Abstract. In this article, the electro-osmotic flow of Oldroyd-B fluid in a circular micro-channel with slip boundary condition is considered. The corresponding fractional system is represented by using a newly defined time-fractional Caputo-Fabrizio derivative without singular kernel. Closed form solutions for the velocity field are acquired by means of Laplace and finite Hankel transforms. Additionally, Stehfest’s algorithm is used for inverse Laplace transform. The solutions for fractional Maxwell, ordinary Maxwell and ordinary Newtonian fluids are obtained as limiting cases of the obtained solution. Finally, the influence of fractional and some important physical parameters on the fluid flow are spotlighted graphically.

Keywords: Electro-osmotic flow; Slip boundary condition; Oldroyd-B fluid; Time-fractional Caputo-Fabrizio derivative; Stehfest’s algorithm.

1. Introduction

Electro-osmosis phenomenon refers to bulk movement of an aqueous solution past a stationary solid surface due to an externally applied electric field [1]. It is used intensified in the context of micro and nano fluidics and in enormous scientific applications such as lab-on-a-chip technologies, soil analysis and chemical analysis [2]. Towards its potential applications, many theoretical [3-9], numerical [10, 11] and experimental [12, 13] studies on different electro-osmotic flow models are carried out by many authors for Newtonian and non-Newtonian fluids.

The subject of fractional calculus deals with the integrals and derivatives of any arbitrary real number. It has powerful applications in physics, engineering and many scientific areas [14-16]. The fractional derivative has many definitions, namely, Riemann-Liouville time-fractional derivative, Caputo time-fractional derivative [17], more recently, Caputo and Fabrizio [18]. The first two definitions, Riemann-Liouville time-fractional derivative, Caputo time-fractional derivative have a singular kernel while the new definition of the Caputo and Fabrizio time-fractional derivative is without singular kernel. Losada and Nieto [19] introduced the fractional integer corresponding to the fractional Caputo-Fabrizio derivative and studied its related fractional differential equations. In addition, Alsaedi et al. [20] found the solutions for a coupled system of time-fractional differential equations including continuous functions and the Caputo-Fabrizio fractional derivative. Moreover, Baleanu et al. [21] applied the variational homotopic perturbation and q-homotopic analysis methods to make a comparison between Caputo and Caputo-Fabrizio derivatives for the time-fractional advection equation. They indicated that rough answers for both derivatives are similar and the Caputo-Fabrizio derivative is faster than the Caputo derivative in terms of CPU speed up.
2. Formulation of the Problem and Governing Equation

The continuity equation for an incompressible viscoelastic fluid is

$$\nabla \cdot \mathbf{V} = 0$$

We consider the flow of an Oldroyd-B fluid in a straight circular cylinder. The velocity field is defined as

$$\mathbf{V} = v_r \hat{e}_r + v_\theta \hat{e}_\theta + v_z \hat{e}_z, \quad \text{with} \quad v_r = 0, v_\theta = 0, v_z = u(r,t)$$

The constitutive equation is

$$\left(1 + \lambda \frac{\partial}{\partial t}\right) \tau(r,t) = \mu \left(1 + \lambda_r \frac{\partial}{\partial r}\right) \frac{\partial u(r,t)}{\partial r}$$

where, $\tau = T_{zz}$ is the shear stress, $\lambda$ and $\lambda_r$ are the relaxation and retardation time, respectively, and $\mu$ is the constant viscosity. The start-up from rest of the electro-osmotic flow of a viscoelastic fluid in a circular microchannel of radius $R$. The dielectric constant of the fluid is $\varepsilon$. It is assumed that the channel wall is uniformly charged with a zeta potential $\psi_w$, and the liquid solution is a viscoelastic fluid whose behavior can be described by the fractional Oldroyd-B (Jeffrey’s) equation (3). When the external electric field $E_0\hat{e}_z$ is imposed along the axial direction, then the fluid in the micro-channel sets in motion due to the electro-osmosis.

