

The Development and Application of the RCW Method for the Solution of the Blasius Problem

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Abstract. In this research, a numerical algorithm is employed to investigate the classical Blasius equation which is the governing equation of boundary layer problem. The base of this algorithm is on the development of RCW (Rahmanzadeh-Cai-White) method. In fact, in the current work, an attempt is made to solve the Blasius equation by using the sum of Taylor and Fourier series. While, in the most common numerical methods, the answer is considered only as a Taylor series. It should be noted that in these algorithms which use Taylor expansion, the values of the truncation error are considerable. However, adding the Fourier series to the Taylor series leads to reduce the amount of the truncation error. Nevertheless, the results of this research show the RCW method has the ability to achieve the accuracy of analytical solution. Moreover, it is well illustrated that the accuracy of RCW method is higher than the Runge-Kutta one.

Keywords: Boundary layer, Blasius equation, Initial value problems, RCW method.

1. Introduction

The viscous fluid flow over a semi-infinite flat plate is one of the important and benchmark problems in fluid mechanics [1-5]. Using the boundary layer theory, the governing partial differential equations of this problem (Navier-Stokes equations) can be become simpler [6-10]. In many cases, these simplified partial differential equations can be transformed into a nonlinear ordinary differential equation, using a similarity parameter. In fact, using this strategy, the hydrodynamic behaviors of fluid flow over a flat plate can be determined by solving an ordinary differential equation [11-14]. For example, the governing equation for the steady laminar flow over a semi-infinite flat plate can be described as follows [15]:

$$y^{(3)}(x) + \frac{1}{2} y(x) y''(x) = 0, \quad x \geq 0 \quad (1)$$
$$y(0) = 0, \quad y'(0) = 0, \quad y'(\infty) = 1$$

The above equation is well-known as the Blasius problem which is one of the basic and favorite problems in fluid mechanics. For the first time, the following analytical solution was presented for this problem by Blasius [16]:

$$y(x) = \sum_{k=1}^N \left(-\frac{1}{2} \right)^k \frac{A_k \times 0.332057336^{k+1}}{(3k+2)!} x^{3k+2} \quad (2)$$



In this equation, $A_0 = A_1 = 1$ and

$$A_k = \sum_{r=0}^{k-1} \binom{3k-1}{3r} A_r A_{k-r-1}, \quad k \geq 2 \quad (3)$$

However, it is noteworthy that the convergence of the analytical solution is difficult, especially for $x \geq 5$. Due to this difficulty, this problem has been extensively analyzed by many researchers. These researches include many numerical and analytical solutions.

Most of the numerical solutions of this problem are based on the algorithms proposed for solving the ordinary differential equations. In most of these approaches, Taylor expansion (a polynomial) is used as an answer [17-19]. Sometimes, it is somehow possible to guess the overall form of the answer from physics of the problem or boundary conditions. But it should be noted that in most numerical approaches which use Taylor expansion, this guess cannot be applied [20-22].

One of the most attractive and effective methods for numerical solution of ordinary differential equations is RCW method. This method was presented for the first time in 2013 by Rahmanzadeh, Cai and White [23]. To understand how to solve differential equations with this method, consider the following differential equation:

$$y^n = f(x, y, y', \dots, y^{n-1}) \quad (4)$$

To estimate the answer of this ordinary differential equation, a polynomial of degree n is used as follows:

$$y(x) = a_n(x-x_0)^n + a_{n-1}(x-x_0)^{n-1} + \dots + a_m(x-x_0)^m + \dots + a_0 \quad (5)$$

The polynomial coefficients in Eq. (5) are divided into two groups: the free coefficients and the fixed coefficients. The numbers of $m+1$ coefficients are considered as fixed coefficients and can be obtained using the boundary conditions (a_0, a_1, \dots, a_m) .

