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Research Paper

A Modified Energy Balance Method to Obtain Higher-order Approximations to the Oscillators with Cubic and Harmonic Restoring Force

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Abstract. This article analyzes a strongly nonlinear oscillator with cubic and harmonic restoring force and proposes an efficient analytical technique based on the modified energy balance method (MEBM). The proposed method incorporates higher-order approximations. After applying the proposed MEBM, a set of complicated higher-order nonlinear algebraic equations are obtained. Higher-order nonlinear algebraic equations are cumbersome to investigate especially in the case of a large initial oscillation amplitude. This limitation is overcome in the proposed method by using the iterative procedure based on the homotopy perturbation method. The approximated results agree well with the numerically obtained exact solutions. These third-order approximate solutions are found to be almost the same as exact solutions, which cannot be found using the existing energy balance method. Highly accurate result and simple solution procedure are advantages of this proposed method, which could be applied to other nonlinear oscillatory problems arising in nonlinear science and engineering.

Keywords: Analytical technique; Strongly nonlinear oscillator; Nonlinear algebraic equations; Energy balance method; Iterative procedure; Homotopy perturbation method.

1. Introduction

The nonlinear behavioural phenomenon is commonly found in various nonlinear systems including the area of physical sciences, applied mathematics, mechanical structures and engineering [1, 2]. The appearance of nonlinear phenomena is often an indication of an emerging hazardous state, which needs to be avoided or controlled. This has led to intensive research on the behaviour of nonlinear systems, using various approaches over many decades. Most nonlinear phenomena can be described by a set of nonlinear differential equations (NDEs). The methodology for solving linear differential equations is comparatively simple and well established. In contrast, the methodology for solving NDEs is very limited and is mostly based on linearization techniques. More specifically in some cases, it is possible to approximate a nonlinear equation by a closely related linear equation, and such approximation may provide the desired results. However, linearization of NDEs may not always be feasible in which case the original nonlinear equation must be solved directly. In this situation, NDEs have been the subject of comprehensive studies in various branches of nonlinear science and engineering.

The numerical methods, e.g., the Runge-Kutta fourth-order method, are frequently used to obtain approximate solutions for nonlinear systems. However, for stiff differential equations, chaotic differential equations and hyperchaotic differential equations, numerical methods do not always yield accurate results. In the applications of numerical analysis, it is a big challenge still now. As an alternative, some researchers have introduced analytical approximation techniques. Traditional perturbation methods [3, 4, 5, 6, 7, 8, 9, 10, 11] are the most widely used analytical methods for solving nonlinear equations, which is the most versatile tool available for nonlinear analysis of engineering problems. However, traditional perturbation methods have several limitations depending on the degree of nonlinearity of the equations, especially for strongly nonlinear differential equations [12].

In the recent past, some other approximation techniques have been investigated. These include the Akbari-Ganji's [13], the cubication [14], the pseudospectral [15], the frequency-amplitude formulation [16], the rational energy balance [17], the rational variational [18] and the closed-form numerical [19] methods and the iteration method [20, 21, 22]. In addition, the global error minimization [23], the harmonic balance [24, 25, 26, 27], the rational harmonic balance [28], the global and modified multi-level residue harmonic balance [29, 30], the parameter-expansion [31, 32, 33, 34] methods and the VIM-Pade technique [35] have been used to derive periodic solutions to strongly nonlinear oscillatory problems. In fact, to the best of our knowledge, of most of these methods, only the first-order approximation has ever been considered. This does not result in sufficient accuracy.

The energy balance method (EBM) [36, 37, 38, 39, 40] offers a general technique for approximating periodic solutions to strongly nonlinear systems, where the nonlinear terms are not small. He [36] was the first to use the energy balance method on the strongly nonlinear differential equations. This method, a variational principle for the nonlinear oscillation is established, then a Hamiltonian is constructed, from which the angular frequency can be readily obtained by collocation method. But most of the published energy balance methods have been considered only first-order approximations, which leads to insufficient accuracy. Applying the proposed modified energy balance method (MEBM), a set of complicated nonlinear algebraic equations are obtained. It is extremely difficult to solve these complicated nonlinear higher-order algebraic equations analytically, especially in the case of a large initial oscillation amplitudes. In this paper, these limitations are removed. An iterative procedure based on the homotopy perturbation method has been used to solve the set of complicated nonlinear algebraic equations which is valid for whole ranges of initial oscillation amplitudes. Higher-order approximations (mainly third-order approximation) have been obtained for the strongly nonlinear oscillator with cubic and harmonic restoring force. Comparison between the obtained results and exact solutions shows that the MEBM is effective and convenient. The advantage of the MEBM is that the solution gives more correct results than many corresponding existing solutions, and it is easy, direct, concise, and simple to implement compared to existing methods.

The rest of this paper is organized as follows: In section 2, we outline the energy balance method and the proposed modified energy balance method. In section 3, we describe the strongly nonlinear oscillator with cubic and harmonic restoring force both geometrically and mathematically. In section 4, we implement the MEBM to the strongly nonlinear oscillator with cubic and harmonic restoring force. The results and detailed discussions are explained in section 5. Concluding remarks are given in section 6.

2. The Solution Procedures

2.1 The Energy Balance Method (EBM)

First, the basic idea of energy balance method (EBM) is reviewed to describe the proposed modified energy balance method. A second-order nonlinear differential equation [36] is given as

$$\ddot{x} = -f(x, \dot{x}), \text{ with initial conditions } x(0) = A_0, \dot{x}(0) = 0 \tag{1}$$

in which x and t represent dimensionless displacement and time respectively, and the dot denotes the derivative with respect to time t.
The variational principle of Eq. (1) can be expressed as

$$J(x) = \int_0^t \left(-\frac{1}{2}\dot{x}^2 + F(x)\right) dt \tag{2}$$

where $F(x) = \int f(x, \dot{x})\, dx$.
Its Hamiltonian can be written in the following form

$$H = \frac{1}{2}\dot{x}^2 + F(x) = F(A_0) \tag{3}$$

or

$$R(t) = \frac{1}{2}\dot{x}^2 + F(x) - F(A_0) = 0 \tag{4}$$

The following trial solution is used to obtain the angular frequency

$$x = A_0\cos(\omega t) \tag{5}$$

Substituting Eq. (5) into Eq. (4), gives the following residual equation

$$R(t) = \frac{1}{2}A_0^2\omega^2 sin^2(\omega t) + F(A_0\cos(\omega t)) - F(A_0) = 0 \tag{6}$$

Since Eq. (5) is only an approximation of the exact solution, Eq. (6) cannot be made zero everywhere. Collocation at $\omega t = \frac{\pi}{4}$ gives

$$\omega(A_0) = \frac{2}{A_0}\sqrt{F(A_0) - F\left(\frac{A_0}{\sqrt{2}}\right)} \tag{7}$$

Its period can be determined by using the relation $T = \frac{2\pi}{\omega}$ as

$$T = \frac{2\pi}{\frac{2}{A_0}\sqrt{F(A_0) - F\left(\frac{A_0}{\sqrt{2}}\right)}} \tag{8}$$

2.2. The Proposed Modified Energy Balance Method (MEBM)

A general N-th order periodic solution of Eq. (1) can be assumed in the following form

$$x = A_0(\rho\cos(\omega t) + u\cos(3\omega t) + v\cos(5\omega t) + w\cos(7\omega t) + z\cos(9\omega t) + \cdots), \tag{9}$$

where A_0, ρ and ω are constants. If $\rho = 1 - u - v - \cdots$, Eq. (9) readily satisfies the initial conditions given in Eq. (1). Substituting Eq. (9) into Eq. (4), the following residual equation can be obtained

$$A_0^2\omega^2\left[\frac{1}{4} - \frac{u}{2} + \frac{5u^2}{2} + \cdots + \left(-\frac{1}{4} + 2u - \frac{7u^2}{4} + \cdots\right)cos(2\omega t) + \left(-\frac{3u}{2} + \frac{3u^2}{2} + \cdots\right)cos(4\omega t) + \cdots\right] \tag{10}$$

$$= F(A_0) - F\left[A_0\left((1 - u - v - \cdots)\,cos(\omega t) + u\,cos(3\omega t) + v\,cos(5\omega t) + \cdots\right)\right]$$

