Vibration and Static Analysis of Functionally Graded Porous Plates

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Abstract

This research deals with free vibration and static bending of a simply supported functionally graded (FG) plate with the porosity effect. Material properties of the plate which are related to its change are position-dependent. Governing equations of the FG plate are obtained by using the Hamilton's principle within first-order shear deformation plate theory. In solving the problem, the Navier solution is also used. In this study, the effect of the porosity and material distribution parameters on the static and vibration responses of the FG plate is presented and discussed.

Keywords: Functionally Graded Plate, Porosity; Static Analysis; Vibration Analysis.

1. Introduction

Functionally graded materials (FGMs) are a new generation of composites where the volume fraction of the FGM constituents varies gradually, giving a non-uniform microstructure with continuously graded macro properties such as elasticity modulus, density, heat conductivity, etc.. FGMs have many practical applications, such as biomedical sectors, aircrafts, and space vehicles. In the last few decades, much more attention has been given to the mechanical behavior of FG Plates ([1]-[28]).

In recent years, some studies about the porosity effect in the FG structures have been published in the literature; Wattanasakulpong and Ungbhakorn [29] conducted the vibration analysis of porous FG beams. Mechab et al. [30, 31] examined the free vibration analysis of a FG nano-plate resting on elastic foundations with the porosity effect. Şimşek and Aydin [32] examined the forced vibration of FG microplates with porosity effects based on the modified couple stress theory. Jahwari and Naguib [33] investigated FG viscoelastic porous plates with a higher order plate theory and a statistical based model of cellular distribution. Vibration characteristics of FG beams with porosity effect and various thermal loadings are investigated by [34-35].

In this study, the free vibration and the static bending analysis of a simply supported FG plate are studied with porosity effect by using Navier solution within the first-order shear deformation plate theory. The effects of the porosity and material distribution parameters on the static and vibration responses of the FG plate are investigated. The linear/ nonlinear analysis of buckling and the vibration of FG graphene beams reinforced by the porous nano-composite are investigated by Chen et al. [36] and Kitipornchai et al. [37].

2. Theory and Formulations

Consider a simply supported rectangular FG porous plate with thickness h in X3 direction. The lengths of LX and LY in X1 and X2 directions are shown in Figure 1, respectively. The FG plate is subjected to a uniformly distributed transverse load (q). The FG plate considered in numerical examples is made of Aluminum and Zirconia. The bottom surface of the FG plate is Aluminum and the top surface is Zirconia.
The effective material properties of the FG plate, $P$, such as the Young's modulus $E$, the Poisson's ratio $\nu$, the mass density $\rho$ and the shear modulus $G$ vary continuously in the thickness direction ($X_3$ axis) according to a power-law function as follows:

$$ P(X_3) = (P_T - P_B) \left( \frac{X_3 + \frac{1}{2}}{h} \right)^k + P_B - \left( P_T + P_B \right) \frac{a}{2} $$

(1)

Where $P_T$ and $P_B$ are the material properties of the top surface and the bottom surface materials of the plate, $k$ is the non-negative power-law exponent which determines the material variation profile using the thickness of the plate. It is clear from Eq. (1) that when $X_3 = -h/2$, $P = P_B$, and when $X_3 = h/2$, $P = P_T$. According to Eq. (1), when $k = 0$ (full bottom material) or $k = \infty$ (full top material) the material of the plate is homogeneous.

Two porosity models (even and uneven) are used in the porosity effect which was proposed by Wattanasakulpong and Unghbhor [29] for the power law distribution. In the even model, the porosity spreads uniformly through height direction and in the uneven model, the porosity spreads functionally through height direction. The distribution of the even and uneven porosity models is shown in Fig. 2.

According to the power law distribution, the effective material property for the even porosity can be expressed as follows:

$$ P(X_3) = (P_T - P_B) \left( \frac{X_3 + \frac{1}{2}}{h} \right)^k + P_B - \left( P_T + P_B \right) \frac{a}{2} $$

(2)

Where $a (a<1)$ is the volume fraction of porosities. When $a=0$, the plate becomes a perfect FGM. For uneven porosity distribution and according to the power law distribution, the effective material property can be expressed as follows:

$$ P(X_3) = (P_T - P_B) \left( \frac{X_3 + \frac{1}{2}}{h} \right)^k + P_B - \left( P_T + P_B \right) \frac{a}{2} \left( 1 - \frac{2}{h} X_3 \right) $$

(3)