To choose the free coefficients $(a_n, a_{n-1}, \dots, a_{m+1})$, the error function is first introduced as:

$$e(a_n, a_{n-1}, \dots, a_{m+1}) = \int_{x_0}^{x_0+h} R(x, a_n, a_{n-1}, \dots, a_{m+1})^2 dx \quad (6)$$

Then, an attempt is made to minimize this function. In the above equation, the function of R is called the residual function and is defined and calculated as the following:

$$R(x, a_n, a_{n-1}, \dots, a_{m+1}) = y^n(x) - f(x, y'(x), \dots, y^{n-1}(x)) \quad (7)$$

In fact, the residual function (R) is difference between exact solution and approximation solution. Therefore, square of this function (R^2) must be integrated in the Eq. (5). It should be mentioned that the error function can be minimized by using the different optimization methods, such as the Nelder-Mead simplex and genetic algorithms [24].

According to what was said above, an attempt is made in this paper to solve the Blasius problem by using the RCW method. To reach this goal, it is first necessary to develop this method. In fact, the proposed and used answer in this paper for solving the Blasius problem is as follows:

$$y(x) = a_n(x-x_0)^n + a_{n-1}(x-x_0)^{n-1} + \dots + a_0 + f(x) \quad (8a)$$

where

$$f(x) = A \sin(x-x_0) + B \cos(x-x_0) \quad (8b)$$

As it is seen from the above equations, the answer of Blasius problem is considered as the sum of Taylor and Fourier series. In fact, the main intention of this research is to show that the RCW method has the ability to use other functions unless polynomial function with the help of error function. The full details of this method are presented in the next sections.

2. Theory

In this section, an attempt is made to develop the RCW method for solving the Blasius problem. Consider the presented Blasius equation with their boundary conditions (Eq. (1)). In this equation, the values of $y''(0)$ is not clear. Since the RCW method is presented to solve the initial value problems; therefore, it is first necessary to determine the $y''(0)$ value. To find the $y''(0)$ value, it is assumed that at a certain point ($x=8$), the following conditions are consistent, simultaneously:

$$y' \rightarrow 1 \quad \text{and} \quad y'' \rightarrow 0 \quad (9)$$

According to the above explanations, and by applying the Shooting method [25], the boundary value problems in Eq. (1) can be converted to an initial value problem as follows:

$$y^{(3)} + 0.5y''y = 0 \quad I. V.: \quad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = t_k \tag{10}$$

$$z^{(3)} + 0.5y''z + 0.5yz'' = 0 \quad I. V.: \quad z(0) = 0, \quad z'(0) = 0, \quad z''(0) = 1 \tag{11}$$

It should be mention that the t_k is guessed value of $y''(0)$ which can be modified by using the Newton's method as follows:

$$t_k = t_{k-1} - \frac{y'(8)_{t_{k-1}} - 1}{z'(8)_{t_{k-1}}} \tag{12}$$

In fact, Eqs. (10) to (12) are coupled together and must be solved simultaneously. To solve the Eqs. (10) and (11) by using RCW method [26], the following answers are considered:

$$y(x) = A \sin(x - x_0) + B \cos(x - x_0) + \sum_{n=0}^4 a_{ny} (x - x_0)^n \tag{13}$$

$$z(x) = a_{4z} (x - x_0)^4 + a_{3z} (x - x_0)^3 + a_{2z} (x - x_0)^2 + a_{1z} (x - x_0) + a_{0z} \tag{14}$$

The $a_{3y}, a_{2y}, a_{1y}, a_{0y}, a_{3z}, a_{2z}, a_{1z}$ and a_{0z} coefficients are the fixed coefficients which are determined by using the initial conditions. In order to obtain other coefficients (the free coefficients), it is necessary that the functions of $y(x)$ and $z(x)$ be substituted into Eqs. (10) and (11). Then, by using the residual function according to Eq. (7), the following relationships can be presented:

$$R_1(x, A, B, a_{4y}, a_{4z}) = y^{(3)}(x) + 0.5y''(x)y(x) \tag{15}$$

$$R_2(x, A, B, a_{4y}, a_{4z}) = z^{(3)}(x) + 0.5y''(x)z(x) + 0.5y(x)z''(x) \tag{16}$$

