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Research Paper

Variational Principle for the Generalized KdV-Burgers Equation with Fractal Derivatives for Shallow Water Waves

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Abstract. The unsmooth boundary will greatly affect motion morphology of a shallow water wave, and a fractal space is introduced to establish a generalized KdV-Burgers equation with fractal derivatives. The semi-inverse method is used to establish a fractal variational formulation of the problem, which provides conservation laws in an energy form in the fractal space and possible solution structures of the equation.

Keywords: Continuum assumption, Two scale transform, Fractal dimension, Variational derivative.

1. Introduction

This paper considers the following generalized KdV-Burgers equation [1-4]

$$\frac{\partial u}{\partial T}+au\frac{\partial u}{\partial X}+b\frac{\partial^2 u}{\partial X^2}+c\frac{\partial^3 u}{\partial X^3}=0 \tag{1}$$

where *a, b* and *c* are constants.

When $a=1, b=0$ and $c=1$, Eq. (1) is the KdV equation, and when $a=1, b=1$ and $c=0$, we obtain the Burgers equation. There are many analytical methods to solve Eq. (1), among which the exp-function method [5-7], the semi-inverse variational method [8-11], the Taylor series method [12], He's frequency formulation for fast insight into the periodic property of a nonlinear equation [13], the homotopy perturbation method [14-19] and the variational iteration method [20-22] have been caught much attention.

Eq. (1) describes a shallow water wave, however an unsmooth boundary will greatly affect the solitary properties, so the smooth space (X,T) should be replaced by a fractal space (X^β, T^α), where β and α are, respectively, fractal dimensions in space and time. In the fractal space Eq. (1) can be modified as

$$\frac{\partial u}{\partial T^\alpha}+au\frac{\partial u}{\partial X^\beta}+b\frac{\partial^2 u}{\partial X^{2\beta}}+c\frac{\partial^3 u}{\partial X^{3\beta}}=0 \tag{2}$$

where the fractal derivatives are defined as [23, 24]

$$\frac{\partial u}{\partial T^{\alpha}}(T_0,X)=\Gamma(1+\alpha)\lim_{\substack{T-T_0\to\Delta T\\ \Delta T\neq 0}}\frac{u(T,X)-u(T_0,X)}{(T-T_0)^{\alpha}} \tag{3}$$

$$\frac{\partial u}{\partial X^{\beta}}(T,X_0)=\Gamma(1+\alpha)\lim_{\substack{X-X_0\to\Delta X\\ \Delta X\neq 0}}\frac{u(T,X)-u(T,X_0)}{(X-X_0)^{\beta}} \tag{4}$$

We have the following chain rule

$$\frac{\partial^2}{\partial X^{2\beta}}=\frac{\partial}{\partial X^{\beta}}\frac{\partial}{\partial X^{\beta}} \tag{5}$$

$$\frac{\partial^3 u}{\partial X^{3\beta}}=\frac{\partial}{\partial X^{\beta}}\frac{\partial}{\partial X^{\beta}}\frac{\partial}{\partial X^{\beta}} \tag{6}$$

In the definitions given in Eqs. (3) and (4), ΔX and ΔT are, respectively, the smallest spatial scale for discontinuous boundary and the smallest temporal scale for watching the solitary wave . When the spatial scale is larger than ΔX , the boundary is considered as a smooth one, and traditional continuum mechanics works, on the scale of ΔX , the boundary is discontinuous, and it is considered a fractal curve. When we watch the solitary wave on a scale larger than ΔT , a smooth wave morphology is predicted, however, when we observe the wave on the scale of ΔT , discontinuous wave morphology can be found [24].

In the fractal space, all variables depend upon the scales used for observation and the fractal dimensions of the discontinuous boundary. For example, the velocity difference (Δu)Δu) across a distance (ΔX) or a period (ΔT) can be written in the forms [24]

$$\Delta u \sim L^{\alpha} \qquad \Delta u \propto (\Delta X)^{\beta} \tag{7}$$

$$\Delta u \propto (\Delta T)^{\alpha} \tag{8}$$

The fractal derives are widely used in applications [24-30] for discontinuous media.

2. Variation Principle

In a fractal space, the physical laws should be also be followed. Wang et al. [30] established a variational principle for traveling wave in a fractal space by the semi-inverse method [31].
According to the basic properties given in Eqs. (7) and (8), we have the following two-scale transform [32, 33]

$$t=T^{\alpha} \tag{9}$$

$$x=X^{\beta} \tag{10}$$

Eq. (2) becomes

$$u_t+auu_x+bu_{xx}+cu_{xxx}=0\,, \tag{11}$$

In order to use the semi-inverse method [31] to establish a variational formulation for Eq. (11), we write Eq. (11) in the form

$$u_t+(\frac{1}{2}au^2+bu_x+cu_{xx})_x=0 \tag{12}$$

According to Eq. (12), we can introduce a function φ satisfying

$$\varphi_x = u \tag{13}$$

$$\varphi_t = -(\frac{1}{2}au^2 + bu_x + cu_{xx}) \tag{14}$$

We want to establish a variational formulation for the problem

$$J(u,\varphi) = \iint L(u,u_t,u_x,u_{xx},u_{xxx},\varphi,\varphi_x,\varphi_{xx},\varphi_{xxx})dxdt \tag{15}$$

where L is the trial-Lagrange function.
By the semi-inverse method [31], we assume that the trial-Lagrange function can be written in the form

$$L = u\varphi_t + (\frac{1}{2}au^2 + bu_x + cu_{xx})\varphi_x + F \tag{16}$$

where F is an unknown function of u and/or φ and/or their derivatives. If F is free from φ and its derivatives, the stationary condition with respect to φ is Eq. (2). The semi-inverse method is a useful mathematical tool to establishment of a needed variational formulation from governing equations [8-11, 34-39].
The stationary condition with respect to u reads