According to the first-order shear deformation plate theory, the axial and the displacement fields are expressed as:
where \( u, v, w \) are \( X_1, X_2 \) and \( X_3 \) components of the displacements, respectively, \( \phi_{X_1} \) and \( \phi_{X_2} \) are rotations of transverse normal on the middle plane around \( X_1 \) and \( X_2 \) directions, and \( t \) indicates time. By using equations (4-6), the strain-displacement relation can be obtained as follows:

\[
\varepsilon_{x_1} = \frac{\partial u}{\partial x_1} = \frac{\partial u_0}{\partial x_1} + x_3 \frac{\partial \phi_{x_1}}{\partial x_1} + \frac{\partial \phi_{x_2}}{\partial x_1} + \frac{\partial \phi_{x_3}}{\partial x_1} \tag{7a}
\]

\[
\varepsilon_{x_2} = \frac{\partial v}{\partial x_2} = \frac{\partial v_0}{\partial x_2} + x_3 \frac{\partial \phi_{x_1}}{\partial x_2} + \frac{\partial \phi_{x_2}}{\partial x_2} + \frac{\partial \phi_{x_3}}{\partial x_2} \tag{7b}
\]

\[
\varepsilon_{x_3} = \frac{1}{2} \left( \frac{\partial u_0}{\partial x_2} + \frac{\partial v_0}{\partial x_1} + \frac{\partial \phi_{x_1}}{\partial x_2} + \frac{\partial \phi_{x_2}}{\partial x_1} + \frac{\partial \phi_{x_3}}{\partial x_1} \right) \tag{7c}
\]

The constitutive equations of the FG plate are as follows:

\[
\sigma_{ij} = \frac{E(X_3)}{1 - v(X_3)^2} \left[ \varepsilon_{ij} \varepsilon_{ij} (1 - v(X_3)) + (1 + v(X_3)) \varepsilon_{ij} \right] \tag{8}
\]

The elastic strain energy \( (U_i) \) and the kinetic energy \( (T) \) of the porous FG plate are expressed as follows:

\[
U_i = \frac{1}{2} \int \sigma_{ij} \varepsilon_{ij} dV \tag{9a}
\]

\[
T = \frac{1}{2} \int \rho(X_3) \left[ \left( \frac{\partial u}{\partial t} \right)^2 + \left( \frac{\partial v}{\partial t} \right)^2 + \left( \frac{\partial w}{\partial t} \right)^2 \right] dV \tag{9b}
\]

The Hamilton’s principle of the problem is

\[
\frac{\partial}{\partial t} \left[ T - U_i \right] dt = 0 \tag{10}
\]

After using the Hamilton’s principle, the governing equations of the FG porous plate can be obtained as follows:

\[
\frac{\partial N_{x_1}}{\partial x_1} + \frac{\partial N_{x_1}}{\partial x_2} = J_1 \frac{\partial^2 u_0}{\partial t^2} + J_2 \frac{\partial^2 \phi_{x_1}}{\partial t^2} \tag{11a}
\]

\[
\frac{\partial N_{x_2}}{\partial x_1} + \frac{\partial N_{x_2}}{\partial x_2} = J_1 \frac{\partial^2 v_0}{\partial t^2} + J_2 \frac{\partial^2 \phi_{x_2}}{\partial t^2} \tag{11b}
\]

\[
\frac{\partial M_{x_1}}{\partial x_1} + \frac{\partial M_{x_1}}{\partial x_2} - V_{x_1} = J_1 \frac{\partial^2 \phi_{x_1}}{\partial t^2} + J_2 \frac{\partial^2 u_0}{\partial t^2} \tag{11c}
\]
\[
\frac{\partial M_{X_i}}{\partial X_i} + \frac{\partial M_{X_i}}{\partial x} - V_{X_i} = J_3 \frac{\partial^2 \phi_{X_i}}{\partial t^2} + J_2 \frac{\partial^2 V_0}{\partial t^2}
\]  
(11d)

\[
\frac{\partial V_{X_i}}{\partial X_i} + \frac{\partial V_{X_i}}{\partial x} + q = J_3 \frac{\partial^2 W_0}{\partial t^2}
\]  
(11e)

where \( N_{X_i}, N_{X_i}, M_{X_i}, M_{X_i}, V_{X_i}, V_{X_i} \) are the stress resultants defined as follows:

\[
\begin{bmatrix}
N_{X_1} \\
N_{X_2} \\
M_{X_1} \\
M_{X_2}
\end{bmatrix} = \int_{0.5a}^{0.5b} \begin{bmatrix}
Q_{11} & Q_{12} & X_1 Q_{11} & X_1 Q_{12} \\
Q_{21} & Q_{22} & X_2 Q_{11} & X_2 Q_{12} \\
X_1 Q_{11} & X_1 Q_{12} & X_1^2 Q_{11} & X_1^2 Q_{12} \\
X_2 Q_{11} & X_2 Q_{12} & X_2^2 Q_{11} & X_2^2 Q_{12}
\end{bmatrix} \, dX_3
\]