According to the theory of electrostatics, the net change density $\rho_e$ is expressed by a potential distribution $\psi$, which is given by the following equation of Poisson type

$$\Delta^2 \psi = -\frac{\rho_e}{\varepsilon} \Leftrightarrow \Delta \psi = -\frac{\rho_e}{\varepsilon}$$

The corresponding boundary conditions of the zeta potential are

$$\psi(R,\theta) = \psi_w, \quad \frac{\partial \psi}{\partial r} \bigg|_{r=0} = 0.$$
where \( n_0 \) is the bulk number concentration, \( z_v \) is the valence of ions, \( e \) is the fundamental charge, \( k_B \) is the Boltzmann constant, \( T \) is the absolute temperature.

\[
\Delta f = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}
\]

if \( f = f(r,t) \) (in our problem all functions depend only of \( (r,t) \)). Implies that

\[
\Delta f = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial f}{\partial r} \right) = \frac{1}{r} \left( \frac{\partial f}{\partial r} + r \frac{\partial^2 f}{\partial r^2} \right) = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r} \frac{\partial f}{\partial r}
\]

By using (6) and (8) in (4) we have

\[
\frac{\partial^2 \psi}{\partial r^2} + \frac{\partial \psi}{r \partial r} - k^2 \psi = 0, \quad \text{or} \quad r^2 \frac{\partial^2 \psi}{\partial r^2} + \frac{\partial \psi}{r \partial r} - (k^2 r^2 + 0^2) \psi = 0
\]

Eq. (10) is the modified Bessel equation with the general solution

\[
\psi(r) = C_1 I_0(kr) + C_2 K_0(kr)
\]

By using

\[
\frac{d}{dr} I_0(u(r)) = I_1(u(r))u'(r), \quad \frac{d}{dr} K_0(u(r)) = -K_1(u(r))u'(r),
\]

\[
\frac{\partial \psi}{\partial r} = C_1 k I_1(kr) - C_2 k K_1(kr)
\]

Since, \( \lim_{r \to 0} K_1(kr) = \infty, I_1(0) = 0 \), \( C_2 \) must be zero.

\[
\psi(r) = C_1 I_0(kr)
\]

Finally, the solution of (3) and (4) is obtained as

\[
\psi(r) = \psi_w \frac{I_0(kR)}{I_0(kR)}
\]

The relevant equation of the linear momentum is

\[
\rho \frac{\partial u}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} (r \tau_w) + \rho_s E_o, \quad \text{or} \quad \rho \frac{\partial u}{\partial t} = \frac{1}{r} \tau_w + \frac{\partial \tau_w}{\partial r} + \rho_s E_o
\]

But, \( \tau_w = -(2z_v e^2 n_0) \psi / (k_B T) = -\varepsilon k \psi \), \( \psi = I_0(kr) / I_0(kR) \), then, Eq. (15) becomes

\[
\rho \frac{\partial u}{\partial t} = \frac{1}{r} \tau + \frac{\partial \tau}{\partial r} - \varepsilon k \psi E_o I_0(kr) / I_0(kR)
\]

The constitutive and linear momentum equations for ordinary Oldroyd-B fluid are:

The constitutive equation:

\[
\left( 1 + \lambda \frac{\partial}{\partial t} \right) \tau = \mu \left( 1 + \lambda \frac{\partial}{\partial r} \right) \frac{\partial u}{\partial r}
\]

The linear momentum equation:

\[
\rho \frac{\partial u}{\partial t} = \frac{1}{r} \frac{\partial (r \tau)}{\partial r} - \varepsilon k \psi E_o I_0(kr) / I_0(kR)
\]
the initial conditions are written as
\[ u(r,0) = 0, \quad \frac{\partial u(r,t)}{\partial t} \bigg|_{t=0} = 0, \]  
(19)

the boundary conditions are
\[ \frac{\partial u(r,t)}{\partial r} \bigg|_{r=0} = 0, \quad u(R,t) + d \frac{\partial u(r,t)}{\partial r} \bigg|_{r=R} = 0. \]  
(20)