Since Eq. (9) is only an approximation to the exact solution, Eq. (10) cannot be made zero everywhere. Now, taking collocation at $\cdots, \frac{\pi}{8}, \frac{\pi}{4}, \frac{3\pi}{8}, \cdots$ the following nonlinear algebraic equations can be obtained

$$A_0^2\omega^2\left[\frac{1}{4} - \frac{1}{4\sqrt{2}} - \frac{u}{2} + \sqrt{2}u + \frac{5u^2}{2} - \frac{v}{2} + \frac{3v}{\sqrt{2}} + 8uv + \frac{13\ ^2}{2} + \cdots\right] = \tag{11}$$

$$F(A_0) - F[A_0((1 - u - v - \cdots)\,cos(\pi/8) + u\,cos(3\pi/8) + v\,cos(5\pi/8) + \cdots)]$$

$$A_0^2\omega^2\left[\frac{1}{4} + u + u^2 - 3v - 6uv + 9v^2 + \cdots\right] = \tag{12}$$

$$F(A_0) - F[A_0(-u/\sqrt{2} - v/\sqrt{2} + (1 - u - v - \cdots)/\sqrt{2} + \cdots)]$$

$$A_0^2\omega^2\left[\frac{1}{4} + \frac{1}{4\sqrt{2}} - \frac{u}{2} - \sqrt{2}u + \frac{5u^2}{2} - \frac{v}{2} - \frac{3v}{\sqrt{2}} + 8uv + \frac{13v^2}{2} + \cdots\right] = \tag{13}$$

$$F(A_0) - F[A_0((1 - u - v - \cdots)\,cos(3\pi/8) + u\,cos(9\pi/8) + v\,cos(15\pi/8) + \cdots)]$$

Eliminating ω^2 from Eqs. (12)-(13) with use of Eq. (11), Eqs. (12)-(13) take the following form

$$A_0^2\left[\frac{F(A_0)}{4\sqrt{2}} + \frac{3}{2}uF(A_0) - \sqrt{2}uF(A_0) - \frac{3}{2}u^2F(A_0) - \frac{5}{2}vF(A_0) - 14uv - \frac{7}{2\sqrt{2}}v^2F(A_0) + \cdots\right] \tag{14}$$

$$-\frac{1}{4}A_0^2F[A_0((1 - u - v - \cdots)\,cos(\pi/8) + u\,cos(3\pi/8) + v\,cos(5\pi/8) + \cdots)] + \cdots = 0$$

$$A_0^2\left[\frac{F(A_0)}{2\sqrt{2}} - 2\sqrt{2}uF(A_0) - \frac{1}{\sqrt{2}}u^2F(A_0) - 3\sqrt{2}vF(A_0) - 3\sqrt{2}uvF(A_0) - \frac{7}{\sqrt{2}}v^2F(A_0) + \cdots\right] \tag{15}$$

$$-\frac{1}{4}A_0^2F[A_0((1 - u - v - \cdots)\,cos(\pi/8) + u\,cos(3\pi/8) + v\,cos(5\pi/8) + \cdots)] + \cdots = 0$$

Now applying the iterative procedure based on the homotopy perturbation method (see Appendix A for details), one can obtain the values of u and v from Eqs. (14)-(15), which are

$$u = u_0 + u_1 + u_2 + u_3 + \cdots, \tag{16}$$

$$v = v_0 + v_1 + v_2 + v_3 + \cdots, \tag{17}$$

where u_0 and v_0 are the initial approximations and the unknowns $u_1, u_2, u_3, \cdots$ and $v_1, v_2, v_3, \cdots$ are

$$u_1 = -\frac{f(u_0)}{f'(u_0)};\ v_1 = -\frac{f(v_0)}{f'(v_0)},$$

$$u_2 = -\frac{f''(u_0)}{f'(u_0)}\left(\frac{f(u_0)}{f'(u_0)}\right)^2;\ v_2 = -\frac{f''(v_0)}{f'(v_0)}\left(\frac{f(v_0)}{f'(v_0)}\right)^2,$$

$$u_3 = \frac{1}{f'(u_0)}\left(\frac{1}{6}\left(\frac{f(u_0)}{f'(u_0)}\right)^3\right)f'''(u_0) + \frac{f(u_0)}{f'(u_0)}\left(-\frac{f''(u_0)}{f'(u_0)}\left(\frac{f(u_0)}{f'(u_0)}\right)^2\right);$$

$$v_3 = \frac{1}{f'(v_0)}\left(\frac{1}{6}\left(\frac{f(v_0)}{f'(v_0)}\right)^3\right)f'''(v_0) + \frac{f(v_0)}{f'(v_0)}\left(-\frac{f''(v_0)}{f'(v_0)}\left(\frac{f(v_0)}{f'(v_0)}\right)^2\right),$$

....

Finally, substituting the values of $u, v, \cdots$ from Eqs. (16)-(17) into Eq. (11), the angular frequency ω is determined. This completes the determination of all related unknowns for the N-th order periodic solution as given in Eq. (9).

3. Formulation and Mathematical Modelling of the Problem

A strongly nonlinear oscillator with a cubic and harmonic restoring force equation represents a system consisting of a mass resting on a spring with cubic and quintic nonlinearity as shown in Fig.-1, where M is the mass, k is the linear spring stiffness coefficient, $bsin(x)$ is the driving force and $x(t)$ is the system response.

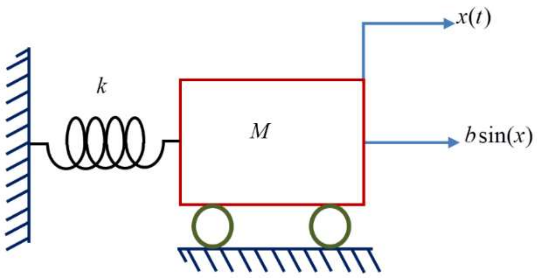

Fig. 1. Geometric structure of the problem.

The strongly nonlinear oscillator with cubic and harmonic restoring force is modelled mathematically by the following nonlinear differential equation [9, 26, 35] as

$$\ddot{x} + x + ax^3 + bsin(x) = 0, \tag{18}$$

where a and b are constants and the double dot denotes a derivative with respect to time t. The initial conditions are given by

$$x(0) = A_0,\ \dot{x}(0) = 0.$$

4. Application

The Variational and Hamiltonian formulations of Eq. (18) can be expressed as

$$J(x) = \int_0^t \left(-\frac{\dot{x}^2}{2} - \frac{x^2}{2} - \frac{ax^4}{4} + b\cos(x)\right) dt \tag{19}$$

and

$$H = \frac{\dot{x}^2}{2} + \frac{x^2}{2} + \frac{ax^4}{4} - b\cos(x) - \frac{A_0^2}{2} - \frac{aA_0^4}{4} + b\cos(A_0) = 0 \tag{20}$$

From Eq. (9), the first-order approximate solution can be expressed as

$$x = A_0\cos(\omega_1 t), \tag{21}$$

where ω_1 is the angular frequency to be determined. Substituting Eq. (21) into Eq. (20), the following residual equation can be obtained

$$\begin{aligned} &-\frac{A_0^2}{4} - \frac{5aA_0^4}{32} + \frac{A_0^2\omega_1^2}{4} + b\cos(A_0) + \frac{1}{4}A_0^2\cos(2\omega_1 t) + \frac{1}{8}aA_0^4\cos(2\omega_1 t) \\ &-\frac{1}{4}A_0^2\omega_1^2\cos(2\omega_1 t) + \frac{1}{32}aA_0^4\cos(4\omega_1 t) - b\cos(A_0\cos(\omega_1 t)) = 0 \end{aligned} \tag{22}$$