\( (12a) \)

\[
\begin{bmatrix}
\frac{\partial u_0}{\partial X_1} \\
\frac{\partial v_0}{\partial X_1} \\
\frac{\partial \phi_{X_1}}{\partial X_1} \\
\frac{\partial \phi_{X_1}}{\partial X_2}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\frac{\partial u_0}{\partial X_1} \\
\frac{\partial v_0}{\partial X_1} \\
\frac{\partial \phi_{X_1}}{\partial X_1} \\
\frac{\partial \phi_{X_1}}{\partial X_2}
\end{bmatrix}
\]

\( (12b) \)

\[
V_{X_i} = \int_{0.5a}^{0.5b} \left( \phi_{X_i} + \frac{\partial w_0}{\partial X_i} \right) Q_{nn} \, dX_3, V_{X_i} = \int_{0.5a}^{0.5b} \left( \phi_{X_i} + \frac{\partial w_0}{\partial X_i} \right) Q_{nn} \, dX_3
\]

\( (12c) \)

where \( Q_{11}, Q_{12}, Q_{22}, Q_{44}, Q_{55}, Q_{nn} \) are the elastic coefficients which are presented as follows:

\[
Q_{11} = Q_{22} = \frac{E(X_i)}{1 - \nu(X_i)}, Q_{12} = \nu Q_{11}, Q_{44} = Q_{55} = Q_{nn} = \frac{E(X_i)}{1 + \nu(X_i)}
\]

(13)

In equation 11, \( J_i \) is defined as follows:

\[
J_i = \int_{0.5a}^{0.5b} \rho(X_i) \, dX_i, i = 1, 2, 3
\]

(14)

To solve the problem, the Navier solution is used for the simply supported rectangular porous FG plate. In the Navier solution, the displacement fields and boundary conditions of the plate are expressed as follows:

\[
u_0(x, x, t) = \sum_{n=1}^{N} \sum_{m=1}^{N} A_{mn} \cos(kX_i) \sin(lX_j) e^{-\sigma_i t}
\]

(15a)

\[
y_0(x, x, t) = \sum_{n=1}^{N} \sum_{m=1}^{N} B_{mn}(t) \sin(kX_i) \cos(lX_j) e^{-\sigma_i t}
\]

(15b)

\[
w_0(x, x, t) = \sum_{n=1}^{N} \sum_{m=1}^{N} C_{mn}(t) \sin(kX_i) \cos(lX_j) e^{-\sigma_i t}
\]

(15c)

\[
\varphi_{X_i}(x, x, t) = \sum_{n=1}^{N} \sum_{m=1}^{N} D_{mn}(t) \cos(kX_i) \sin(lX_j) e^{-\sigma_i t}
\]

(15d)
\[ \varphi_{x_1}(x_1, x_2, t) = \sum_{m=1}^{N} \sum_{n=1}^{N} F_{mn}(t) \sin(kX_m) \cos(lX_n) e^{-jwt} \]  

(15e)

\[ v_0 = 0, \quad w_0 = 0, \quad \varphi_{x_1} = 0, \quad M_{x_1} = 0, \quad N_{x_1} = 0, \quad at, \quad x_1 = 0, \quad L_x \]  

(16a)

\[ u_0 = 0, \quad w_0 = 0, \quad \varphi_{x_2} = 0, \quad M_{x_2} = 0, \quad N_{x_2} = 0, \quad at, \quad x_2 = 0, \quad L_y \]  

(16b)

where \( A_{mn}, B_{mn}, C_{mn}, D_{mn}, E_{mn} \) are displacement coefficients, \( k = m\pi / L_x, l = n\pi / L_y, \beta \) is the natural frequency and \( i = \sqrt{-1} \).

In the static bending problem, time and its derivatives are zero in the governing equations. In the Navier solution, the uniform distributed load \( q_0 \) is defined as follows:

\[ q_0 = \sum_{m=1}^{N} \sum_{n=1}^{N} q_{mn}(t) \sin(kX_m) \sin(lX_n) \]  

(17)

where

\[ q_{mn} = \int_{L_x}^{l_x} \int_{L_y}^{l_y} \frac{4q}{L_x L_y} \sin(kX_m) \sin(lX_n) \, dx \, dx \]  

(18)