By introducing the following non-dimensional variables
\[ \psi^* = \frac{\psi}{\psi_w}, \quad u^* = \frac{u}{u_s}, \quad r^* = \frac{r}{R}, \quad t^* = \frac{t}{T}, \quad d^* = \frac{d}{R^2}, \quad \lambda^* = \frac{\lambda}{R^2}, \quad \lambda^* = \frac{\lambda_s}{R^2}, \quad u_s = -\frac{\psi_w E_0}{\mu}, \quad \tau^* = \frac{R \tau}{\mu \alpha_s}, \]  
(21)

after dropping the star notation, we can obtain the basic equation for Oldroyd-B fluid in the following non-dimensional form
\[ \left( 1 + \lambda \frac{\partial}{\partial t} \right) \tau^* = \left( 1 + \lambda \frac{\partial}{\partial r} \right) \frac{\partial u(r,t)}{\partial r} + \frac{1}{r} \frac{\partial \tau^*}{\partial r} + \frac{K^2 I_o(Kr)}{I_o(K)} \frac{\partial u(r,t)}{\partial r} \bigg|_{r=0} = 0, \]  
(22)

2.1. The fractional model with Caputo-Fabrizio derivatives

In this case, the constitutive equation is
\[ (1 + \lambda D_t^\alpha)^\tau(r,t) = (1 + \lambda D_t^\beta) \frac{\partial u(r,t)}{\partial r}, \quad 0 \leq \alpha \leq 1, \quad 0 \leq \beta \leq 1. \]  
(23)

By applying the operator \((1 + \lambda D_t^\alpha)\) to Eq. (22), the obtained result can be written as
\[ (1 + \lambda D_t^\alpha) \frac{\partial u(r,t)}{\partial r} = \frac{1}{r} \frac{\partial \tau^*}{\partial r} + \frac{K^2 I_o(Kr)}{I_o(K)} \frac{\partial u(r,t)}{\partial r} \bigg|_{r=0} = 0, \]  
(24)

By using equation (16) and the property \(D_t^\alpha C = 0, C = \text{Constant}\), the result reduce to the following form,
\[ (1 + \lambda D_t^\alpha) \frac{\partial u(r,t)}{\partial r} = (1 + \lambda D_t^\beta) \frac{1}{r} \frac{\partial \tau^*}{\partial r} + \frac{K^2 I_o(Kr)}{I_o(K)} \frac{\partial u(r,t)}{\partial r} \bigg|_{r=1} = 0. \]  
(25)

Then, Eq. (25) becomes
\[ \frac{\partial u(r,t)}{\partial t} + \lambda D_t^\alpha \frac{\partial u(r,t)}{\partial t} = \frac{1}{r} \frac{\partial \tau^*}{\partial r} + \lambda \frac{1}{r} \frac{\partial \tau^*}{\partial r} \left( \frac{1}{r} D_t^\beta \frac{\partial u(r,t)}{\partial r} + \frac{K^2 I_o(Kr)}{I_o(K)} \frac{\partial u(r,t)}{\partial r} \right) \bigg|_{r=0} = 0, \]  
(26)

\[ \frac{\partial u(r,t)}{\partial r} \bigg|_{r=0} = 0, \quad u(1,t) + d \frac{\partial u(r,t)}{\partial r} \bigg|_{r=1} = 0. \]  
(27)

\[ D_t^\alpha f(r,t) = \int_0^t e^{-(t-s)/\alpha} \frac{\partial f(r,t)}{\partial \tau} d\tau = \left( e^{-(t-s)/\alpha} \right) \left( \frac{e^{-(t-s)/\alpha}}{1-\alpha} \right) \frac{\partial f(r,t)}{\partial t} \]  
(28)

\[ L \left[ D_t^\alpha f(r,t) \right] = L \left[ e^{-(t-s)/\alpha} \right] L \left[ \frac{\partial f(r,t)}{\partial t} \right] = \frac{1}{1-\alpha} \frac{1}{1-\alpha} \left[ sL \left[ f(r,t) \right] - f(r,0) \right] = \frac{sL \left[ f(r,t) \right] - f(r,0)}{(1-\alpha)s + \alpha} \]  
(29)

Then, Eq. (25) becomes
\[ \frac{\partial u(r,t)}{\partial t} + \lambda D_t^\alpha \frac{\partial u(r,t)}{\partial t} = \frac{1}{r} \frac{\partial \tau^*}{\partial r} + \lambda \frac{1}{r} \frac{\partial \tau^*}{\partial r} \left( \frac{1}{r} D_t^\beta \frac{\partial u(r,t)}{\partial r} + \frac{K^2 I_o(Kr)}{I_o(K)} \frac{\partial u(r,t)}{\partial r} \right) \bigg|_{r=0} = 0, \]  
(30a)

\[ L \left[ D_t^\alpha u(r,t) \right] = \frac{sL \left[ u(r,t) \right] - u(r,0)}{(1-\alpha)s + \alpha} = \frac{s \tilde{u}(r,t)}{(1-\alpha)s + \alpha}. \]  
(30b)
where \( \mathcal{L}(r,s) = \int_0^\infty u(r,t) e^{-s t} dt \) is the Laplace transform of the function \( u(r,t) \).