If we collocate at $\omega_1 t = \frac{\pi}{4}$, the first-order approximate angular frequency can be determined as

$$\omega_1 = \sqrt{1 + \frac{3aA_0^2}{4} - \frac{4b}{A_0^2}\cos(A_0) + \frac{4b}{A_0^2}\cos(A_0/\sqrt{2})} \tag{23}$$

A second-order approximate solution can be considered from Eq. (9) is

$$x = A_0\cos(\omega_2 t) + A_0 u(\cos(3\omega_2 t) - \cos(\omega_2 t)) \tag{24}$$

Substituting Eq. (24) into Eq. (20) and then taking the collocation at $\omega_2 t = \frac{\pi}{8}$ and $\omega_2 t = \frac{\pi}{4}$, the following nonlinear algebraic equations can be obtained

$$\begin{aligned} &1 - \frac{1}{\sqrt{2}} + \frac{5aA_0^2}{8} - \frac{aA_0^2}{2\sqrt{2}} - \omega_2^2 + \frac{\omega_2^2}{\sqrt{2}} + 2u + aA_0^2u + \frac{aA_0^2u}{\sqrt{2}} + 2\omega_2^2u - 4\sqrt{2}\omega_2^2u - 2u^2 \\ &+\sqrt{2}u^2 - \frac{3aA_0^2u^2}{2} - 10\omega_2^2u - \sqrt{2}\omega_2^2u^2 + 2aA_0^2u^3 - \sqrt{2}aA_0^2u^3 - \frac{3aA_0^2u^4}{2} + \sqrt{2}aA_0^2u^4 \\ &-\frac{4b}{A_0^2}\cos(A_0) + \frac{4b}{A_0^2}\cos(A_0(1-u)\cos(\pi/8) + u\cos(3\pi/8)) = 0 \end{aligned} \tag{25}$$

$$1+\frac{3aA_0^2}{4}-\omega_2^2+4u+2aA_0^2u-4\omega_2^2u-4u^2-6aA_0^2u^2-4\omega_2^2u^2$$
$$+8aA_0^2u^3-4aA_0^2u^4-\frac{4b}{A_0^2}cos(A_0)+\frac{4b}{A_0^2}cos(A_0(1-2u)/\sqrt{2})=0 \tag{26}$$

After simplification, Eq. (25) can be written into another form as

$$\omega_2^2=\left(1-\frac{1}{\sqrt{2}}+\frac{5aA_0^2}{8}-\frac{4b}{A_0^2}cos(A_0)+\cdots\right)/\left(1-\frac{1}{\sqrt{2}}-2u+4\sqrt{2}u+10u^2+\sqrt{2}u^2\right) \tag{27}$$

Substituting the value of ω_2^2 into Eq. (26) and then simplifying, the following nonlinear algebraic equation of u is obtained

$$f(u):\frac{aA_0^2}{8}-\frac{aA_0^2}{4\sqrt{2}}-4u+4\sqrt{2}u-12u^2+48u^3-32u^4+86aA_0^2u^5-34aA_0^2u^6$$
$$+(2\sqrt{2}+24u-24u^2+\cdots)\frac{b}{A_0^2}cos(A_0)+(4-8u+40u^2+\cdots)\frac{b}{A_0^2}cos\left(A_0\left(\frac{1-2u}{\sqrt{2}}\right)\right)$$
$$-(4+16u+16u^2)\frac{b}{A_0^2}cos(A_0((1-u)\,cos(\pi/8)+u\,cos(3\pi/8)))+\cdots=0 \tag{28}$$

Now applying the iterative procedure based on the homotopy perturbation method (see Appendix A for details), the value of u obtained from Eq. (28) is

$$u=u_0+u_1+u_2+u_3+\cdots, \tag{29}$$

where u_0 is an initial approximation and the unknowns $u_1,u_2,u_3,\cdots$ can be defined as

$$u_1=-\frac{f(u_0)}{f'(u_0)},$$
$$u_2=-\frac{f''(u_0)}{f'(u_0)}\left(\frac{f(u_0)}{f'(u_0)}\right)^2,$$
$$u_3=\frac{1}{f'(u_0)}\left(\frac{1}{6}\left(\frac{f(u_0)}{f'(u_0)}\right)^3\right)f'''(u_0)+\frac{f(u_0)}{f'(u_0)}\left(-\frac{f''(u_0)}{f'(u_0)}\left(\frac{f(u_0)}{f'(u_0)}\right)^2\right),$$
....

Now substituting the value of u from Eq. (29) into Eq. (27), the second-order approximate angular frequency ω_2 is determined as the following

$$\omega_2=\sqrt{\left(1-\frac{1}{\sqrt{2}}+\frac{5aA_0^2}{8}-\frac{4b}{A_0^2}cos(A_0)+\cdots\right)/\left(1-\frac{1}{\sqrt{2}}-2u+4\sqrt{2}u+10u^2+\sqrt{2}u^2\right)} \tag{30}$$

Thus, the second-order approximate solution of Eq. (18) is $x=A_0\,cos(\omega_2 t)+A_0u(cos(3\omega_2 t)-cos(\omega_2 t))$ where u and ω_2 are respectively given by Eqs. (29)-(30).
Higher-order approximations have been obtained by using the proposed modified energy balance method. In this study, a third-order approximation is considered as

$$x=A_0\cos(\omega_3 t)+A_0u\,(\cos(3\omega_3 t)-\cos(\omega_3 t))+A_0v(\cos(5\omega_3 t)-\cos(\omega_3 t)) \tag{31}$$

Substituting Eq. (31) into Eq. (20) and then taking collocation at $\omega_3 t=\frac{\pi}{8}$, $\omega_3 t=\frac{\pi}{4}$ and $\omega_3 t=\frac{3\pi}{8}$, the following nonlinear algebraic equations are obtained

$$1-\frac{1}{\sqrt{2}}+\frac{5aA_0^2}{8}-\omega_3^2+\frac{\omega_3^2}{\sqrt{2}}+2u+aA_0^2u+2u\omega_3^2-4\sqrt{2}u\omega_3^2-2u^2-\frac{3aA_0^2u^2}{2}-10u^2\omega_3^2-\sqrt{2}u^2\omega_3^2$$
$$+2aA_0^2u^3-\sqrt{2}aA_0^2u^3-\frac{3aA_0^2u^4}{2}+2v+2\sqrt{2}v+2aA_0^2v+\frac{3aA_0^2v}{\sqrt{2}}+2v\omega_3^2-6\sqrt{2}v\omega_3^2-2\sqrt{2}uv$$
$$-3aA_0^2uv-32uv\omega_3^2-6\sqrt{2}uv\omega_3^2+3\sqrt{2}aA_0^2u^2v+2aA_0^2u^3v-2v^2-\sqrt{2}v^2-\frac{9aA_0^2v^2}{2}-26v^2\omega_3^2$$
$$-7\sqrt{2}v^2\omega_3^2+6aA_0^2uv^2-3aA_0^2u^2v^2+4aA_0^2v^3-2aA_0^2uv^3-\frac{4b}{A_0^2}cos(A_0)$$
$$+\frac{4b}{A_0^2}cos(A_0((1-u-v)\,cos(\pi/8)+u\,cos(3\pi/8)+v\,cos(5\pi/8)))+\cdots=0 \tag{32}$$

$$1+\frac{3aA_0^2}{4}-\omega_3^2+4u+2aA_0^2u-4u\omega_3^2-4u^2-6aA_0^2u^2-4u^2\omega_3^2+8aA_0^2u^3-4aA_0^2u^4+4v$$
$$+2aA_0^2v+12v\omega_3^2-8uv-12aA_0^2uv+24uv\omega_3^2+24aA_0^2u^2v-16aA_0^2u^3v-4v^2-6aA_0^2v^2$$
$$-36v^2\omega_3^2+24aA_0^2uv^2-24aA_0^2u^2v^2+8aA_0^2v^3-16aA_0^2uv^3-4aA_0^2v^4-\frac{4b}{A_0^2}cos(A_0)$$
$$+\frac{4b}{A_0^2}cos\left(A_0\left(\frac{1-u-v}{\sqrt{2}}-\frac{u}{\sqrt{2}}-\frac{v}{\sqrt{2}}\right)\right)=0 \tag{33}$$