These residual functions are used to determine an error function based on the Eq. (6), as follows:

$$e(A, B, a_{4y}, a_{4z}) = \int_{x_0}^{x_0+h} (R_1^2 + R_2^2) dx \tag{17}$$

The next step to obtain the A, B, a_{4y} and a_{4z} coefficients is the minimizing of error function. To minimize the error function, it is needed to derive from this function on the A, B, a_{4y} and a_{4z} variables as follows:

$$de_1 = \int_{x_0}^{x_0+h} (R_1 \frac{\partial R_1}{\partial A} + R_2 \frac{\partial R_2}{\partial A}) dx \tag{18}$$

$$de_2 = \int_{x_0}^{x_0+h} (R_1 \frac{\partial R_1}{\partial B} + R_2 \frac{\partial R_2}{\partial B}) dx \tag{19}$$

$$de_3 = \int_{x_0}^{x_0+h} (R_1 \frac{\partial R_1}{\partial a_{4y}} + R_2 \frac{\partial R_2}{\partial a_{4y}}) dx \tag{20}$$

$$de_4 = \int_{x_0}^{x_0+h} (R_1 \frac{\partial R_1}{\partial a_{4z}} + R_2 \frac{\partial R_2}{\partial a_{4z}}) dx \tag{21}$$

Eqs. (18) to (21) include a nonlinear algebraic system of equations that can be solved by Newton method by using the initial approximations for A, B, a_{4y} and a_{4z} as follows:

$$\begin{bmatrix} A \\ B \\ a_{4y} \\ a_{4z} \end{bmatrix} = -J^{-1} \cdot \begin{bmatrix} de_1 \\ de_2 \\ de_3 \\ de_4 \end{bmatrix} + \begin{bmatrix} A \\ B \\ a_{4y} \\ a_{4z} \end{bmatrix} \tag{22}$$

where J is:



$$J = \begin{bmatrix} \frac{\partial de_1}{\partial A} & \frac{\partial de_1}{\partial B} & \frac{\partial de_1}{\partial a_{4y}} & \frac{\partial de_1}{\partial a_{4z}} \\ \frac{\partial de_2}{\partial A} & \frac{\partial de_2}{\partial B} & \frac{\partial de_2}{\partial a_{4y}} & \frac{\partial de_2}{\partial a_{4z}} \\ \frac{\partial de_3}{\partial A} & \frac{\partial de_3}{\partial B} & \frac{\partial de_3}{\partial a_{4y}} & \frac{\partial de_3}{\partial a_{4z}} \\ \frac{\partial de_4}{\partial A} & \frac{\partial de_4}{\partial B} & \frac{\partial de_4}{\partial a_{4y}} & \frac{\partial de_4}{\partial a_{4z}} \end{bmatrix} \tag{23}$$

3. Results

Based on what is said in the previous section, the following algorithm can be used to solve the Blasius equation:

- 1- An initial value for t_k is guessed, such that $y''(0) = t_k$
- 2- The fixed coefficients of $y(x)$ are calculated by using the boundary conditions as follows:

$$y(x_0) = A \sin(0) + B \cos(0) + a_{4y}(0)^4 + \dots + a_{0y} \rightarrow a_{0y} = y(x_0) - B \tag{24}$$

$$y'(x_0) = A \cos(0) - B \sin(0) + 4a_{4y}(0)^3 + \dots + a_{1y} \rightarrow a_{1y} = y'(x_0) - A \tag{25}$$

$$y''(x_0) = -A \sin(0) - B \cos(0) + 12a_{4y}(0)^2 + \dots + 2a_{2y} \rightarrow a_{2y} = \frac{y''(x_0) + B}{2} \tag{26}$$

$$\begin{cases} y^3(x_0) = -0.5y''(x_0) \cdot y(x_0) \\ y^3(x_0) = -A \cos(0) + B \sin(0) + 24a_{4y}(0) + 6a_{3y} \end{cases} \rightarrow a_{3y} = \frac{y^{(3)}(x_0) + A}{6} \tag{27}$$

