(r_{h-1})\} \\ L(r) &= \int P_h(r)^{-1} C(r) dr \end{aligned} \tag{36}$$

It is observered from Eq. (33) that solution has effectively reduced to the constants u_h and σ_h for $h = 0, 1, 2,$

…*m*-1 and thus it is implied that the remaining stress components for the layer *h* also reduce to constants u_h and σ_h. Thus, it is sufficient to satisfy the following boundary and continuity conditions:

$\sigma_{rr} = 0$ at r = a, b

σ_{rr} and u are continuous at $r = r_h$ for h = 0, 1, 2,…m-1

The algebraic relation, which results from applying these conditions, is straightforward but too lengthy for the reader to be meaningful and presented in this section.

7. Conclusion

An analytical solution was obtained for the two-dimensional axisymmetric thermoelastic problem of a transversely isotropic functionally graded hybrid composite cylindrical shell during post-solidification cooling. The solution was further verified by applying it to a multilayered functionally graded cylindrical shell. During the analysis, it was observed that the nature of all stresses was exponential and magnitude, therefore, they were increased as the material parameter *m* increased. The method of the solution presented in this study was useful in the analysis of the functionally graded hybrid composite cylindrical shell to optimize the design in terms of the material usage and performance. For an isotropic material, the required formulation could be employed as $c_{11} = c_{33} = \lambda + 2\mu$, $c_{12} = c_{13} = \lambda$, $c_{44} = \mu$, where λ and μ are Lame constants.

Nomenclature

r	radial coordinate (dimensionless form $\bar{r} = r/b$)
z	axial coordinate (dimensionless form $\bar{z} = z/b$)
a , b	inner and outer radii of cylindrical shell
n , m	material parameters
u_r	radial displacement (dimensionless form $\bar{u}_r = u_r/(\bar{b}\ \alpha_1^0 T)$)
σ_{rr}	radial stress component (dimensionless form $\sigma_{rr}^* = \sigma_{rr}/(c_{11}^0\alpha_1^0 T)$)
$\sigma_{\theta\theta}$	tangential stress component (dimensionless form $\sigma_{\theta\theta}^* = \sigma_{\theta\theta}/(c_{11}^0\alpha_1^0 T)$)
σ_{zz}	axial stress component (dimensionless form $\sigma_{zz}^* = \sigma_{zz}/(c_{11}^0\alpha_1^0 T)$)
$C_i \ (i = 1,2)$	integration constants
$F(\alpha,\beta,\delta;x)$	hypergeometric function, where α, β and δ are independent of x

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Appendix A. A Short Note on Hypergeometric Functions

For $\delta \neq 0, -1, -2, -3, ..$ the hypergeometric function $F(\alpha, \beta, \delta; x)$ described in equation (13), which, *a fortiori*, is convergent for $|x| \leq 1$ and it is one of the two linearly independent solutions of the standard form of homogenous differential equation [24]

$$x(1-x)y'' + \{\delta - (\alpha + \beta + 1)x\}y' - \alpha\beta y = 0 \tag{A1}$$

If δ is not an integer, the general solution of the hypergeometric equation has the form:

$$y = C_1 F(\alpha, \beta, \delta; x) + C_2 x^{1-\delta} F(\alpha - \delta + 1, \beta - \delta + 1, 2 - \delta; x) \tag{A2}$$

Appendix B. Solution of Equation (11)

Introducing a new variable $x = n(r/b)^m$ and applying the transformations $u_r(r) = r u_r^*(x)$, $u_\theta(\theta) = 0$ and $u_z(z) = G z$, the homogenous equation is transformed into equation (11). Equation (11) is the standard form of the homogenous hypergeometric differential equation with the solution shown in equation (12). Back transforming using $r u_r^*(x) = u_r(r)$ we obtain

$$\begin{aligned} u_r(r) &= \hat{C}_1 r F(\alpha, \beta, \delta; n(r/b)^m) + \hat{C}_2 r (n(r/b)^m)^{-2/m} (-1)^{-2/m} F(\alpha - \delta + 1, \beta - \delta + 1, 2 - \delta; n(r/b)^m) \\ &= C_1 r F(\alpha, \beta, \delta; n(r/b)^m) + C_2 r^{-1} F(\alpha - \delta + 1, \beta - \delta + 1, 2 - \delta; n(r/b)^m) \end{aligned} \tag{B1}$$

Hence,

$$\begin{aligned} P(r) &= r F(\alpha, \beta, \delta; n(r/b)^m) \\ Q(r) &= r^{-1} F(\alpha - \delta + 1, \beta - \delta + 1, 2 - \delta; n(r/b)^m) \end{aligned} \tag{B2}$$