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Research Article

Moving Mesh Non-standard Finite Difference Method for Non-linear Heat Transfer in a Thin Finite Rod

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Abstract. In this paper, a moving mesh technique and a non-standard finite difference method are combined, and a moving mesh non-standard finite difference (MMNSFD) method is developed to solve an initial boundary value problem involving a quartic nonlinearity that arises in heat transfer with thermal radiation. In this method, the moving spatial grid is obtained by a simple geometric adaptive algorithm to preserve stability. Moreover, it uses variable time steps to protect the positivity condition of the solution. The results of this computational technique are compared with the corresponding uniform mesh non-standard finite difference scheme. The simulations show that the presented method is efficient and applicable, and approximates the solutions well, while because of producing unreal solution, the corresponding uniform mesh non-standard finite difference fails.

Keywords: Non-standard finite difference; Positivity; Moving mesh; Heat conduction equation.

1. Introduction

Many of partial differential equations (PDEs), which describe physical phenomena, do not have exact solutions. Therefore, the application of numerical methods to obtain their solutions is controversial. In the process of simulation, it is desirable that numerical results reflect the original system behavior. For example, in the heat conduction problem, as described by diffusion equation, the solution must be non-negative.

In order to solve PDEs numerically, especially non-linear PDEs, sometimes traditional standard finite difference methods fail by generating oscillation and producing unreal solutions. However, the non-standard finite difference methods developed by Mickens [6, 7, 8, 9, 11] have better performance than the classical methods in terms of numerical stability in such a way that they can be constructed flexibly to preserve some important properties. For this reason, the non-standard finite difference (NSFD) schemes have been developed to solve biological problems and dynamic systems in recent years. (see [4 & 5]).

The general basic rules to construct NSFDs are as follows [7]:

● The order of the discrete derivatives should be equal to the order of the corresponding derivatives presented in the differential equation.
● In general, discrete representations for derivatives have nontrivial denominator functions.
● Nonlinear terms can be replaced by nonlocal discrete representations.
● Special conditions that hold for either the differential equation or its solutions should also hold for the difference equation model or its solutions.

In this study, the aim is to apply NSFD scheme for the non-linear heat transfer in a thin rod. This problem has been solved by NSFD on the uniform mesh in some studies [1, 2, &3], and in some cases, some oscillations appear in the obtained solution. We

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intend to apply the moving mesh for NSFD to solve aforementioned problem such that the grid points concentrate on the particular part of the domain where the heat source is located and the large temperature variation occurs.
The present study is organized as follows:
First, some NSFD schemes for the non-linear heat transfer in a thin finite rod based on uniform mesh is reviewed in section 2. In section 3, by a simple adaptive technique, NSFD scheme on moving grids is developed such that positivity and stability are satisfied for the aforementioned problem. Next, in section 4, the new method is examined for some examples and the results are compared to some NSFD schemes on the uniform mesh. Finally, section 5 provides some concluding remarks.

2. Problem presentation and some non-standard schemes to solve it

Consider the following initial-boundary problem [1, 2]:

$$\begin{cases} T_\tau = kT_{XX} - \beta_0(T^4 - T_\infty^4), (X,\tau) \in (0,l)\times(0,\infty) \\ T(0,\tau) = T_1, T(l,\tau) = T_2, \tau \geq 0 \\ T(X,0) = T_0 \sin\left(\dfrac{X\pi}{l}\right), X \in (0,l) \end{cases} \tag{1}$$

Where $T = T(\tau, X)$ denote the temperature distribution in a very thin, homogeneous, and thermally conducting solid rod. This rod includes constant cross-sectional area A, perimeter p, length l, and constant thermal diffusivity $\kappa > 0$, that occupies the open interval $(0, l)$ along the X −axis of Cartesian coordinate system. $T_0 sin\left(\frac{X\pi}{l}\right)$ is the rod's initial temprature and T_1 and T_2 are boundary conditions. Moreover, T_∞ is the surrounding temperature. Using the following dimensionless variables:

$$u = \frac{T}{T_0}, x = \frac{X}{l}, t = \frac{\tau k}{l^2}, u_\infty = \frac{T_\infty}{T_0}, \beta = \frac{T_0^3 l^3 \beta_0}{k}$$

IBVP Eq. (1) can be rewritten in the dimensionless form as:

$$\begin{cases} u_t = u_{xx} - \beta(u^4 - u_\infty^4), (x,t) \in (0,1)\times(0,\infty) \\ u(0,t) = U_1, u(1,t) = U_2, t\rangle 0 \\ u(x,0) = \sin(\pi x), x \in (0,1), \end{cases} \tag{2}$$

