A Novel Approach for Korteweg-de Vries Equation of Fractional Order

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Abstract. In this study, the local fractional variational iteration method (LFVIM) and the local fractional series expansion method (LFSEM) are utilized to obtain approximate solutions for Korteweg-de Vries equation (KdVE) within local fractional derivative operators (LFDOs). The efficiency of the considered methods is illustrated by some examples. The results reveal that the suggested algorithms are very effective and simple and can be applied for linear and nonlinear problems in mathematical physics.

Keywords: Local fractional operators; Local fractional variational iteration method; Local fractional series expansion method; Korteweg-de Vries.

1. Introduction

The local fractional calculus was successfully utilized to describe the PDEs arising in mathematical physics, such as the diffusion equations [1-4], the gas dynamic equation [5], the telegraph equation [6], the wave equation [7], the Fokker Planck equation [8,9], the Laplace equation [10], the Klein-Gordon equations [11,12], the Helmholtz equation, [13,14] and the Goursat problem [15] on Cantor sets. Recently, the KdVE with LFDOs was given by [16, 17]:

\[
\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + \frac{\partial^\alpha u(x,t)}{\partial x^\alpha} + \frac{\partial^\alpha u(x,t)}{\partial x^{2\alpha}} = 0, \quad 0 < \alpha \leq 1. \tag{1}
\]

subjected to initial condition as:

\[
u(x,0) = \phi(x) \quad . \tag{2}
\]


It is worth mentioning that entropy plays an important role in the analysis of anomalous diffusion processes and fractional diffusion equations. These fractional novel entropy indices and fractional operators allow their implementation in complex dynamical systems [23-25]. Another application is related to local fractional wave equations under fixed entropy arising in fractal hydrodynamics [26].

In the present study, the local fractional variational iteration method and the local fractional series expansion method were
applied to solve the local fractional Korteweg-de Vries equation. The advantage of these methods with respect to other numerical methods is that they don’t need discretization.

2. Basic Definitions of Local Fractional Calculus

In this section, some basic definitions and properties of the fractional calculus theory [10-13] are provided.

Definition 1. A function \( f(x) \) is local fractional continuous at \( x = x_0 \), if it holds

\[
| f(x) - f(x_0) | < \varepsilon \alpha \, , \, 0 < \alpha \leq 1
\]

with \( |x - x_0| < \delta \), for \( \varepsilon, \delta > 0 \), and \( \varepsilon, \delta \in \mathbb{R} \). For \( x \in (a,b) \), it is so called local fractional continuous on \( (a,b) \), denoted by \( f(x) \in C_{\alpha}(a,b) \).

Definition 2. Setting \( f(x) \in C_{\alpha}(a,b) \), the local fractional derivative of \( f(x) \) at \( x = x_0 \) is defined as:

\[
D_{x}^{\alpha} f(x) = \lim_{x \to x_0} \frac{\Delta^{\alpha}(f(x) - f(x_0))}{(x - x_0)^\alpha},
\]

where \( \Delta^{\alpha}(f(x) - f(x_0)) \geq \Gamma(\alpha + 1)(f(x) - f(x_0)) \).

Not that the local fractional derivative of high order is written in the following form:

\[
D_{x}^{\alpha} f(x) = f^{(k\alpha)}(x) = D_{x}^{\alpha} D_{x}^{\alpha} \cdots D_{x}^{\alpha} f(x),
\]

and the local fractional partial derivative of high order as:

\[
\frac{\partial^{k\alpha} f(x,y)}{x^{\alpha}} = \frac{\partial^{\alpha}}{\partial x^{\alpha}} \frac{\partial^{\alpha}}{\partial x^{\alpha}} \cdots \frac{\partial^{\alpha}}{\partial x^{\alpha}} f(x,y).
\]