- 3- The fixed coefficients of $y(x)$ are replaced in Eq. (13), such that:

$$y(x) = A \sin(x - x_0) + B \cos(x - x_0) + a_{4y}(x - x_0)^4 + \frac{y^{(3)}(x_0) + A}{6}(x - x_0)^3 + \frac{y''(x_0) + B}{2}(x - x_0)^2 + (y'(x_0) - A)(x - x_0) + y(x_0) - B \tag{28}$$

- 4- The residual function and the error function derivatives for $y(x)$ function are formed as:

$$R_1(A, B, a_{4y}, x) = y^{(3)}(x) + 0.5y''(x) \cdot y(x) \tag{29}$$

$$de_1 = R_1 \cdot \frac{\partial R_1}{\partial A} \tag{30}$$

$$de_2 = R_1 \cdot \frac{\partial R_1}{\partial B} \tag{31}$$

$$de_3 = R_1 \cdot \frac{\partial R_1}{\partial a_{4y}} \tag{32}$$

- 5- The initial values of A, B, and a_{4y} in $y(x)$ are guessed. Then, the new values of these variables are obtained, until convergence is reached:

$$\begin{bmatrix} A_i \\ B_i \\ a_{4yi} \end{bmatrix} = \begin{bmatrix} A_{i-1} \\ B_{i-1} \\ a_{4yi-1} \end{bmatrix} - \begin{bmatrix} \int_{x_0}^{x_0+h} \frac{\partial e_1}{\partial A} dx & \dots & \int_{x_0}^{x_0+h} \frac{\partial e_1}{\partial a_{4y}} dx \\ \cdot & \cdot & \cdot \\ \int_{x_0}^{x_0+h} \frac{\partial e_3}{\partial A} dx & \dots & \int_{x_0}^{x_0+h} \frac{\partial e_3}{\partial a_{4y}} dx \end{bmatrix} \begin{bmatrix} \int_{x_0}^{x_0+h} \partial e_1 dx \\ \cdot \\ \int_{x_0}^{x_0+h} \partial e_3 dx \end{bmatrix} \tag{33}$$

The above integrals can be calculated numerically. After calculating them, the free coefficients of $y(x)$ function (A , B and a_{4y}) are obtained during a step length. For the next step length, calculations can be done from the third step.

6- Now, it is necessary to evaluate the guessed amount of t_k . For this purpose, Eq. (11) is solved by using the described method in above. After calculating and replacing the fixed coefficients of $z(x)$ in Eq. (14), the function of $z(x)$ is obtained as follows:

$$z(x) = a_{4z}(x - x_0) + \frac{z^{(3)}(x_0)}{6}(x - x_0)^3 + \frac{z''(x_0)}{2}(x - x_0)^2 + z'(x_0)(x - x_0) + z(x_0) \tag{34}$$

7- The residual function and the error function derivative for $z(x)$ function are formed as follows:

$$R_2 = z^{(3)}(x) + 0.5y''(x) \cdot z(x) + 0.5y(x) \cdot z''(x) \tag{35}$$

$$de_4 = R_2 \cdot \frac{\partial R_2}{\partial a_{4z}} \tag{36}$$

8- An initial value for a_{4y} is guessed and then the new value of it is obtained as follows:

$$a_{4zi} = a_{4zi-1} - \left[\int_{x_0}^{x_0+h} \frac{\partial e_z}{\partial a_{4z}} dx \right]^{-1} \cdot \left[\int_{x_0}^{x_0+h} e_z dx \right] \tag{37}$$

9- The new value of t_k can be obtained from the Eq. (12).