$$1+\frac{1}{\sqrt{2}}+\frac{5aA_0^2}{8}+\frac{aA_0^2}{2\sqrt{2}}-\omega_3^2-\frac{\omega_3^2}{\sqrt{2}}+2u+aA_0^2u-\frac{aA_0^2u}{\sqrt{2}}+2u\omega_3^2+4\sqrt{2}u\omega_3^2-2u^2-\sqrt{2}u^2-\frac{3aA_0^2u^2}{2} \quad (34)$$
$$-10u^2\omega_3^2+\sqrt{2}u^2\omega_3^2+2aA_0^2u^3+\sqrt{2}aA_0^2u^3-\frac{3aA_0^2u^4}{2}-\sqrt{2}aA_0^2u^4+2v-2\sqrt{2}v+2aA_0^2v-\frac{3aA_0^2v}{\sqrt{2}}$$
$$+2v\omega_3^2+6\sqrt{2}v\omega_3^2+2\sqrt{2}uv-3aA_0^2uv+3\sqrt{2}aA_0^2uv-32uv\omega_3^2+6\sqrt{2}uv\omega_3^2-3\sqrt{2}aA_0^2u^2v$$
$$+2aA_0^2u^3v+2\sqrt{2}aA_0^2u^3v-2v^2+\sqrt{2}v^2-\frac{9aA_0^2v^2}{2}+3\sqrt{2}aA_0^2v^2-26v^2\omega_3^2+7\sqrt{2}v^2\omega_3^2+6aA_0^2uv^2$$
$$-3\sqrt{2}aA_0^2uv^2-3aA_0^2u^2v^2+4aA_0^2v^3-3\sqrt{2}aA_0^2v^3-2aA_0^2uv^3+2\sqrt{2}aA_0^2uv^3-\frac{3aA_0^2v^4}{2}+\sqrt{2}aA_0^2v^4$$
$$-\frac{4b}{A_0^2}cos(A_0)+\frac{4b}{A_0^2}cos(A_0((1-u-v)cos(3\pi/8)+u\,cos(9\pi/8)+v\,cos(15\pi/8)))=0$$

From Eq. (32), it can be easily written in another form as

$$\omega_3^2=\left(1+2u-\frac{4b}{A_0^2}cos(A_0)+\frac{4b}{A_0^2}cos(A_0((1-u-v)cos(\pi/8)+u\,cos(3\pi/8)+v\,cos(5\pi/8)))+\cdots\right)/ \quad (35)$$
$$\left(1-\frac{1}{\sqrt{2}}-2u+4\sqrt{2}u+10u^2+\sqrt{2}u^2-2v+6\sqrt{2}v+32uv+6\sqrt{2}uv+26v^2+7\sqrt{2}v^2\right)$$

By eliminating ω_3^2 from Eqs. (33)-(34) with the help of Eq. (35) and then simplifying, the following nonlinear algebraic equations of u and v can be expressed

$$f(u):\frac{aA_0^2}{8}-\frac{aA_0^2}{4\sqrt{2}}-4u-3aA_0^2u-\frac{15aA_0^2u^2}{2}+\frac{43aA_0^2u^2}{2\sqrt{2}}+48u^3+40aA_0^2u^3-32u^4-\frac{161aA_0^2u^4}{2} \quad (36)$$
$$+86aA_0^2u^5-34aA_0^2u^6+12v+6aA_0^2v+48uv+26aA_0^2uv+208u^2v+154aA_0^2u^2v-256u^3v$$
$$-394aA_0^2u^3v+558aA_0^2u^4v-332aA_0^2u^5v+4v^2+\frac{31aA_0^2v^2}{2}+240uv^2+164aA_0^2uv^2-320u^2v^2$$
$$-741aA_0^2u^2v^2+1236aA_0^2u^3v^2-742aA_0^2u^4v^2+16v^3-384uv^3+1652aA_0^2u^2v^3-1480aA_0^2u^3v^3$$
$$-32v^4+153\sqrt{2}aA_0^2v^4+782aA_0^2uv^4$$
$$+\left(2\sqrt{2}+24u-16\sqrt{2}u-24u^2-4\sqrt{2}u^2-40v-224uv+\cdots\right)\frac{b}{A_0^2}cos(A_0)$$
$$+\left(4-2\sqrt{2}-8u+16\sqrt{2}u+40u^2-8v+\cdots\right)\frac{b}{A_0^2}cos\left(A_0\left(\frac{1-u-v}{\sqrt{2}}-\frac{u}{\sqrt{2}}-\frac{v}{\sqrt{2}}\right)\right)$$
$$-(4+16u+\cdots)\frac{b}{A_0^2}cos(A_0(1-u-v)cos(\pi/8)+u\,cos(3\pi/8)+v\,cos(5\pi/8))+\cdots=0$$

$$f(v):\frac{aA_0^2}{4\sqrt{2}}-4\sqrt{2}u-2\sqrt{2}aA_0^2u-28\sqrt{2}u^2-\frac{71aA_0^2u^2}{2\sqrt{2}}+8\sqrt{2}u^3+20\sqrt{2}aA_0^2u^3+24\sqrt{2}u^4-\frac{17aA_0^2u^4}{\sqrt{2}} \quad (37)$$
$$-16\sqrt{2}aA_0^2u^5+23\sqrt{2}aA_0^2u^6-4\sqrt{2}v-\frac{3aA_0^2v}{\sqrt{2}}-96\sqrt{2}uv-\frac{137aA_0^2uv}{\sqrt{2}}+40\sqrt{2}u^2v+106\sqrt{2}aA_0^2u^2v$$
$$+48\sqrt{2}u^3v-70\sqrt{2}aA_0^2u^3v-22\sqrt{2}aA_0^2u^4v+38\sqrt{2}aA_0^2u^5v-76\sqrt{2}v^2-\frac{249aA_0^2v^2}{2\sqrt{2}}+104\sqrt{2}uv^2$$
$$+192\sqrt{2}aA_0^2uv^2-261\sqrt{2}aA_0^2u^2v^2+168\sqrt{2}aA_0^2u^3v^2-73\sqrt{2}aA_0^2u^4v^2+104\sqrt{2}v^3-144\sqrt{2}uv^3$$
$$+388\sqrt{2}aA_0^2u^2v^3-\left(4\sqrt{2}-32\sqrt{2}u-8\sqrt{2}u^2-48\sqrt{2}v-48\sqrt{2}uv-56\sqrt{2}v^2\right)\frac{b}{A_0^2}cos(A_0)$$
$$+(4-8u+\cdots)\frac{b}{A_0^2}cos(A_0(1-u-v)cos(\pi/8)+u\,cos(3\pi/8)+v\,cos(5\pi/8))$$
$$-(4+\cdots)\frac{b}{A_0^2}cos(A_0(1-u-v)cos(3\pi/8)+u\,cos(9\pi/8)+v\,cos(15\pi/8))+\cdots=0$$

Now solving Eqs. (36)-(37), by applying the iterative procedure based on the homotopy perturbation method (see Appendix A for details), the values of u and v can be obtained as

$$u=u_0+u_1+u_2+u_3+\cdots, \quad (38)$$
$$v=v_0+v_1+v_2+v_3+\cdots, \quad (39)$$

where u_0 and v_0 are the initial approximations and the unknowns $u_1,u_2,u_3,\cdots$ and $v_1,v_2,v_3,\cdots$ are

$$u_1=-\frac{f(u_0)}{f'(u_0)};\ v_1=-\frac{f(v_0)}{f'(v_0)},$$
$$u_2=-\frac{f''(u_0)}{f'(u_0)}\left(\frac{f(u_0)}{f'(u_0)}\right)^2;\ v_2=-\frac{f''(v_0)}{f'(v_0)}\left(\frac{f(v_0)}{f'(v_0)}\right)^2,$$

$$u_3 = \frac{1}{f'(u_0)}\left(\frac{1}{6}\left(\frac{f(u_0)}{f'(u_0)}\right)^3\right)f'''(u_0) + \frac{f(u_0)}{f'(u_0)}\left(-\frac{f''(u_0)}{f'(u_0)}\left(\frac{f(u_0)}{f'(u_0)}\right)^2\right);$$

$$v_3 = \frac{1}{f'(v_0)}\left(\frac{1}{6}\left(\frac{f(v_0)}{f'(v_0)}\right)^3\right)f'''(v_0) + \frac{f(v_0)}{f'(v_0)}\left(-\frac{f''(v_0)}{f'(v_0)}\left(\frac{f(v_0)}{f'(v_0)}\right)^2\right),$$

....