Definition 3. Let’s denote a partition of the interval \([a,b]\) as \((t_j, t_{j+1})\), \( j = 0, \ldots, N-1 \), and \( t_N = b \) with \( \Delta t_j = t_{j+1} - t_j \) and \( \Delta t = \max\{\Delta t_0, \Delta t_1, \ldots\} \). The local fractional integral of \( f(x) \) in the interval \([a,b]\) is given by:

\[
I_{a}^{b} f(x) = \frac{1}{\Gamma(1+\alpha)} \int_{a}^{b} f(t)(dt)^\alpha = \frac{1}{\Gamma(1+\alpha)} \lim_{N \to \infty} \sum_{j=0}^{N-1} f(t_j)(\Delta t_j)^\alpha.
\]

Definition 4. In fractal space, the Mittage Leffler function, sine function, and cosine function are defined as:

\[
E_{\alpha}(x^{\alpha}) = \sum_{k=0}^{\infty} \frac{x^{k\alpha}}{\Gamma(1+k\alpha)} , \quad 0 < \alpha \leq 1
\]

\[
\sin_{\alpha}(x^{\alpha}) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{(2k+1)\alpha}}{\Gamma[1+(2k+1)\alpha]} , \quad 0 < \alpha \leq 1
\]

\[
\cos_{\alpha}(x^{\alpha}) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k\alpha}}{\Gamma[1+2k\alpha]} , \quad 0 < \alpha \leq 1
\]

The following results are valid:

\[
\frac{d^{\alpha} x^{k\alpha}}{dx^{\alpha}} = \frac{\Gamma(1+k\alpha)}{\Gamma(1+(k-1)\alpha)} x^{(k-1)\alpha}
\]

\[
\frac{d^{\alpha} E_{\alpha}(x^{\alpha})}{dx^{\alpha}} = E_{\alpha}(x^{\alpha})
\]

\[
\frac{d^{\alpha} E_{\alpha}(kx^{\alpha})}{dx^{\alpha}} = kE_{\alpha}(kx^{\alpha})
\]

\[
\frac{1}{\Gamma(1+\alpha)} \int_{a}^{b} E_{\alpha}(x^{\alpha})(dx)^\alpha = E_{\alpha}(b^{\alpha}) - E_{\alpha}(a^{\alpha})
\]
\[ \frac{1}{\Gamma(1+\alpha)} \int_{a}^{b} \sin_{\alpha}(x^{\alpha})(dx)^{\alpha} = \cos_{\alpha}(a^{\alpha}) - \cos_{\alpha}(b^{\alpha}) \]  

(15)

### 3. Analysis of the Methods

A general nonlinear local fractional partial differential equation is considered as follows:

\[ L_{\alpha}u(x,t) + R_{\alpha}u(x,t) + N_{\alpha}u(x,t) = f(x,t) \]  

(16)

where \( L_{\alpha} = \frac{\partial^{\alpha}}{\partial t^{\alpha}} \) and \( R_{\alpha} \) are linear local fractional derivative operators of order \( \alpha \), \( N_{\alpha} \) denotes nonlinear local fractional operator, and \( f(x,t) \) is the source term.

#### 3.1. Analysis of the LFVIM

According to the rule of local fractional variational iteration method, the correction local fractional functional for (3.1) is constructed as:

\[ u_{n+1}(t) = u_{n}(t) + \sum_{i=0}^{n} \left( \frac{\lambda^{\alpha}}{\Gamma(1+\alpha)} \right)^{(i)} \left[ L_{\alpha}u_{n}(\xi) + R_{\alpha}u_{n}(\xi) + N_{\alpha}u_{n}(\xi) - f(\xi) \right] \]  

(17)

where \( \lambda^{\alpha} / \Gamma(1+\alpha) \) is a fractal Lagrange multiplier. Making the local fractional variation of (17) yields:

\[ \delta^{\alpha}u_{n+1}(t) = \delta^{\alpha}u_{n}(t) + \sum_{i=0}^{n} \left( \frac{\lambda^{\alpha}}{\Gamma(1+\alpha)} \right)^{(i)} \left[ L_{\alpha}u_{n}(\xi) + R_{\alpha}u_{n}(\xi) + N_{\alpha}u_{n}(\xi) - f(\xi) \right] \]  

(18)

The extremum condition of \( u_{n+1} \) is given by:

\[ \delta^{\alpha}u_{n+1}(t) = 0 \]  

(19)