3. Solution of the Problem

By applying the Laplace transform to Eq. (18), the solution in transform domain is

\[
(1 - \alpha + \lambda)s^2 + as \mathcal{L}(r,s) = \left(1 - \beta + \lambda s + \beta r \right) \partial \left( \frac{\partial \mathcal{L}(r,s)}{\partial r} \right) + \frac{K^2 I_0(K)}{I_0(K)} \cdot \mathcal{L}(r,s),
\]

or

\[
\mathcal{L}(r,s) = \left[ \frac{1 - \beta + \lambda}{(1 - \beta + \lambda)s + \beta} \right] \left(1 - \alpha + \lambda \right) s^2 + a s \mathcal{L}(r,s) \left( \frac{1}{1 - \alpha + \lambda} \right) + \left[ \frac{(1 - \beta + \lambda)s + \alpha}{(1 - \alpha + \lambda)s^2 + a s} \right] \frac{K^2 I_0(K)}{I_0(K)} \cdot \mathcal{L}(r,s).
\]

By applying Hankel transform and rearrange the following results is obtained

\[
\mathcal{U}_n(r_n, s) = \frac{J_0(r_n)}{K_0^2 + K n^2},
\]

where

\[
\mathcal{U}_n(s) = \left[ \frac{(1 - \alpha)s + \alpha}{s \left[ (1 - \alpha + \lambda)s + \alpha \right]} \right] \left( a_0 s^2 + b_0 s + c_0 \right) \gamma_0 + \left( a_n = (1 - \beta)(1 - \alpha + \lambda), \right.

b_n = a(n - \beta) + \beta(1 - \alpha + \lambda), \quad c_n = a \beta, \quad a_\infty = a(n - \beta) + \beta(n - \alpha)(1 - \alpha + \lambda),

a_{2n} = a(n - \beta) + \beta(n - \alpha)(1 - \alpha + \lambda), \quad a_{2n} = a(n - \beta) + \beta(n - \alpha)(1 - \alpha + \lambda), \quad a_{2n} = a(n - \beta) + \beta(n - \alpha)(1 - \alpha + \lambda),

\]

Rearrange Eq. (23), such that \( u(r,t) \) satisfies the initial and the boundary conditions. First consider the auxiliary function

\[
h(r) = \frac{I_0(K)}{I_0(K)} - \frac{dK}{I_0(K)} - 1 \quad \text{(35)}
\]

the Hankel transform

\[
h_n = \frac{J_0(r_n)}{I_0(K)} \frac{I_0(K)}{dK} \frac{I_0(K)}{I_0(K)} - 1 \quad \text{(36)}
\]

and we have

\[
\mathcal{U}_n(r_n, s) = \frac{\Gamma(3)}{s} \cdot h_n + \left( \mathcal{U}_n(s) \frac{J_0(r_n)}{r_n^2 + K^2} - \frac{\Gamma(3)}{s} \cdot h_n \right) \quad \text{(37a)}
\]

or

\[
\mathcal{U}_n(r_n, s) = \frac{\Gamma(3)}{s} \cdot h_n + A_n(s) \quad \text{(37b)}
\]

where \( A_n(s) = \mathcal{U}_n(s) \frac{J_0(r_n)}{r_n^2 + K^2} - \frac{\Gamma(3)}{s} \cdot h_n \) with inverse Laplace transform

\[
a_n(t) = L^{-1} \left[ \mathcal{U}_n(s) \right] = \sum_{j=1}^{\infty} \frac{2}{j} \sum_{j=1}^{\infty} \frac{\Gamma(2)}{j} = \frac{\ln(2)}{t} \sum_{j=1}^{\infty} \frac{\Gamma(2)}{j}.
\]
\[ d_j = (-1)^{j+p} \sum_{i=\lfloor \frac{j+1}{2} \rfloor}^{\min(j,p)} \frac{i^p (2n)!}{(p-i)! (i-1)! (j-i)! (2i-j)!} \]  