By substituting Eqs. (15) and Eqs. (17-18) into Eqs. (11), and then using the matrix procedure, the algebraic equations of the static problem can be expressed as follows:

\[
\begin{bmatrix}
K_{11} & K_{12} & K_{13} & K_{14} & K_{15} \\
K_{21} & K_{22} & K_{23} & K_{24} & K_{25} \\
K_{31} & K_{32} & K_{33} & K_{34} & K_{35} \\
K_{41} & K_{42} & K_{43} & K_{44} & K_{45} \\
K_{51} & K_{52} & K_{53} & K_{54} & K_{55}
\end{bmatrix}
\begin{bmatrix}
A_m \\
B_m \\
C_m \\
D_m \\
E_m
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
q_{mn} \\
0 \\
0
\end{bmatrix}
\]  

(19)

where

\[ K_{11} = k^2 \int_{-0.5L_x}^{0.5L_x} Q_{11} \, dx_3 + l^2 \int_{-0.5L_y}^{0.5L_y} Q_{44} \, dx_3 \]  

(20a)

\[ K_{12} = K_{21} = kl \int_{-0.5L_x}^{0.5L_x} Q_{12} \, dx_3 + kl \int_{-0.5L_y}^{0.5L_y} Q_{42} \, dx_3 \]  

(20b)

\[ K_{13} = K_{31} = 0 \]  

(20c)

\[ K_{14} = K_{41} = k^2 \int_{-0.5L_x}^{0.5L_x} X_3 Q_{11} \, dx_3 + l^2 \int_{-0.5L_y}^{0.5L_y} X_3 Q_{44} \, dx_3 \]  

(20d)

\[ K_{15} = K_{51} = kl \int_{-0.5L_x}^{0.5L_x} X_3 Q_{12} \, dx_3 + kl \int_{-0.5L_y}^{0.5L_y} X_3 Q_{42} \, dx_3 \]  

(20e)

\[ K_{22} = l^2 \int_{-0.5L_y}^{0.5L_y} Q_{22} \, dx_3 + k^2 \int_{-0.5L_x}^{0.5L_x} Q_{44} \, dx_3 \]  

(20f)

\[ K_{23} = K_{32} = 0 \]  

(20g)

\[ K_{24} = K_{42} = kl \int_{-0.5L_x}^{0.5L_x} X_3 Q_{12} \, dx_3 + kl \int_{-0.5L_y}^{0.5L_y} X_3 Q_{42} \, dx_3 \]  

(20h)
For the free vibration problem of the FG porous plate, the algebraic equations expressed as follows:

\[
K_{25} = K_{32} = l^2 \int_{-0.5h}^{0.5h} X_2 Q_{22} dX_3 + k^2 \int_{-0.5h}^{0.5h} X_2 Q_{44} dX_3
\]  
(20j)

\[
K_{33} = k^2 \int_{-0.5h}^{0.5h} Q_{44} dX_3 + l^2 \int_{-0.5h}^{0.5h} Q_{44} dX_3
\]  
(20k)

\[
K_{34} = K_{45} = k \int_{-0.5h}^{0.5h} Q_{44} dX_3
\]  
(20l)

\[
K_{35} = K_{45} = l \int_{-0.5h}^{0.5h} Q_{44} dX_3
\]  
(20m)

\[
K_{44} = k^2 \int_{-0.5h}^{0.5h} X_2 Q_{22} dX_3 + l^2 \int_{-0.5h}^{0.5h} X_2 Q_{44} dX_3 + \int_{-0.5h}^{0.5h} X_2 Q_{44} dX_3
\]  
(20n)

\[
K_{45} = K_{45} = kl \int_{-0.5h}^{0.5h} X_2 Q_{22} dX_3 + kl \int_{-0.5h}^{0.5h} X_2 Q_{44} dX_3
\]  
(20o)

\[
K_{55} = l^2 \int_{-0.5h}^{0.5h} X_2 Q_{22} dX_3 + k^2 \int_{-0.5h}^{0.5h} X_2 Q_{44} dX_3 + \int_{-0.5h}^{0.5h} Q_{44} dX_3
\]  
(20p)