This yields the stationary conditions as:

\[ \left. 1 + \frac{\lambda^{\alpha}}{\Gamma(1+\alpha)} \right|_{\xi=x} = 0, \quad \left. \left( \frac{\lambda^{\alpha}}{\Gamma(1+\alpha)} \right)^{(\alpha)} \right|_{\xi=x} = 0. \]  

(20)

This in turn gives Lagrange multiplier as:

\[ \frac{\lambda^{\alpha}}{\Gamma(1+\alpha)} = -1. \]  

(21)

By substituting this value of the Lagrange multiplier into Eq. (17), one gives the iteration formula as:

\[ u_{n+1}(t) = u_{n}(t) - \sum_{i=0}^{n} \left( \frac{\lambda^{\alpha}}{\Gamma(1+\alpha)} \right)^{(i)} \left[ L_{\alpha}u_{n}(\xi) + R_{\alpha}u_{n}(\xi) + N_{\alpha}u_{n}(\xi) - f(\xi) \right]. \]  

(22)

Finally, from Eq. (22), the solution of Eq. (16) is obtained as follows:

\[ u(x,t) = \lim_{n \to \infty} u_{n}(x,t). \]  

(23)

#### 3.2. Local Fractional SEM

Equation (16) can be written in the following form:

\[ \frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}} = P_{\alpha}u(x,t) \]  

(24)

where \( P_{\alpha} \) is a linear local fractional derivative operator with respect to \( x \) and \( u(x,t) \) is a local fractional continuous function. Suppose that:

\[ u(x,t) = \sum_{i=0}^{\infty} T_{i}(t)U_{j}(x) \]  

(25)

where \( T_{i}(t) \) and \( U_{j}(x) \) are local fractional continuous functions. Regarding Eq. (25), it is considered that:

\[ T_{i}(t) = \frac{t^{i\alpha}}{\Gamma(1+i\alpha)} \]  

(26)

which reduces to the following form:

\[ u(x,t) = \sum_{j=0}^{\infty} \frac{t^{j\alpha}}{\Gamma(1+j\alpha)} U_{j}(x) \]  

(27)
Making use of Eq. (27), we get

\[ \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \sum_{i=0}^{\infty} \frac{t^i}{\Gamma(1+i\alpha)} U_{i+1}(x) \]  \hspace{1cm} (28)

\[ P_t u(x,t) = P_t \left( \sum_{i=0}^{\infty} \frac{t^i}{\Gamma(1+i\alpha)} U_i(x) \right) = \sum_{i=0}^{\infty} \frac{t^i}{\Gamma(1+i\alpha)} (P_t U_i)(x) \]  \hspace{1cm} (29)

Regarding Eqs. (28) and (29), we have

\[ \sum_{i=0}^{\infty} \frac{1}{\Gamma(1+i\alpha)} t^{i\alpha} U_{i+1}(x) = \sum_{i=0}^{\infty} \frac{t^{i\alpha}}{\Gamma(1+i\alpha)} (P_t U_i)(x) \]  \hspace{1cm} (30)

Therefore, from Eq. (30), a recursion is obtained as

\[ U_{i+1}(x) = (P_t U_i)(x) \]  \hspace{1cm} (31)

By using the recursion formula of Eq. (31), the solution of Eq. (24) is obtained as:

\[ u(x,t) = \sum_{i=0}^{\infty} \frac{t^{i\alpha}}{\Gamma(1+i\alpha)} U_i(x) \]  \hspace{1cm} (32)

4. Application

In this section, to give a clear overview of the LFVIM and LFSEM for KdVE within LFDOs, the following example is presented.