Now substituting the values of u and v from Eqs. (38)-(39) into Eq. (35), the third-order approximate angular frequency ω_3 is obtained as the following

$$\omega_3 = \sqrt{\left(1 + 2u - \frac{4b}{A_0^2}cos(A_0) + \frac{4b}{A_0^2}cos(A_0((1-u-v)\,cos(\pi/8) + u\,cos(3\pi/8) + v\,cos(5\pi/8))) + \cdots\right) \Big/ \left(1 - \frac{1}{\sqrt{2}} - 2u + 4\sqrt{2}u + 10u^2 + \sqrt{2}u^2 - 2v + 6\sqrt{2}v + 32uv + 6\sqrt{2}uv + 26v^2 + 7\sqrt{2}v^2\right)} \quad (40)$$

Thus, the third-order approximate solution of Eq. (18) is $x = A_0\cos(\omega_3 t) + A_0 u\,(\cos(3\omega_3 t) - \cos(\omega_3 t)) + A_0 v(\cos(5\omega_3 t) - \cos(\omega_3 t))$ where u, v and ω_3 are given respectively by Eqs. (38)-(40).

5. Results and Discussions

The approximate solutions of Eq. (18) have been compared with the solutions obtained using the existing methods [9, 26, 35] and the exact solutions obtained numerically. The errors (%) of the solutions have been also computed. These solution comparisons are shown in Table 1 to Table 4 for the condition $a = b = 1$ and the initial oscillation amplitude $A_0 = \frac{\pi}{18}$, $A_0 = \frac{\pi}{6}$, $A_0 = 1$ and $A_0 = 2$ respectively. In Table-1 to Table-2, the errors (%) are reported very less for both of the approximations (first-, second- and third-order) for considering the values of initial amplitude as $A_0 = \frac{\pi}{18}$ and $A_0 = \frac{\pi}{6}$ respectively. On the other hand, in Table-3 to Table-4, the errors (%) are increased swiftly only for first- and second-order approximations but for third-order approximation the errors (%) are reported no change for considering the higher values of initial amplitude as $A_0 = 1$ and $A_0 = 2$ respectively.

Table 1. First-, second- and third-order approximate solutions of Eq. (18) are compared with existing results and the corresponding numerical solution (Runge-Kutta fourth-order method) for $a = 1$, $b = 1$ and $A_0 = \frac{\pi}{18}$:

t	$x_{[1stMEBM]}^{[this\ study]}$ $Er(\%)$	$x_{[2ndMEBM]}^{[this\ study]}$ $Er(\%)$	$x_{[3rdMEBM]}^{[this\ study]}$ $Er(\%)$	x_{nu}	$x_{[1stHBM]}^{[26]}$ $Er(\%)$	$x_{[2ndHBM]}^{[26]}$ $Er(\%)$
0	0.174532 0.0000	0.174532 0.0000	0.174532 0.0000	0.174532	0.174532 0.0000	0.174532 0.0000
0.5	0.132306 0.0673	0.132217 0.0000	0.132217 0.0000	0.132217	0.132306 0.0673	0.132217 0.0000
1	0.026058 0.1498	0.026019 0.0000	0.026019 0.0000	0.026019	0.026059 0.1537	0.026019 0.0000
1.5	-0.092798 0.1143	-0.092692 0.0000	-0.092692 0.0000	-0.092692	-0.092797 0.1132	-0.092692 0.0000
2	-0.166751 0.0137	-0.166728 0.0000	-0.166728 0.0000	-0.166728	-0.166751 0.0137	-0.166728 0.0000
2.5	-0.160016 0.0243	-0.159977 0.0000	-0.159977 0.0000	-0.159977	-0.160017 0.0250	-0.159977 0.0000
3	-0.075852 0.1254	-0.075757 0.0000	-0.075757 0.0000	-0.075757	-0.075853 0.1267	-0.075757 0.0000
3.5	0.045015 0.1535	0.044946 0.0000	0.044947 0.0022	0.044946	0.045014 0.1512	0.044946 0.0000
4	0.144101 0.0513	0.144027 0.0000	0.144027 0.0000	0.144027	0.144100 0.0506	0.144027 0.0000
4.5	0.173458 0.0017	0.173455 0.0000	0.173455 0.0000	0.173455	0.173458 0.0017	0.173455 0.0000
5	0.118882 0.0816	0.118785 0.0000	0.118784 0.0008	0.118785	0.118883 0.0825	0.118785 0.0000

Table 2. First-, second- and third-order approximate solutions of Eq. (18) are compared with the existing results and the corresponding numerical solution (Runge-Kutta fourth-order method) for $a = 1$, $b = 1$ and $A_0 = \frac{\pi}{6}$:

t	$x_{[1stMEBM]}^{[this\ study]}$ $Er(\%)$	$x_{[2ndMEBM]}^{[this\ study]}$ $Er(\%)$	$x_{[3rdMEBM]}^{[this\ study]}$ $Er(\%)$	x_{nu}	$x_{[1stHBM]}^{[26]}$ $Er(\%)$	$x_{[2ndHBM]}^{[26]}$ $Er(\%)$
0	0.523598 0.0000	0.523598 0.0000	0.523598 0.0000	0.523598	0.523598 0.0000	0.523598 0.0000
0.5	0.387776 0.5854	0.385491 0.0072	0.385488 0.0080	0.385519	0.387797 0.5908	0.385528 0.0023
1	0.050773 0.9885	0.050267 0.0179	0.050207 0.1372	0.050276	0.050838 1.1178	0.050273 0.0059

Table 2. Continued

t	$x^{[this\ study]}_{[1stMEBM]}$ $Er(\%)$	$x^{[this\ study]}_{[2ndMEBM]}$ $Er(\%)$	$x^{[this\ study]}_{[3rdMEBM]}$ $Er(\%)$	x_{nu}	$x^{[26]}_{[1stHBM]}$ $Er(\%)$	$x^{[26]}_{[2ndHBM]}$ $Er(\%)$
1.5	-0.312570 0.9198	-0.309673 0.0154	-0.309795 0.0238	-0.309721	-0.312492 0.8946	-0.309725 0.0012
2	-0.513751 0.0621	-0.513428 0.0007	-0.513458 0.0050	-0.513432	-0.513726 0.0572	-0.513434 0.0003
2.5	-0.448396 0.3057	-0.447019 0.0022	-0.446934 0.0212	-0.447029	-0.448480 0.3245	-0.447039 0.0022
3	-0.150412 0.9009	-0.149048 0.0140	-0.148869 0.1341	-0.149069	-0.150597 1.0250	-0.149063 0.0040
3.5	0.225606 1.3167	0.222615 0.0264	0.222882 0.0934	0.222674	0.225402 1.2251	0.222674 0.0000
4	0.484580 0.2434	0.483386 0.0035	0.483505 0.0211	0.483403	0.484482 0.2232	0.483411 0.0016
4.5	0.492151 0.1023	0.491650 0.0004	0.491538 0.0223	0.491648	0.492250 0.1224	0.491654 0.0012
5	0.244392 0.7374	0.242582 0.0086	0.242294 0.1273	0.242603	0.244678 0.8553	0.242600 0.0012