Where U_1 and U_2 are the dimensionless forms of T_1 and T_2, respectively. To obtain NSFD schemes on a uniform mesh, consider the fixed step size Δx and Δt on x-axis and t-axis, respectively, and denote the approximate solution of $u(x,t)$ at point $(i\Delta x, n\Delta t), i = 0,1,\cdots,m$ and $n = 0,1,\cdots$ by u_i^n. Accordingly, two non-standard explicit finite difference schemes for Eq. (2) are proposed as follows [1]:

$$u_i^{n+1} = \frac{(1-2R)u_i^n + R(u_{i+1}^n + u_{i-1}^n) + \beta u_\infty^4 \Delta t}{1 + \beta (u_i^n)^3 \Delta t} \tag{3}$$

$$u_i^{n+1} = \frac{(1-2R)u_i^n + R(u_{i+1}^n + u_{i-1}^n) + \beta u_\infty^4 \Delta t}{1 + \beta \Delta t \left[(u_{i+1}^n)^3 + (u_{i-1}^n)^3\right]/2} \tag{4}$$

Where $R = \frac{\Delta t}{\Delta x}$. Above-mentioned schemes are obtained by replacing the quartic term given in Eq. (2) with the following nonlinear forms:

$$\beta(u^4 - u_\infty^4) \to \beta(u_i^n)^3 u_i^{n+1} - \beta u_\infty^4 \tag{5}$$

$$\beta(u^4 - u_\infty^4) \to \beta\left[\frac{(u_{i+1}^n)^3 + (u_{i-1}^n)3}{2}\right] u_i^{n+1} - \beta u_\infty^4 \tag{6}$$

As can be seen, both schemes show instability when both values of β and u_∞ become large. In one of the previous studies [2], the representation of the quartic term is replaced by the following nonlinear term:

$$\beta(u^4 - u_\infty^4) \to \beta\left[(u_i^n)^2 + (u_\infty)^2\right](u_i^n + u_\infty)(u_i^{n+1} - u_\infty) \tag{7}$$

Therefore, a new non-standard finite difference scheme with a good stability can be obtained as follows:

$$u_i^{n+1} = \frac{(1-2R)u_i^n + R(u_{i+1}^n + u_{i-1}^n) + \beta\Delta t\left[(u_i^n)^2 + (u_\infty)^2\right](u_i^n + u_\infty)u_\infty}{1 + \beta\Delta t\left[(u_i^n)^2 + (u_\infty)^2\right](u_i^n + u_\infty)}$$

It can be seen that the truncation errors in above-mentioned schemes are of order $O(\Delta t + (\Delta x)^2)$. It is clear that the schemes also satisfy the positivity condition:

$$u_i^n \geq 0 \Rightarrow u_i^{n+1} \geq 0,$$

when $(1-2R) \geq 0$. In a study [3], the researchers employed Crank-Nicholson method and proposed the following nonstandard implicit finite difference schemes:

$$-Ru_{i-1}^{n+1} + \left(2+2R+\beta\Delta t(u_i^n)^3\right)u_i^{n+1} - Ru_{i+1}^{n+1} = Ru_{i-1}^n + (2-2R)u_i^n + Ru_{i+1}^n + \beta\Delta t u_\infty^4, \tag{8}$$

$$-Ru_{i-1}^{n+1} + \left(2+2R+\frac{\beta\Delta t(u_{i+1}^n)^3 + (u_{i-1}^n)^3}{2}\right)u_i^{n+1} - Ru_{i+1}^{n+1} = Ru_{i-1}^n + (2-2R)u_i^n + Ru_{i+1}^n + \beta\Delta t u_\infty^4, \tag{9}$$

$$\begin{aligned} &-Ru_{i-1}^{n+1} + (2+2R+\beta\Delta t\left[(u_i^n)^2 + u_\infty^2\right](u_i^n + u_\infty))u_i^{n+1} - Ru_{i+1}^{n+1} = \\ &Ru_{i-1}^n + (2-2R+\beta\Delta t u_\infty)((u_i^n)^2 + u_i^n u_\infty + u_\infty^2)) + Ru_{i+1}^n + \beta\Delta t u_\infty^4 \end{aligned} \tag{10}$$

The truncation errors in above schemes are of order $O((\Delta t)^2 + (\Delta x)^2)$.