(38b)

where, \( \lfloor (j+1)/2 \rfloor \) is the integer part of \( (j+1)/2 \) and \( p \) is a positive integer number. By applying the inverse Laplace transform to Eq. (25), results that

\[ \bar{u}_h (r_e, t) = t^2 h_e + a_e (t) \]  

(39)

By applying the inverse Hankel transform to Eq. (27), results that

\[ u (r_e, t) = t^2 h (r) + 2 \sum_{n=1}^{\infty} \frac{r_e^2 J_0 (r_e)}{\left( r_e^2 + \frac{1}{d^2} \right)} a_n (t) \]  

(40)

3.1. Some observations

1. It is observed that \( a_e (0) = 0 \) and \( \frac{da_e (t)}{dt} \bigg|_{t=0} = 0 \), implies that \( u (r, 0) = 0 \) and \( \frac{du (r, t)}{dt} \bigg|_{t=0} = 0 \).

2. Using \( h (r) \) and \( \frac{dJ_0 (r_e)}{dr} \bigg|_{r=0} = 0 \), implies that \( \frac{\partial (r, t)}{\partial r} \bigg|_{r=0} = 0 \).

3. Using \( \left[ h (r) + d \frac{d h (r)}{dr} \right]_{r=1} = 0 \) and \( J_0 (r_e) + d \frac{d J_0 (r_e)}{dr} = J_0 (r_e) - d r J_1 (r_e) = 0 \), implies that \( u (r, t) + d \frac{\partial h (r, t)}{\partial r} \bigg|_{r=1} = 0 \).

So these observations imply that the obtained solution in Eq. (28) satisfies the initial and the boundary conditions.

3.2. Particular cases

3.2.1. Ordinary Oldroyd-B fluid

When taking limit \( \alpha, \beta \to 1 \) of Eqs. (38) and (40), the obtained result can be write as:

\[ u (r, t) = t^2 h (r) + 2 \sum_{n=1}^{\infty} \frac{r_e^2 J_0 (r_e)}{\left( r_e^2 + \frac{1}{d^2} \right)} a_n (t) \]  

(41)

with inverse Laplace transform we have

\[ a_n (t) = L^{-1} \left[ A_n (s) \right] \]  

\[ d_j = (-1)^{j+p} \sum_{i=\lfloor \frac{j+1}{2} \rfloor}^{\min(j,p)} \frac{i^p (2n)!}{(p-i)! (i-1)! (j-i)! (2i-j)!} \]  

(42)

where, \( \lfloor (j+1)/2 \rfloor \) is the integer part of \( (j+1)/2 \) and \( p \) is a positive integer number which gives the dimensionless velocity expression for the electro-osmotic flow of the ordinary Oldroyd-B fluid.

3.2.2. Fractional Maxwell fluid

By taking the limit \( \lambda_e \to 0 \) of Eqs. (38) and (40), the obtained result can be write as:

\[ u (r, t) = t^2 h (r) + 2 \sum_{n=1}^{\infty} \frac{r_e^2 J_0 (r_e)}{\left( r_e^2 + \frac{1}{d^2} \right)} a_n (t) \]  

(43)

where

\[ \bar{U}_e (s) = \frac{\left[ (1-\alpha)s + \alpha \right] a_0 s^2 + b_0 s + c_0}{s \left[ (1-\alpha+\lambda) s + \alpha \right] a_0 s^3 + a_0 s^2 + a_2 s + a_3} \]  