Table 3. First-, second- and third-order approximate solutions of Eq. (18) are compared with the existing results and the corresponding numerical solution (Runge-Kutta fourth-order method) for $a = 1$, $b = 1$ and $A_0 = 1$:

t	$x^{[this\ study]}_{[1stMEBM]}$ $Er(\%)$	$x^{[this\ study]}_{[2ndMEBM]}$ $Er(\%)$	$x^{[this\ study]}_{[3rdMEBM]}$ $Er(\%)$	x_{nu}	$x^{[9]}_{[OIM]}$ $Er(\%)$
0	1.000000 0.0000	1.000000 0.0000	1.000000 0.0000	1.000000	1.000000 0.0000
0.5	0.688898 2.0731	0.674955 0.0072	0.674910 0.0005	0.674906	0.672997 0.2828
1	-0.050838 8.0762	-0.045466 3.3440	-0.047041 0.0042	-0.047039	-0.046277 1.6199
1.5	-0.758942 2.0131	-0.742016 0.2619	-0.743969 0.0005	-0.743965	-0.741376 0.3480
2	-0.994830 0.0024	-0.995120 0.0315	-0.994806 0.0000	-0.994806	-0.994866 0.0060
2.5	-0.611732 1.9614	-0.602381 0.4028	-0.599967 0.0005	-0.599964	-0.599003 0.1601
3	0.151988 7.9674	0.136096 3.3216	0.140777 0.0035	0.140772	0.138513 1.6047
3.5	0.821141 1.8384	0.802801 0.4360	0.806322 0.0006	0.806317	0.803346 0.3684
4	0.979377 0.0066	0.980557 0.1271	0.979312 0.0000	0.979312	0.979549 0.0242
4.5	0.528241 1.5910	0.525065 0.9802	0.519971 0.0005	0.519968	0.520164 0.0376
5	-0.251567 7.7540	-0.225811 3.2780	-0.233473 0.0038	-0.233464	-0.229784 1.5762

In figures Fig.-2, Fig.-4 and Fig.-6, the approximate solutions have been compared with numerical solutions (Runge-Kutta fourth-order method) for the condition $a = b = 1$ and the initial oscillation amplitudes $A_0 = 5$, $A_0 = 10$ and $A_0 = 50$ respectively. In the same cases, the phase plane trajectories have also been illustrated in Fig.-3, Fig.-5 and Fig.-6 respectively. In all Tables and Figures, the proposed method provides approximate solutions with much higher accuracy than the existing methods [9, 24, 35]. In a third-order approximation with initial oscillation amplitude $A_0 = 2$, the maximum error (%) is reported as 0.0070% at time $t = 5$. In the same initial oscillation amplitude, the much higher error 0.7147% has been found using the optimal iteration method [9]. It is notable that the solution procedure of the existing methods [9, 24, 35] are more laborious and cumbersome, especially for obtaining the higher-order approximations than the modified energy balance method proposed in this paper. Therefore, this proposed method not only provides a simpler and efficient solving technique but also a highly accurate solution.

Table 4. First-, second- and third-order approximate solutions of Eq. (18) are compared with the existing results and the corresponding numerical solution (Runge-Kutta fourth-order method) for $a = 1$, $b = 1$ and $A_0 = 2$:

t	$x^{[this\ study]}_{[1stMEBM]}$ $Er(\%)$	$x^{[this\ study]}_{[2ndMEBM]}$ $Er(\%)$	$x^{[this\ study]}_{[3rdMEBM]}$ $Er(\%)$	x_{nu}	$x^{[9]}_{[OIM]}$ $Er(\%)$
0	2.000000 0.0000	2.000000 0.0000	2.000000 0.0000	2.000000	2.000000 0.0000

Table 4. Continued

t	$x_{[1stMEBM]}^{[this\ study]}$ $Er(\%)$	$x_{[2ndMEBM]}^{[this\ study]}$ $Er(\%)$	$x_{[3rdMEBM]}^{[this\ study]}$ $Er(\%)$	x_{nu}	$x_{[OIM]}^{[9]}$ $Er(\%)$
0.5	0.961787 6.5686	0.904695 0.2426	0.902647 0.0157	0.902505	0.896649 0.6488
1	-1.074964 9.8627	-0.946665 3.2495	-0.978658 0.0201	-0.978461	-0.972382 0.6212
1.5	-1.995675 0.0928	-1.999243 0.0858	-1.997527 0.0001	-1.997529	-1.997524 0.0002
2	-0.844451 2.3017	-0.862431 4.4799	-0.825535 0.0101	-0.825451	-0.819890 0.6736
2.5	1.183491 12.3739	0.988317 6.1580	1.053420 0.0235	1.053172	1.046951 0.5906
3	1.982719 0.3724	1.996974 0.3437	1.990122 0.0005	1.990132	1.990109 0.0011
3.5	0.723464 3.2071	0.819899 9.6950	0.747455 0.0026	0.747435	0.742234 0.6958
4	-1.286900 14.2407	-1.029620 8.5985	-1.126780 0.0265	-1.126481	-1.120205 0.5571
4.5	-1.961189 0.8423	-1.993198 0.7759	-1.977827 0.0011	-1.977850	-1.977790 0.0030
5	-0.599347 10.3553	-0.777122 16.2345	-0.668534 0.0070	-0.668581	-0.663802 0.7147

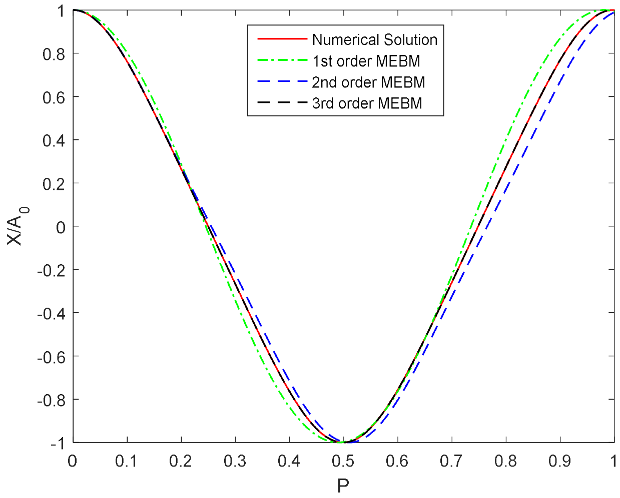

Fig. 2. Comparison of the approximate solution of Eq. (18) with its numerical solution in the case of $a = b = 1$ and $A_0 = 5$.

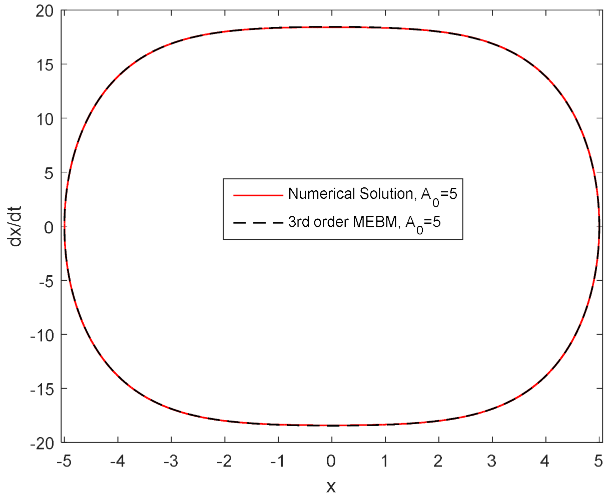

Fig. 3. Comparison of the approximate solution of Eq. (18) with its numerical solution in terms of phase plane in the case of $a = b = 1$ and $A_0 = 5$.