3. Moving mesh non-standard finite difference method

As mentioned earlier, sometimes using NSFD schemes on the uniform meshes leads to instability in the solutions. To resolve this problem, increasing the number of mesh points is a conventional way, though this approach is not always practical. In this case, the mesh adaptation is an imperative tool for using in the numerical solution. Such methods use nonuniform spatial meshes and as time proceeds, they automatically concentrate on the grid points in spatial regions of high activity. Moving mesh methods or adaptive mesh methods are divided into static and dynamic states. In the dynamic state, the mesh equation and the original differential equation are solved simultaneously, and the discretization of the PDE and the grid selection are intrinsically coupled. However, in static status, the grid moves only at discrete time levels and no intrinsic coupling exists between the discretization of the PDE and the grid selection. In this stage, the solution is advanced in time on a fixed non-uniform grid and by a suitable finite difference or a finite element scheme. After that, a remeshing procedure is carried out, the values for new mesh points are approximated, and this process continues repeatedly.

3.1. Application of NSFD on non-uniform mesh

To solve Eq. (2) on a non-uniform mesh, suppose $\{x_i^n\}$ represents mesh points on time level t_n ,where $\Delta t_n = t_{n+1} - t_n$ is the distance between time steps t_n and t_{n+1} , and u_i^n denotes the finite difference approximation of $u(x_i^n, t_n)$. Using Eq. (5) and discreting Eq. (2) on a non-uniform mesh yields:

$$\frac{\tilde{u}_i^{n+1} - u_i^n}{\Delta t_n} = \frac{2}{x_{i+1}^n - x_{i-1}^n}\left(\frac{u_{i+1}^n - u_i^n}{x_{i+1}^n - x_i^n} - \frac{u_i^n - u_{i-1}^n}{x_i^n - x_{i-1}^n}\right) - \beta((u_i^n)^3\tilde{u}_i^{n+1} - u_\infty^4),$$

Where $\tilde{u}_i^{n+1} \approx u(x_i^n, t_{n+1})$. The above-mentioned equation can be rewritten by introducing $\Delta x_i^n = x_{i+1}^n - x_i^n$ in the following form:

$$\begin{aligned} (1+\beta\Delta t_n(u_i^n)^3)\tilde{u}_i^{n+1} &= \frac{2\Delta t_n}{\Delta x_{i-1}^n(\Delta x_i^n + \Delta x_{i-1}^n)}u_{i-1}^n + \left(1 - \frac{2\Delta t_n}{\Delta x_i^n \Delta x_{i-1}^n}\right)u_i^n \\ &+ \frac{2\Delta t_n}{\Delta x_i^n(\Delta x_i^n + \Delta x_{i-1}^n)} + \beta u_\infty^4 \Delta t_n \end{aligned} \tag{11}$$

It is clear that if

$$1 - \frac{2\Delta t_n}{\Delta x_i^n \Delta x_{i-1}^n} \geq 0,$$

Then

$$u_i^n \geq 0 \Rightarrow \tilde{u}_i^{n+1} \geq 0, i = 0,1,...,m.$$

Therefore, to ensure this condition, the temporal step size Δt_n is selected such that:

$$\Delta t_n \leq \frac{\Delta x_i^n \Delta x_{i-1}^n}{2}, I = 0,1,...M-1.$$

Similarly, according to Eq. (6), the following non-standard finite difference scheme is obtained:

$$\begin{aligned} \left(1 + \frac{\beta\Delta t_n\left[(u_{i+1}^n)^3 + (u_{i-1}^n)^3\right]}{2}\right)\tilde{u}_i^{n+1} &= \frac{2\Delta t_n}{\Delta x_{i-1}^n(\Delta x_i^n + \Delta x_{i-1}^n)}u_{i-1}^n + \left(1 - \frac{2\Delta t_n}{\Delta x_i^n \Delta x_{i-1}^n}\right)u_i^n \\ &+ \frac{2\Delta t_n}{\Delta x_i^n(\Delta x_i^n + \Delta x_{i-1}^n)}u_{i+1}^n + \beta u_\infty^4 \Delta t_n \end{aligned} \tag{12}$$

In the next subsection, selecting grids and time steps are provided.