Example: Let us consider the following KdV equation involving LFDOs:

\[ \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + \frac{\partial u(x,t)}{\partial x^\alpha} + \frac{\partial^3 u(x,t)}{\partial x^{3\alpha}} = 0, \quad 0 < \alpha \leq 1. \]  \hspace{1cm} (33)

subjected to the initial condition

\[ u(x,0) = E_\alpha(x^\alpha). \]  \hspace{1cm} (34)

1. Busing LFVIM:

From Eqs. (22) and (33), the local fractional iteration algorithm can be written as follows:

\[ u_{n+1}(t) = u_n(t) - \frac{1}{\Gamma(1+\alpha)} \int_0^t \left( \frac{\partial^\alpha u_n(\theta)}{\partial \theta^\alpha} + \frac{\partial u_n(\theta)}{\partial x^\alpha} + \frac{\partial^3 u_n(\theta)}{\partial x^{3\alpha}} \right) (d \theta)^\alpha \]  \hspace{1cm} (35)

Therefore, from Eqs. (34) and (35), the components are provided as follows:

\[ u_0(t) = u_0(t) - \frac{1}{\Gamma(1+\alpha)} \int_0^t \left( \frac{\partial^\alpha u_0(\theta)}{\partial \theta^\alpha} + \frac{\partial u_0(\theta)}{\partial x^\alpha} + \frac{\partial^3 u_0(\theta)}{\partial x^{3\alpha}} \right) (d \theta)^\alpha \]

\[ = E_\alpha(x^\alpha) - \frac{1}{\Gamma(1+\alpha)} \int_0^t 2E_\alpha(x^\alpha) (d \theta)^\alpha \]

\[ = E_\alpha(x^\alpha) - \frac{2t^\alpha}{\Gamma(1+\alpha)} E_\alpha(x^\alpha) \]

\[ = E_\alpha(x^\alpha) \left[ 1 - \frac{2t^\alpha}{\Gamma(1+\alpha)} \right], \]

\[ u_1(t) = u_1(t) - \frac{1}{\Gamma(1+\alpha)} \int_0^t \left( \frac{\partial^\alpha u_1(\theta)}{\partial \theta^\alpha} + \frac{\partial u_1(\theta)}{\partial x^\alpha} + \frac{\partial^3 u_1(\theta)}{\partial x^{3\alpha}} \right) (d \theta)^\alpha \]

\[ = E_\alpha(x^\alpha) - \frac{2t^\alpha}{\Gamma(1+\alpha)} E_\alpha(x^\alpha) + \frac{1}{\Gamma(1+\alpha)} \int_0^t \frac{4\theta^\alpha}{(1+\alpha)} E_\alpha(x^\alpha) (d \theta)^\alpha \]

\[ = E_\alpha(x^\alpha) - \frac{2t^\alpha}{\Gamma(1+\alpha)} E_\alpha(x^\alpha) + \frac{4t^{2\alpha}}{\Gamma(1+2\alpha)} E_\alpha(x^\alpha) \]

\[ = E_\alpha(x^\alpha) \left[ 1 - \frac{2t^\alpha}{\Gamma(1+\alpha)} + \frac{4t^{2\alpha}}{\Gamma(1+2\alpha)} \right]. \]
\[ u_0(t) = u_0(t) = \frac{1}{\Gamma(1+\alpha)} \int_0^t \left( \frac{\partial^n u_0(\theta)}{\partial t^n} + \frac{\partial^n u_0(\theta)}{\partial x^n} + \frac{\partial^n u_0(\theta)}{\partial x^{3n}} \right) (d \theta)^n \]

\[ = E_\alpha(x^n) \left[ 1 - \frac{22^n}{\Gamma(1+\alpha)} + \frac{42^{2n}}{\Gamma(1+2\alpha)} - \frac{82^{3n}}{\Gamma(1+3\alpha)} \right], \]

\[ u_1(t) = u_1(t) = \frac{1}{\Gamma(1+\alpha)} \int_0^t \left( \frac{\partial^n u_1(\theta)}{\partial t^n} + \frac{\partial^n u_1(\theta)}{\partial x^n} + \frac{\partial^n u_1(\theta)}{\partial x^{3n}} \right) (d \theta)^n \]

\[ = E_\alpha(x^n) \left[ 1 - \frac{22^n}{\Gamma(1+\alpha)} + \frac{42^{2n}}{\Gamma(1+2\alpha)} - \frac{82^{3n}}{\Gamma(1+3\alpha)} + \frac{162^{4n}}{\Gamma(1+4\alpha)} \right], \]

\[ u_n(t) = E_\alpha(x^n) \sum_{i=0}^n (-2)^i \frac{t^{i\alpha}}{\Gamma(1+r\alpha)}. \]