For the free vibration problem of the FG porous plate, the algebraic equations expressed as follows:

\[
\begin{bmatrix}
K_{11} & K_{12} & K_{13} & K_{14} & K_{15} \\
K_{21} & K_{22} & K_{23} & K_{24} & K_{25} \\
K_{31} & K_{32} & K_{33} & K_{34} & K_{35} \\
K_{41} & K_{42} & K_{43} & K_{44} & K_{45} \\
K_{51} & K_{52} & K_{53} & K_{54} & K_{55}
\end{bmatrix}
\begin{bmatrix}
M_{11} \\
M_{12} \\
M_{13} \\
M_{14} \\
M_{15} \\
M_{21} \\
M_{22} \\
M_{23} \\
M_{24} \\
M_{25} \\
M_{31} \\
M_{32} \\
M_{33} \\
M_{34} \\
M_{35} \\
M_{41} \\
M_{42} \\
M_{43} \\
M_{44} \\
M_{45} \\
M_{51} \\
M_{52} \\
M_{53} \\
M_{54} \\
M_{55}
\end{bmatrix}
= \begin{bmatrix}
A_m \\
B_m \\
C_m \\
D_m \\
E_m
\end{bmatrix}
\]  
(21)

where

\[
M_{11} = J_1, M_{12} = M_{21} = 0, M_{13} = M_{31} = 0, M_{14} = M_{41} = 0, M_{15} = M_{51} = 0, M_{22} = J_2, M_{23} = M_{32} = 0, M_{24} = M_{42} = 0, M_{25} = M_{52} = 0, M_{33} = J_3, M_{34} = 0, M_{35} = M_{44} = 0, M_{45} = 0
\]  
(22)

By solving the eigenvalue problem which given in eq. (21), the fundamental frequencies can be determined and the dimensionless quantities can be expressed as follows:

\[
w = \frac{w E_i h^3}{q L^4}, \omega = \beta \sqrt{ \frac{\rho_i L^2}{E_i h^2} }
\]  
(23)

where \( w \) is the dimensionless deflection and \( \omega \) is the dimensionless fundamental frequency.

3. Results And Discussion

In the numerical study, the natural frequencies and the static bending deflections of the simply supported FG plate are calculated and presented in figures for different porosity parameters, porosity models and material distributions. The FG porous plate considered in numerical examples is made of Aluminum (Al; \( E=70 \) GPa, \( \nu=0.3 \), \( \rho=2702 \) kg/m$^3$) and Zirconia (\( E=151 \)GPa, \( \nu=0.3 \), \( \rho=3000 \) kg/m$^3$). The top surface material of the FG plate is Zirconia, the bottom surface material of the FG plate is Aluminum. According to Eq. (2) and Eq. (3), when \( k=0 \) and \( k=\infty \), the material of the plate gets homogeneous Aluminum and homogeneous Zirconia, respectively. The dimensions of the FG plate in the numerical examples are considered as follows: \( h = 0.3 \) m, \( L_X =10h \) and \( L_Y=10h \).

Figures 3 and 4 display the effect of the material distribution parameter \( k \) on the dimensionless bending deflections and fundamental frequencies of the porous FG plate for different porosity parameters and porosity models, respectively.
It can be seen from Figures 3 and 4 that by an increase in the power-law exponent k the dimensionless bending deflections increase and also the dimensionless fundamental frequencies decrease. The reason is that, according to equation (1), the elasticity modulus and bending rigidity of the FG plate decrease with an increase in the k. As can be seen in Figures 3 and 4, by an increase in k parameter, the difference between the even and uneven porosity models increases as well. In small values of k parameter, this difference can be neglected and this shows that the material distribution plays an important role in the porosity effect.

In order to investigate the effect porosity parameters on the bending and vibration characteristics, the dimensionless bending deflections and fundamental frequencies of the porous FG plate versus the porosity parameter a for different porosity models for k=2 were obtained in Figures 5 and 6, respectively.

Fig. 5. The effect of porosity parameter (a) and porosity models on the dimensionless bending deflections $\overline{w}$ of the FG porous plate.
As can be seen in Figure 5, by an increase in the porosity parameter $a$ in both even and uneven porosities, the dimensionless bending deflections of the FG plate also increase. The reason is that, by an increase in the porosity parameter, the strength of the material decreases. In Figure 6, by an increase in the porosity parameter $a$ in the even porosity model, the dimensionless fundamental frequencies decrease. However, by an increase in the porosity parameter $a$ in the uneven porosity model, the dimensionless fundamental frequencies increase. Moreover, it can be seen in Figures 5 and 6 that, by increasing porosity parameter ($a$), the difference between the even and uneven porosity models increases significantly. Porosity parameters have a very important role in the static and vibration behavior of the FG porous plates.

4. Conclusions

Free vibration and static bending responses of a simply supported FG plate are investigated through the porosity effect and based on the first-order shear deformation theory by using Navier solution. According to power-law distributions and within different porosity models, the material properties of the plate are assumed to vary though the height direction. The effects of material distribution and porosity parameters on the vibration characters and bending deflections of the FG porous plate are presented within the different porosity models. It is observed in the results that material distribution and porosity parameters have very important roles in the static and vibration responses of the FG porous plates. The difference between the even and uneven porosity models increases with an increase in $k$ and $a$ parameters.

References