For nonlinear wave problems, Sanz-Serna [10] has suggested a simple adaptive technique, independent of time steps and in accordance with the equidistribution principle, that we can use this algorithm to move mesh points within each time step. The Grid movement is based on the equidistribution principle as follows:

Suppose that at time level t_{n+1}, we have $\tilde{u}_i^{n+1}$ (from Eqs. (11 or 12)), $i = 0,1,\cdots m$. The following algorithm is used to obtain $x_i^{n+1}, i = 0,1,\cdots,m$, where it distributes the arc-length of $u(x,t_{n+1})$. Remeshing algorithm [11]:

1: Given $x_i^n, \tilde{u}_i^{n+1}, i = 0,1,...,m$ and $\Delta x_i^n = x_{i+1}^n - x_i^n, i = 0,1,...,m-1$.

2: $S_0 = 0$.

3: For $i = 1,...,m, S_i = S_{i-1} + \sqrt{(\Delta x_i^n)^2 + \left\|\tilde{u}_i^{n+1} - \tilde{u}_{i-1}^{n+1}\right\|_2^2}$. Next i.

4: $\delta = \dfrac{S_m}{m}, k = 1, x_0^{n+1} = x_0^n, x_m^{n+1} = x_m^n, u_0^{n+1} = \tilde{u}_0^{n+1}, u_m^{n+1} = \tilde{u}_m^{n+1}, i = 1$.

5: $B = i\delta$

6: If $B \le S_k$, go to (8).

7: $k = k+1$. Go to (6).

8: $x_i^{n+1} = x_{k-1}^n + \dfrac{(B - S_{k-1})\Delta x_k^n}{S_k - S_{k-1}}$.

9: $\Delta x_{i-1}^{n+1} = x_i^{n+1} - x_{i-1}^{n+1}$.

10: If $i < m$, then $i = i+1$ and go to (5).

11: Return $x_i^{n+1}, i = 0,1,...,m$ and $\Delta x_i^{n+1}, i = 0,1,...,m-1$.

3.3. MMNSFD algorithm

Considering the advancement in solving the target problem up to the time level t_n, $x_i^n, u_i^n, i = 0,1,\cdots,m$ is obtained. To compute x_i^{n+1} and u_i^{n+1} at the time level t_{n+1}, the following steps have to be performed:

1) Select the suitable time step Δt_n such that the positivity condition is satisfied for all Δx_i^n.

2) Advance the solution on the fixed non-uniform grid x_i^n using Eqs. (11 or 12) and obtain approximations $\tilde{u}_i^{n+1}$.

3) Use remeshing algorithm to define x_i^{n+1} (to perform this stage, use the described moving mesh algorithm as explained in section 3.2).

4) Compute u_i^{n+1} as an approximation to $u(x_i^{n+1}, t_{n+1})$ by means of a suitable interpolation (for example, by Spline interpolation) of the values $x_i^{n+1}, \tilde{u}_i^{n+1}$.

In the above-mentioned algorithm, $x_i^0, i = 0,1,\cdots,m$, are chosen so that they form a uniform partition of $[0,1]$ and $u_i^0, i = 0,1,\cdots,m$ are obtained by the initial condition. Stages of the above-mentioned algorithm are shown in Fig.1.

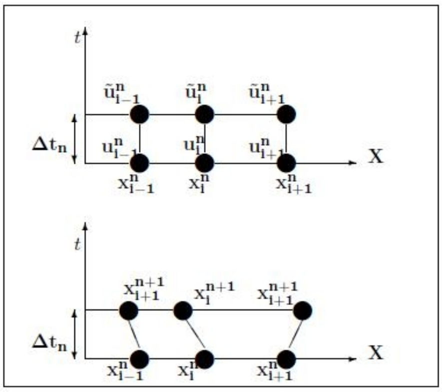

Fig. 1. Adaptive mesh strategy. Steps 1 and 2 are shown in up and steps 3 and 4 in down.

4. Numerical experiments

In this section, $U_1 = U_2 = 0$ for numerical simulation and $\Delta x = 0.02$ for uniform mesh schemes were considered.

At first, $\beta = u_\infty = 2$ was chosen. In this case, the solution of Eq. (2) by using NSFD scheme Eq. (3) was computed. This scheme is unstable for $\Delta t = 0.0002$ and the solution begins to oscillate (Fig.2.a), while the MMNSFD algorithm is stable (Fig. 2b(*)). Moreover, the moving mesh trajectory of this case up to $t = 1$ was plotted as in Fig. 3.