10- Steps 2–9 are repeated until the convergence of the t_k is obtained

In Table 1, the obtained values of A , B , and a_{4y} in different length steps are shown. It should be mentioned that in the presented calculations, the step length is considered to be $h = 1.25$. Also, the value of $y''(x_0)$ is calculated to 0.332057336.

Table 1. The obtained values of A , B and a_{4y} in different length steps

x	A	B	a_{4y}
0-1.25	-0.05370927	0.01248	-0.0005228
1.25-2.5	-0.02922	-0.08582	0.001124
2.5-3.75	0.079824	-0.00682	-0.0009045
3.75-5.0	0.017207	0.050786	3.78×10^{-5}

To verify the accuracy of computations of RCW method in solving the Blasius equation, the values of $y(x)$ are compared with the analytical data of Blasius [16] and the approximate analytical data of Ref. [27] in Table 2.

Table 2. Comparison of $y(x)$ values with the analytical data of Blasius [16] and the approximate analytical data of Ref. [27]

x	RCW method	Analytical solution of Blasius [16]	Approximate analytical data of Ref. [27]
0.4	0.0266	0.0266	0.0266
0.8	0.1061	0.1061	0.1061
1.2	0.2379	0.2379	0.2379
1.6	0.4203	0.4203	0.4203
2	0.6500	0.6500	0.6500
2.4	0.9223	0.9223	0.9223
2.8	1.2310	1.2310	1.2311
3.6	1.9295	1.9295	1.9297
4.4	2.6924	2.6924	2.6922
5.0	3.2833	3.2833	3.2827

Table 2 shows that the results obtained by RCW method are in excellent agreement with analytical data even for the high values of h . In order to have a comparison between the results of RCW method and another numerical solution, the Blasius equation is solved by Runge-Kutta (RK) method with the same step length ($h = 1.25$).

In order to show the difference between the results of RCW and RK methods, the amount of Error in computing the values of $y(x)$ are presented in figure 1. It should be mentioned that the Error is defined as:

$$Error = \log |y_{cal} - y_{analytical}| \tag{38}$$

where, y_{cal} denotes the numerical solutions of $y(x)$ and $y_{analytical}$ denotes the analytical solution of $y(x)$.

As it is shown in Fig. 1, the amounts of Error in RCW method are much lower compared to the amounts of Error in RK method.



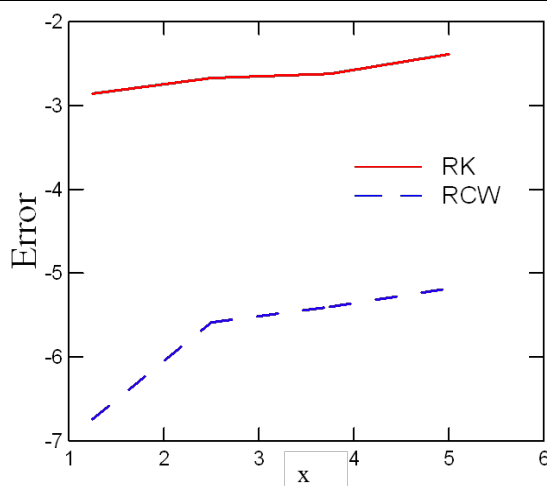


Fig. 1. Comparison of the results of RCW and RK methods in solving the Blasius equation

4. Conclusions

The main goal of this paper was to develop the RCW method for solving the Blasius problem. In the modified RCW method, the answer of this problem was considered as the sum of the Fourier series and a polynomial of degree 4. In this method, the coefficients of answer were divided to free and fixed groups. The fixed coefficients were obtained from boundary conditions, while the free coefficients were calculated by minimizing the error function. The results of this work showed that there is an excellent agreement between the RCW approach and analytical solution.

Conflict of Interest

The authors declared no potential conflicts of interest with respect to the research, authorship and publication of this article.

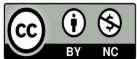
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