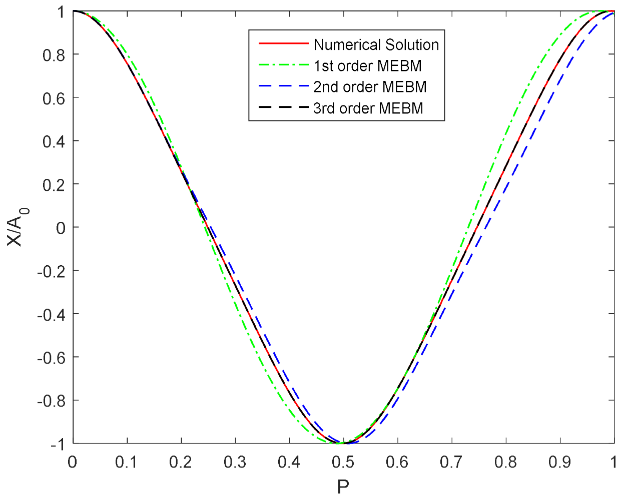

Fig. 4. Comparison of the approximate solution of Eq. (18) with its numerical solution in the case of $a = b = 1$ and $A_0 = 10$.

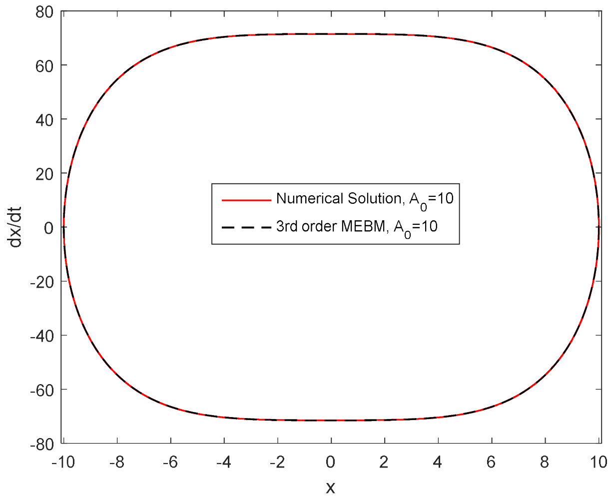

Fig. 5. Comparison of the approximate solution of Eq. (18) with its numerical solution in terms of phase plane in the case of $a = b = 1$ and $A_0 = 10$.

In Table 1 to Table 4, $x_{[1stME\ \ ]}^{[thi\ \ study]}$, $x_{[2ndMEBM]}^{[this\ study]}$ and $x_{[3rdMEBM]}^{[this\ study]}$ respectively denote first-, second- and third-order approximate solutions obtained by using the proposed modified energy balance method. x_{nu} represents the numerical solutions which is obtained by using the Runge-Kutta fourth-order method. $x_{[1stHBM]}^{[26]}$ and $x_{[2ndHB\ \ ]}^{[26]}$ signify the first- and second-order approximate solutions which have been obtained by using the harmonic balance method [26]. $x_{[OIM]}^{[9]}$ describes the existing approximate solutions that have been determined by using the optimal iterative method [9]. $Er(\%)$ denotes the percentage errors which are obtained from the relation $\left|\frac{x_{nu}-x_{[i]}}{x_{nu}} \times 100\right|$ where $i = 1,2,3$.

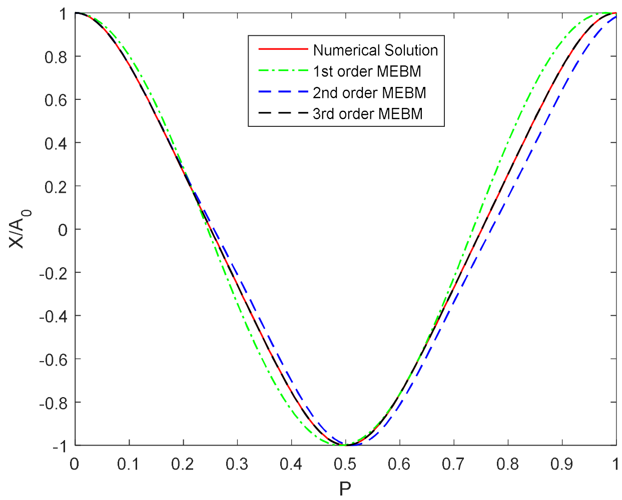

Fig. 6. Comparison of the approximate solution of Eq. (18) with its numerical solution in the case of $a = b = 1$ and $A_0 = 50$.

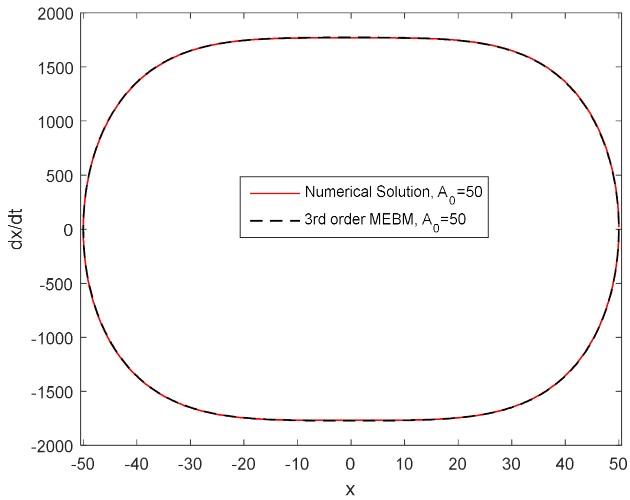

Fig. 7. Comparison of the approximate solution of Eq. (18) with its numerical solution in terms of phase plane in the case of $a = b = 1$ and $A_0 = 50$.

6. Conclusion

In this paper, a new efficient analytical technique has been developed based on the modified energy balance method to determine approximate periodic solutions for strongly nonlinear oscillator with cubic and harmonic restoring forces. Compared with the existing methods, the determination procedure of the proposed method is straightforward and simple, and it surpasses them in terms of solution accuracy. Remarkably, it can clearly be seen that the third-order approximate solutions are almost identical to the exact solutions. Furthermore, the proposed method opens the way of incorporating any (N-th) order of approximation for solving even stronger nonlinear systems. In conclusion, the proposed method has great potential as an efficient alternative to the existing methods.

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Conflict of Interest

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Appendix A

A higher-order nonlinear algebraic equation is of the form

$$f(x) = 0 \tag{A.1}$$

Consider the nonlinear algebraic equation Eq. (A1), and we construct a homotopy $H: R \times [0,\ 1] \rightarrow R$ which satisfy

$$H(x,p) = f(x) - f(x_0) + pf(x_0) = 0\,, \tag{A.2}$$

$$x \in R, p \in [0,\ 1]$$

where p is embedding parameter, x_0 is an initial approximation of Eq. (A.1). Hence, it is obvious that

$$H(x,\ 0) = f(x) - f(x_0) = 0 \tag{A.3}$$

$$H(x, 1) = f(x) = 0 \tag{A.4}$$

and the changing process of p from 0 to 1, refers to $H(x,p)$ from $H(x,0)$ to $H(x,1)$. Applying the perturbation technique [21], due to the fact that $0 \leq p \leq 1$ can be considered as a small parameter, we can assume that the solution of Eq. (A.2) can be express as a series in p

$$x = x_0 + x_1 p + x_2 p^2 + x_3 p^3 + \cdots \tag{A.5}$$

When $p \to 1$, Eq. (A.2) corresponds to Eq. (A.1) and Eq. (A.5) becomes the approximate solution of Eq. (A.1), that is [21]

$$x = \lim_{p\to 1} x = x_0 + x_1 + x_2 + x_3 + \cdots \tag{A.6}$$

and in [21] the unknowns are

$$x_1 = -\frac{f(x_0)}{f'(x_0)} \tag{A.7}$$

$$x_2 = -\frac{f''(x_0)}{f'(x_0)}\left(\frac{f(x_0)}{f'(x_0)}\right)^2 \tag{A.8}$$

$$x_3 = \frac{1}{f'(x_0)}\left(\frac{1}{6}\left(\frac{f(x_0)}{f'(x_0)}\right)^3\right)f'''(x_0) + \frac{f(x_0)}{f'(x_0)}\left(-\frac{f''(x_0)}{f'(x_0)}\left(\frac{f(x_0)}{f'(x_0)}\right)^2\right) \tag{A.9}$$

$\dots$.