(44)

\[ a_0 = (1-\beta)(1-\alpha+\lambda), \]
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\[ b_0 = \alpha(1-\beta) + \beta(1-\alpha + \lambda), \quad c_0 = \alpha \beta, \]

\[ a_n = \alpha(1-\beta) + \beta(1-\alpha + \lambda)(1-\alpha + \lambda) r_n^2, \]

\[ a_{2n} = \alpha \beta + [\beta(1-\alpha) + \alpha(1-\beta + \lambda)] r_n^2, \quad a_{2n} = \alpha \beta r_n^2, \quad \gamma_0 = \frac{K^2}{dI_a(K)} + I_a(K). \]

\[ A_n(s) = \mathcal{O}_n(s) J_0(r_n) / (r_n^2 + K^2) - h_s \Gamma(3) / s^3 \]

with inverse Laplace transform we have

\[ a_n(t) = L^{-1} \{ A_n(s) \} = \frac{\ln(2)}{t} \sum_{j=1}^{\infty} \frac{d^j A_n(j \ln(2))}{d s^j}, \]

\[ d_j = (-1)^{j+p} \sum_{i=0}^{\min(j,p)} \frac{i^p (2n)!}{(p-i)! (i-1)! (j-i)! (2i-j)!} \]

where, \((j+1)/2\) is the integer part of \((j+1)/2\) and \(p\) is a positive integer number which gives the dimensionless velocity expression for the electro-osmotic flow of Maxwell fluid with fractional derivative.

### 3.2.3. Ordinary Maxwell fluid

By taking the limit \(\lambda \to 0\) and \(\alpha, \beta \to 1\) of Eqs. (38) and (40), the obtained result can be write as:

\[ u(r,t) = t^2 h(r) + 2 \sum_{n=1}^{\infty} \frac{r_n^2 J_0(r_n)}{r_n^2 + d^2} a_n(t) \]

where \(A_n(s) = \frac{J_0(r_n)}{r_n^2 + K^2 s^3} \left[ 2s^2 + r_n^2 \right], b_n = \frac{2 \left( r_n^2 - r_n^2 - K^2 \right) I_0(K) - dK^2 I_1(K)}{dr_n^2 I_0(K)}\), with inverse Laplace transform one can obtain

\[ a_n(t) = L^{-1} \{ A_n(s) \} = \frac{\ln(2)}{t} \sum_{j=1}^{\infty} \frac{d^j A_n(j \ln(2))}{d s^j}, \]

\[ d_j = (-1)^{j+p} \sum_{i=0}^{\min(j,p)} \frac{i^p (2n)!}{(p-i)! (i-1)! (j-i)! (2i-j)!} \]

where, \([(j+1)/2]\) is the integer part of \((j+1)/2\) and \(p\) is a positive integer number which gives the dimensionless velocity expression for the electro-osmotic flow of the ordinary Maxwell fluid.

### 3.2.4. Ordinary Newtonian fluid

By taking the limit \(\lambda, \lambda \to 0\) and \(\alpha, \beta \to 1\) of Eqs. (38) and (40), the obtained result can be write as

\[ u(r,t) = t^2 h(r) + 2 \sum_{n=1}^{\infty} \frac{r_n^2 J_0(r_n)}{r_n^2 + d^2} a_n(t) \]

where \(A_n(s) = \frac{J_0(r_n)}{r_n^2 + K^2 s^3} \left[ 2s^2 + r_n^2 \right], b_n = \frac{2 \left( r_n^2 - r_n^2 - K^2 \right) I_0(K) - dK^2 I_1(K)}{dr_n^2 I_0(K)}\), with inverse Laplace transform one can obtain

\[ a_n(t) = L^{-1} \{ A_n(s) \} = \frac{\ln(2)}{t} \sum_{j=1}^{\infty} \frac{d^j A_n(j \ln(2))}{d s^j}, \]

\[ d_j = (-1)^{j+p} \sum_{i=0}^{\min(j,p)} \frac{i^p (2n)!}{(p-i)! (i-1)! (j-i)! (2i-j)!} \]
where, \([\lfloor j+1/2 \rfloor] \) is the integer part of \((j+1)/2\) and \(p\) is a positive integer number which gives the dimensionless velocity expression for the electro-osmotic flow of the ordinary Newtonian fluid.

4. Numerical Results and Discussion

In this section we present the graphical analysis of the electro-osmotic flow of an Oldroyd-B fluid with a slip boundary condition in a circular micro-channel by using a newly defined time-fractional Caputo-Fabrizio derivative without singular kernel. Numerical results are given to demonstrate the effects of pertinent parameters such as \(\alpha, \beta, d, K, \lambda\) and \(\lambda_0\) on the fluid flow velocity.