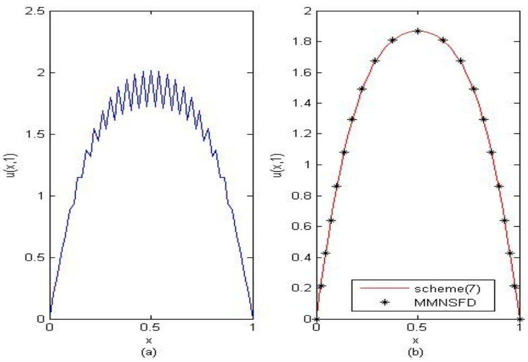

Fig. 2. (a) Numerical solution at t=1 obtained by Eq. (3) for Δt=0.0002, Δx=0.02 and (b) Numerical solution obtained by Eqs. (11&7) for $\beta = u_\infty = 2, m = 20$.

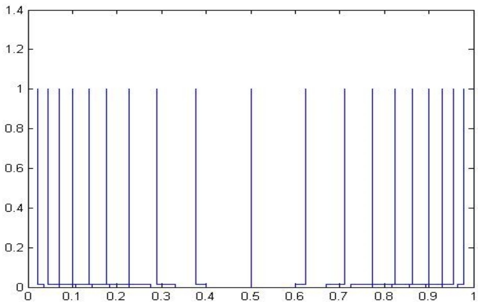

Fig. 3. The corresponding mesh trajectories with m=20 mesh points, for $\beta = u_\infty = 2$.

Then, $\beta = u_\infty = 6$ is considered. For this case, the computed solution by Eq. (3) as plotted in Fig. 4.a shows instability. The solution by MMNSFD is plotted and compared in Fig. 4.b($*$). Moreover, the moving mesh trajectory for this case is represented in Fig. 5.

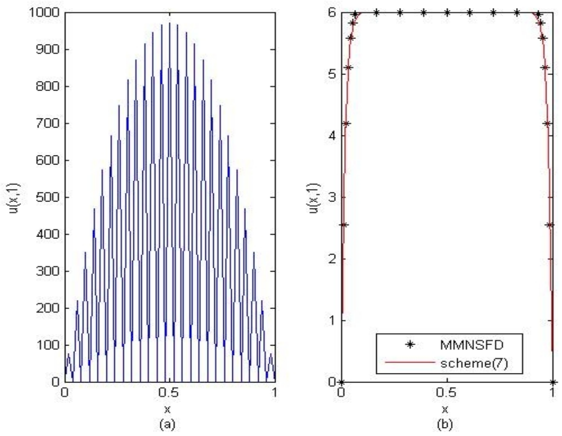

Fig. 4. Numerical solution at t=1 obtained by Eq. (3) for $\mathbf{\Delta t = 0.0002, \Delta x = 0.02}$ and (b) Numerical solution obtained by Eqs. (11 & 7) for $\boldsymbol{\beta = u_\infty = 6, m = 20}$.

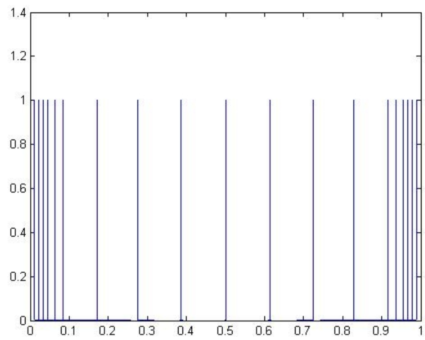

Fig. 5. The corresponding mesh trajectories with m=20 mesh points, for $\boldsymbol{\beta = u_\infty = 6}$.

Considering $\beta = u_\infty = 10$, for $\Delta t = 0.0001$, the computed solution by NSFD scheme (Eq. (3)) is unstable and solution begins to oscillate (Fig. 6). For $\Delta t = 0.0002$, it produces a non-acceptable solution (Fig.7.a), while the MMNSFD algorithm is stable (Fig. 7.b($*$)).

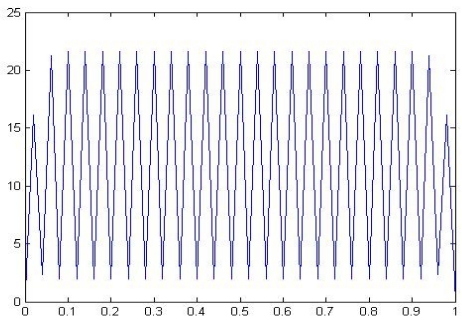

Fig. 6. Numerical solution obtianed by Eq. (3) on uniform mesh when $\boldsymbol{\beta = u_\infty = 10}$ and $\boldsymbol{\Delta t = 0.0001, \Delta x = 0.02}$.