$$\varphi_t + au\varphi_x - b\varphi_{xx} + c\varphi_{xxx} + \frac{\delta F}{\delta u} = 0 \tag{17}$$

where $\delta F / \delta u$ is the variational derivative defined as

$$\frac{\delta F}{\delta u} = \frac{\partial F}{\partial u} - \frac{\partial}{\partial t}(\frac{\partial F}{\partial u_t}) - \frac{\partial}{\partial x}(\frac{\partial F}{\partial u_x}) + \frac{\partial^2}{\partial t^2}(\frac{\partial F}{\partial u_{tt}}) + \frac{\partial^2}{\partial t \partial x}(\frac{\partial F}{\partial u_{tx}}) + \frac{\partial^2}{\partial x^2}(\frac{\partial F}{\partial u_{xx}}) - \cdots \tag{18}$$

In view of Eqs. (13) and (14), we have

$$\begin{aligned}\frac{\delta F}{\delta u} &= -\varphi_t - au\varphi_x + b\varphi_{xx} - c\varphi_{xxx} \\ &= \frac{1}{2}au^2 + bu_x + cu_{xx} - au^2 + bu_x - cu_{xx} \\ &= -\frac{1}{2}au^2 + 2bu_x\end{aligned} \tag{19}$$

From Eq. (19), we cannot identify F, so we have to modify the trial-Lagrange function in the form [37,38]

$$L = Au\varphi_t + B\varphi_x\varphi_t + (\frac{1}{2}au^2 + bu_x + cu_{xx})\varphi_x + F \tag{20}$$

The Euler-Lagrange equations are

$$-Au_t - 2B\varphi_{xt} - (\frac{1}{2}au^2 + bu_x + cu_{xx})_x + \frac{\delta F}{\delta \varphi} = 0 \tag{21}$$

$$A\varphi_t + au\varphi_x - b\varphi_{xx} + c\varphi_{xxx} + \frac{\delta F}{\delta u} = 0 \tag{22}$$

In view of Eqs. (13) and (14), we have

$$\frac{\delta F}{\delta \varphi} = Au_t + 2B\varphi_{xt} + (\frac{1}{2}au^2 + bu_x + cu_{xx})_x = (A+2B)u_t + (\frac{1}{2}au^2 + bu_x + cu_{xx})_x \tag{23}$$

$$\begin{aligned}\frac{\delta F}{\delta u} &= -A\varphi_t - au\varphi_x + b\varphi_{xx} - c\varphi_{xxx} \\ &= A(\frac{1}{2}au^2 + bu_x + cu_{xx}) - au^2 + bu_x - cu_{xx} \\ &= (\frac{1}{2}A - 1)au^2 + (A+1)bu_x + (A-1)cu_{xx}\end{aligned} \tag{24}$$

Setting

$$\frac{\delta F}{\delta \varphi} = 0 \tag{25}$$

and

$$A + 2B = 1 \tag{26}$$

Eq. (21) turns out to be Eq. (12). Setting the coefficient of u_x to be zero in Eq. (24)

$$A + 1 = 0 \tag{27}$$

we obtain

$$\frac{\delta F}{\delta u} = -\frac{3}{2}au^2 - 2cu_{xx} \tag{28}$$

From Eq. (28), F can be identified as

$$F = -\frac{1}{2}au^3 + c(u_x)^2 \tag{29}$$

Finally we obtain the following Lagrange function

$$J(u,\varphi) = \iint \left\{ -u\varphi_t + \varphi_x\varphi_t + (\frac{1}{2}au^2 + bu_x + cu_{xx})\varphi_x - \frac{1}{2}au^3 + c(u_x)^2 \right\} dxdt \tag{30}$$

which is subject to Eq. (13).

Proof. The Euler-Lagrange equations of Eq. (30) are

$$u_t - 2\varphi_{xt} - (\frac{1}{2}au^2 + bu_x + cu_{xx})_x = 0 \tag{31}$$

$$-\varphi_t + au\varphi_x - b\varphi_{xx} + c\varphi_{xxx} - \frac{3}{2}au^2 - 2cu_{xx} = 0 \tag{32}$$

In view of the constraint, Eq. (13), it is easy to prove that Eqs. (31) and (32) are equivalent to, respectively, Eq. (12) and Eq. (14).

In the fractal space (X^β, T^α), the variational formulation can be written in the form

$$J(u,\varphi) = \iint \left\{ -u\frac{\partial \varphi}{\partial T^\alpha} + \frac{\partial \varphi}{\partial X^\beta}\frac{\partial \varphi}{\partial T^\alpha} + (\frac{1}{2}au^2 + b\frac{\partial u}{\partial X^\beta} + c\frac{\partial^2 u}{\partial X^{2\beta}})\frac{\partial \varphi}{\partial X^\beta} - \frac{1}{2}au^3 + c(\frac{\partial u}{\partial X^\beta})^2 \right\} dX^\beta dT^\alpha \tag{33}$$

which is subject to Eq. (13).

3. Conclusion

This paper established a variational formulation for the generalized KdV-Burgers equation in a fractal space

(X^{β}, T^{α}) by the semi-inverse method. The variational principle suggested possible conservation laws and possible solution structures, and it provided a theoretical basis for both the numerical and analytical methods.

Conflict of Interest

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Nomenclature

a, b, c	Constants	α	Fractal dimension in time
(X, T)	Coordinates on a large space	β	Fractal dimension in space
(x,t)	Coordinates on a small space	φ	Potential function

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