All parameters, variables and functions are considered non-dimensional. In Fig. 1, we present the effect of the fractional parameter \(\alpha\) versus \(r\) on the velocity field for two different values of time \(t\). It is observed that for small value of time in Fig. 1a, near the boundary of the cylinder the velocity increases as the fractional parameter \(\alpha\) increases while after some critical values of \(r\) the velocity decreases as the fractional parameter \(\alpha\) increases. But for large value of time in Fig. 1b, the velocity increases as fractional parameter \(\alpha\) increases. The effect of the fractional parameter \(\beta\) versus \(r\) is presented in Fig. 2. This shows the opposite influence than Fig. 1.

The effects of fractional parameters \(\alpha\) and \(\beta\) versus \(t\) on velocity profile are presented in Fig. 3. From this figure one can

Fig. 1. Profiles of dimensionless velocity against \(r\) for \(\alpha\) variation at \(d=0.002, \lambda=0.5, \lambda_0=0.1, K=20, \beta=0.8\) and two values of time \(t\).

Fig. 2. Profiles of dimensionless velocity against \(r\) for \(\beta\) variation at \(d=0.002, \lambda=0.5, \lambda_0=0.1, K=20, \alpha=0.8\) and two values of time \(t\).

Fig. 3. Profiles of dimensionless velocity against \(t\) for \(\alpha\) variation at \(d=0.002, \lambda=0.5, \lambda_0=0.1, K=20, \beta=0.8\) and \(r=0.1\).

Fig. 4. Profiles of dimensionless velocity against \(r\) for \(\beta\) variation at \(d=0.002, \lambda=0.5, \lambda_0=0.1, K=20, \alpha=0.8\) and \(r=0.1\).

observe that, for these parameters the fluid behavior is changed at many time values.
In Figs. 5, 6, 7, we study the effects of slip, electro-kinetic width and relaxation time, respectively, versus $r$ on fluid flow velocity at two different values of time $t$. From these figures, it is observed that by increasing the values of slip parameter, the electrokinetic width parameter as well as the relaxation time the velocity increases. It is also important to note that by increasing the time $t$ at the boundary layer difference is increasing.

Fig. 5. Profiles of dimensionless velocity against $r$ for $d$ variation at $\lambda = 0.5$, $\lambda_r = 0.1$, $K = 20$, $\alpha = \beta = 0.6$ and two values of time $t$

Fig. 6. Profiles of dimensionless velocity against $r$ for $K$ variation at $\lambda = 0.5$, $\lambda_r = 0.1$, $d = 0.002$, $\alpha = \beta = 0.6$ and two values of time $t$

Fig. 7. Profiles of dimensionless velocity against $r$ for $\lambda_r$ variation at $d = 0.002$, $\lambda = 0.1$, $K = 20$, $\alpha = \beta = 0.6$ and two values of time $t$

Fig. 8. Profiles of dimensionless velocity against $r$ for $\lambda_r$ variation at $d = 0.002$, $\lambda = 0.5$, $K = 20$, $\alpha = \beta = 0.6$ and two values of time $t$

The effect of retardation time $\lambda_r$ on the velocity profile versus $r$ is presented in Fig. 8. It is observed that by increasing the value of retardation time $\lambda_r$ the velocity decreases and much influence is appeared for large values of time $t$. Fig. 8, shows an opposite influence than Figs. 5, 6 and 7.

Figs. 9 and 10 are plotted in order to study the influence of relaxation and retardation time, respectively, versus $t$ at two values of $r$. From these figures, we observe that initially the velocity has minimum value near the boundary while after some values of time (critical values) the velocity has a maximum value near the boundary.

Similar results are obtained from Figs. 11 and 12 which presented the velocity field when both parameters ($r, t$) are simultaneous variable and for two values of fractional parameters. The grid points for the plotting are ($r_i = 0.01i$, $t_j = 0.01j$, $i, j = 1, 2, \ldots, 100$. A comparison between our result and the result of Jiang et al. [29] is presented in Fig. 13.