Consequently, it yields

\[ u(x,t) = \lim_{n \to \infty} u_n(x,t) = E_\alpha(x^n) \sum_{r=0}^\infty (-2)^i \frac{t^{i\alpha}}{\Gamma(1+r\alpha)} \]

(36)

II. By using LFSEM:

From Eq. (31), the following recursive formula is obtained:

\[ U_{i+1}(x) = \left( \frac{\partial^n u_i(x,t)}{\partial x^n} + \frac{\partial^n u_i(x,t)}{\partial x^{3n}} \right) \]

(37)

\[ U_0(x) = E_\alpha(x^n). \]

(38)

Regarding Eqs. (37) and (38), the following equations are obtained:

\[ U_1(x) = \left( \frac{\partial^n u_0(x,t)}{\partial x^n} + \frac{\partial^n u_0(x,t)}{\partial x^{3n}} \right) = -2E_\alpha(x^n), \]

\[ U_2(x) = \left( \frac{\partial^n u_1(x,t)}{\partial x^n} + \frac{\partial^n u_1(x,t)}{\partial x^{3n}} \right) = 4E_\alpha(x^n), \]

\[ U_3(x) = \left( \frac{\partial^n u_2(x,t)}{\partial x^n} + \frac{\partial^n u_2(x,t)}{\partial x^{3n}} \right) = -8E_\alpha(x^n), \]

\[ U_4(x) = \left( \frac{\partial^n u_3(x,t)}{\partial x^n} + \frac{\partial^n u_3(x,t)}{\partial x^{3n}} \right) = 16E_\alpha(x^n), \]

\[ U_5(x) = \left( \frac{\partial^n u_4(x,t)}{\partial x^n} + \frac{\partial^n u_4(x,t)}{\partial x^{3n}} \right) = -32E_\alpha(x^n), \]

\[ \vdots \]

Therefore, we have:

\[ u(x,t) = \sum_{i=0}^n \frac{t^{i\alpha}}{\Gamma(1+i\alpha)} U_i(x) \]

\[ = E_\alpha(x^n) \left[ 1 - \frac{22^n}{\Gamma(1+\alpha)} + \frac{42^{2n}}{\Gamma(1+2\alpha)} - \frac{82^{3n}}{\Gamma(1+3\alpha)} + \cdots \right], \]

(39)

\[ = E_\alpha(x^n) \sum_{r=0}^\infty (-2)^i \frac{t^{i\alpha}}{\Gamma(1+r\alpha)}, \]

\[ = E_\alpha(x^n) E_\alpha(-2t^n). \]

In Figs. 1, 2, 3 and 4, the approximate solutions of Eq. (33) along with initial condition of Eq. (34) are shown for different values of \( \alpha = 1/2, \ln(2)/\ln(3), \ln(3)/\ln(4), 1 \) respectively.
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Fig. 1. The plot of solution to local fractional Korteweg-de Vries equation with fractal dimension $\alpha = 1/2$

Fig. 2. The plot of solution to local fractional Korteweg-de Vries equation with fractal dimension $\alpha = \ln(2)/\ln(3)$

Fig. 3. The plot of solution to local fractional Korteweg-de Vries equation with fractal dimension $\alpha = \ln(3)/\ln(4)$

Fig. 4. The plot of solution to local fractional Korteweg-de Vries equation with fractal dimension $\alpha = 1$

5. Conclusions

The LFVIM and LFSEM are successfully applied to find the approximate analytical solutions for KdVE with LFDOs. The approximate analytical solutions for the Korteweg-de Vries equation on Cantor sets involving local fractional derivatives are successfully developed by recurrence relations resulting in convergent series solutions. The suggested methods are a potential tool for developing approximate solutions for local fractional partial differential equations with fractal initial value conditions, which of course, draws new problems beyond the scope of the present study. LFSEM provides us with less computations as compared with LFVIM.

Conflict of Interest

The authors declare no conflict of interest.

References