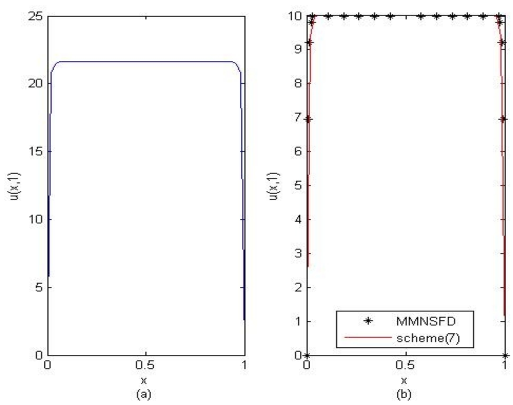

Fig. 7. (a) Numerical solution obtained by Eq. (3) for $\mathbf{\Delta t = 0.0002}$ and (b) Numerical solution obtained by Eqs. (11 & 7) for $\boldsymbol{\beta = u_\infty = 10}$.

Finally, $\beta = u_\infty = 15$ is considered. For this case, the computed solution by NSFD scheme (Eq. (3)), the solution by MMNSFD method and the mesh movement are plotted in Figs. 8.a, 8.b, and 9, respectively. It can be seen the scheme (Eq. (3)) produces an unreasonable solution.

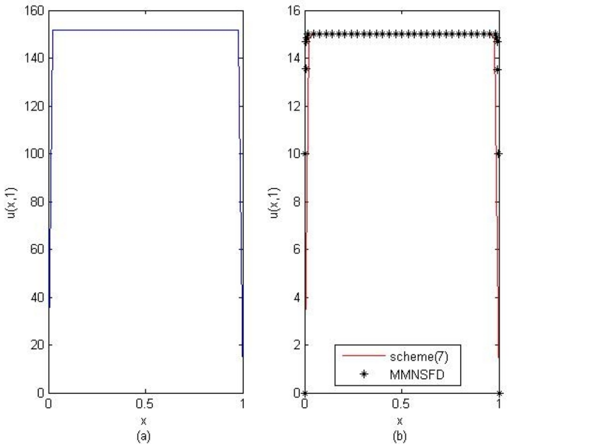

Fig. 8. (a) Unreasonable numerical solution obtained by Eq. (3) for $\mathbf{\Delta t = 0.0002}$ and (b) Numerical solution obtained by Eqs. (11 & 7) for $\boldsymbol{\beta = u_\infty = 15, m = 40}$.

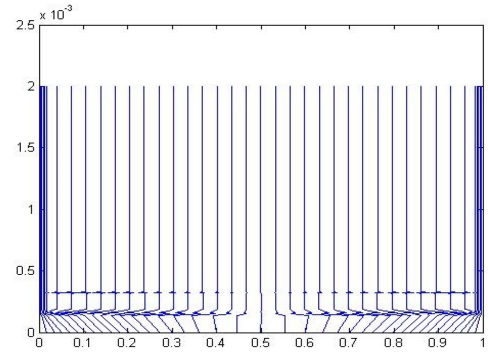

Fig. 9. The corresponding mesh trajectories with m=40 mesh point, for $\boldsymbol{\beta = u_\infty = 15, t = 0.002}$.

5. Conclusion

In this study, we considered the numerical solution of the nonlinear heat transfer in a thin rod using the moving mesh non-standard finite difference method. This problem having some NSFD on the uniform mesh requires the use of an implicit time integration scheme and sometimes a very small Δx and Δt. The numerical results show that the method is able to preserve the stability and concentrate the mesh points near the places where the solution has rapid variations. Moreover, the nonnegativity of solution is maintained via this approach. The results are comparable to those obtained by Dai et al. [2].

Nomenclature

A	Rod's cross sectional area	X	Space variable
$K > 0$	Thermal conductivity of the solid	$\beta_0 = \frac{\kappa\delta\epsilon p}{KA}$	Defined based on others parameters
l	Rod's length	$\delta \approx 5.67 \times 10^{-8}$	Stefan-Boltzmann constant
p	Rod's perimeter	$\epsilon \in [0,1]$	Emissivity of the surface
T	Temperature distribution	κ	Constant thermal diffusivity
T_0	Maximum rod's initial temperature	τ	Temporal variable
T_∞	Surrounding temperature		

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