Fig. 9. Profiles of dimensionless velocity against $t$ for $\lambda_r$ variation at $d = 0.002$, $\lambda = 0.1$, $K = 20$, $\alpha = \beta = 0.6$ and two values of time $r$

Fig. 10. Profiles of dimensionless velocity against $t$ for $\lambda_r$ variation at $d = 0.002$, $\lambda = 0.5$, $K = 20$, $\alpha = \beta = 0.6$ and two values of time $r$
5. Conclusions

The aim of this article is to study the electro-osmotic flow of an Oldroyd-B fluid with slip condition on the boundary in a circular micro-channel by using time-fractional Caputo-Fabrizio derivative without singular kernel. The Laplace and finite Hankel transforms are used to find solutions for the velocity field. In addition, Stehfest’s algorithm is used for inverse Laplace transform. The solutions for fractional Maxwell, ordinary Maxwell and ordinary Newtonian fluids are obtained as limiting cases from the obtained solution. Finally, the influences of the fractional parameter and some important physical parameters on the fluid flow are spotlighted graphically. The following points are observed:

- For small values of time, near the boundary of the cylinder the velocity increased by increasing the values of the fractional parameter $\alpha$ and after some critical values of $r$ the velocity decreased by increasing the values of the fractional parameter $\alpha$.
- For large values of time, the velocity increased by increasing the values of fractional parameter $\alpha$.
For small values of time, near the boundary of the cylinder the velocity decreased by increasing the values of the fractional parameter $\beta$ and after some critical values of $r$ the velocity increased by increasing the values of the fractional parameter $\beta$.

For large values of time, the velocity decreased by increasing the values of the fractional parameter $\beta$.

By slip, electrokinetic width and relaxation time the velocity increases.

By increasing the values of retardation time $\lambda$, the velocity decreased and much influence is appeared for large values of time $t$.

Conflict of Interest

The author(s) declared no potential conflicts of interest with respect to the research, authorship and publication of this article.

Funding

The author Haitao Qi is grateful to National Natural Science Foundation of China (Grant No. 11672163), Natural Science Foundation of Shandong Province, China (Grant Nos. ZR2015AM011) for supporting this work. The author Shaowei Wang is grateful to National Natural Science Foundation of China (Grant No. 11672164) for supporting this work.

Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u$</td>
<td>Velocity of the fluid,</td>
</tr>
<tr>
<td>$E_0$</td>
<td>Electric field strength,</td>
</tr>
<tr>
<td>$d$</td>
<td>Slip length,</td>
</tr>
<tr>
<td>$e$</td>
<td>Electron charge,</td>
</tr>
<tr>
<td>$K$</td>
<td>Dimensionless electrokinetic width,</td>
</tr>
<tr>
<td>$I_0$</td>
<td>Modified Bessel function of the first kind,</td>
</tr>
<tr>
<td>$\tau$</td>
<td>Shear stress,</td>
</tr>
<tr>
<td>$\mu$</td>
<td>Dynamic viscosity,</td>
</tr>
<tr>
<td>$\rho_e$</td>
<td>Electric charge density,</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>Retardation time,</td>
</tr>
<tr>
<td>$\varepsilon$</td>
<td>Dielectric constant,</td>
</tr>
<tr>
<td>$\psi_{\text{Zeta}}$</td>
<td>Zeta potential of the channel wall,</td>
</tr>
<tr>
<td>$k$</td>
<td>Debye-Huckel parameter,</td>
</tr>
<tr>
<td>$R$</td>
<td>Radius of the channel,</td>
</tr>
<tr>
<td>$n_0$</td>
<td>Bulk ionic number concentration,</td>
</tr>
<tr>
<td>$z_0$</td>
<td>Valence of ions,</td>
</tr>
<tr>
<td>$T$</td>
<td>Absolute temperature,</td>
</tr>
<tr>
<td>$J_0$</td>
<td>Bessel function of the first kind,</td>
</tr>
<tr>
<td>$r_+ \gamma$</td>
<td>Positive roots of Bessel function of the first kind,</td>
</tr>
<tr>
<td>$\rho$</td>
<td>Density of the electrolyte solution,</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>Relaxation time,</td>
</tr>
<tr>
<td>$\alpha, \beta$</td>
<td>Fractional order derivative parameters,</td>
</tr>
<tr>
<td>$\psi$</td>
<td>Potential distribution,</td>
</tr>
<tr>
<td>$k_b$</td>
<td>Boltzmann constant,</td>
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</table>

References


References