References

[1] Leissa AW, *Vibration of plates*. Acoustical Society of America, Washington DC, 1993.
[2] Chopra AK, *Dynamic of structures, theory and application to earthquake engineering*. Prentice-Hall, New Jersey, 1995.
[3] Elmas N, Boyaci H, A new perturbation technique in solution of nonlinear differential equations by using variable transformation. *Appl Math Comp*, 227 (2014) 422-427.
[4] El-Naggar AM, Ismail GM, Analytical solution of strongly nonlinear Duffing Oscillators. *Alex Engg J*, 55 (2016) 1581-1585.
[5] Yao S, Cheng Z, The homotopy perturbation method for a nonlinear oscillator with a damping. *J Low Freq Noise Vib Active Control*, (2019) DOI: https://doi.org/10.1177/1461348419836344.
[6] Alam MS, Yeasmin IA, Ahamed MS, Generalization of the modified Lindstedt-Poincare method for solving some strong nonlinear oscillators. *Ain Shams Engg J*, 10 (2019) 195-201.
[7] Suleman M, Wu Q, Comparative solution of nonlinear quintic cubic oscillator using modified homotopy perturbation method. *Ad Math Phy*, 5 (2015) 932905.
[8] Razzak MA, Alam MZ, Sharif MN, Modified multiple time scale method for solving strongly nonlinear damped forced vibration systems. *Res Phy*, 8 (2018) 231-238.
[9] Marinca V, Herisanu N, An optimal iteration method for strongly nonlinear oscillators. *J Appl Math*, 11 (2012) 906341.
[10] Sedighi HM, Shirazi KH, Attarzadeh MA, A study on the quintic nonlinear beam vibrations using asymptotic approximate approaches. *Acta Astronaut*, 91 (2013) 245-250.
[11] Sedighi HM, Shirazi KH, Dynamic pull-in instability of double-sided actuated nano-torsional switches. *Acta Mech Solida Sin*, 28(1) (2015) 91-101.
[12] Akbari M, Ganji DD, Ahmadi A, Kachapi SHH, Analysing the nonlinear vibrational wave differential equation for the simplified model of tower cranes by algebraic method. *Front Mech Engg*, 9(1) (2014) 58-70.
[13] Akbari MR, Nimafar M, Ganji DD, Chalmiani HK, Investigation on non-linear vibration in arched beam for bridges construction via AGM method. *Appl Math Comput*, 298 (2017) 95-110.
[14] Beléndez A, Hernández A, Beléndez T et al, Solutions for conservative nonlinear oscillators using an approximate method based on chebyshev series expansion of the restoring force. *Act Phy Pol A*, 130(3) (2016) 667-678.
[15] Nhat LA, Using differentiation matrices for pseudospectral method solve Duffing Oscillator. *J Non Sci Appl*, 11 (2018) 1331-1336.
[16] Wang Q, Shi X, Li Z, A short remark on Ren-Hu's modification of He's frequency-amplitude formulation and the temperature oscillation in a polar bear hair. *J Low Freq Noise Vib Active Control*, (2019) DOI: 10.1177/1461348419831478.
[17] Daeichin M, Ahmadpoor MA, Askari H, Yildirim A, Rational energy balance method to nonlinear oscillators with cubic term. *Asian-European J Math*, 6(2) (2013) 1350019.
[18] Yazdi MK, Tehrani PH, Rational variational approaches to strong nonlinear oscillations. *Int J Appl Comp Math*, 3(2) (2017) 757-771.
[19] Shui X, Wang S, Closed-form numerical formulae for solutions of strongly nonlinear oscillators. *Int J Non Mech*, 103 (2018) 12-22.
[20] Hoang T, Duhamel D, Foret G et al, Frequency dependent iteration method for forced nonlinear oscillators. *Appl Math Mod*, 42 (2017) 441-448.
[21] Javidi M, Iterative methods to nonlinear equations. *Appl Math Comput*, 193 (2007) 360-365.

[22] Razzak MA, A simple new iterative method for solving strongly nonlinear oscillator systems having a rational and an irrational force. *Alex Engg J*, 57 (2018) 1099-1107.
[23] Yazdi MK, Tehrani PH, Frequency analysis of nonlinear oscillations via the global error minimization. *Non Engg*, 5(2) (2016) 87-92.
[24] Mickens RE, A generalization of the method of harmonic balance. *J Sound Vib*, 111 (1986) 115-518.
[25] Chowdhury MSH, Hosen MA, Ali MY, Ismail AF, An analytical technique to obtain higher-order approximate periods for nonlinear oscillator. *IIUM Engg J*, 19(2) (2018) 182-191.
[26] Hosen MA, Chowdhury MSH, A new reliable analytical solution for strongly nonlinear oscillator with cubic and harmonic restoring force. *Res Phy*, 5 (2015) 111-114.
[27] Hosen MA, Rahman MS, Alam MS, Amin MR, An analytical technique for solving a class of strongly nonlinear conservative systems. *Appl Math Comput*, 218 (2012) 5474-5486.
[28] Belendez A, Gimeno E, Alvarez ML et al, A novel rational harmonic balance approach for periodic solutions of conservative nonlinear oscillators. *Int J Non Sci Num Sim*, 10(1) (2009) 13-26.
[29] Akbarzade M, Farshidianfar A, Nonlinear transversely vibrating beams by the improved energy balance method and the global residue harmonic balance method. *Appl Math Mod*, 45 (2017) 393-404.
[30] Rahman MS, Lee YY, New modified multi-level residue harmonic balance method for solving nonlinearly vibrating double-beam problem. *J Sound Vib*, 406 (2017) 295-327.
[31] Younesian D, Esmailzadeh E, Askari H, *Vibration analysis of oscillators with generalized inertial and geometrical nonlinearities*. In: Dai L., Jazar R. (eds) Nonlinear Approaches in Engineering Applications. Springer, 2018.
[32] Sedighi HM, Shirazi KH, Noghrehabadi A, Application of recent powerful analytical approaches on the non-linear vibration of cantilever beams. *Int J Nonlinear Sci Numer Simul*, 13(7-8) (2012) 487-494.
[33] Sedighi HM, Size-dependent dynamic pull-in instability of vibrating electrically actuated microbeams based on the strain gradient elasticity theory. *Acta Astronaut*, 95 (2014) 111-123.
[34] Sedighi HM, Shirazi KH, Vibrations of micro-beams actuated by an electric field via Parameter Expansion Method. *Acta Astronaut*, 85 (2013) 19-24.
[35] Junfeng L, Li M, The VIM-Pade technique for strongly nonlinear oscillators with cubic and harmonic restoring force. *J Low Freq Noise Vib Active Control*, (2018) DOI: 10.1177/1461348418813612.
[36] He JH, Preliminary report on the energy balance for nonlinear oscillations. *Mech Res Comm*, 29 (2002) 107-111.
[37] Hosen MA, Chowdhury MSH, Ali MY, Ismail AF, An analytical approximation technique for the duffing oscillator based on the energy balance method. *Italian J Pur Appl Math*, 37 (2017) 455-466.
[38] Askari H, Saadatnia Z, Esmilzadeh E, Younesian D, Multi-frequency excitation of stiffened triangular plates for large amplitude oscillations. *J Sound Vib*, 333 (2014) 5817-5835.
[39] Molla MHU, Alam MS, Higher accuracy analytical approximations to nonlinear oscillators with discontinuity by energy balance method. *Res Phy*, 7 (2017) 2104-2110.
[40] Koudahoun LH, Kpomahou YJF, Adjaï DKK, *Periodic solutions for nonlinear oscillations in elastic structures via energy balance method*. (2016), http://vixra.org/abs/1611.0214.

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