Numerical Solution of Time Fractional Cable Equation via the Sinc-Bernoulli Collocation Method

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Abstract. An important equation usually used in modeling neuronal dynamics is cable equation. In this work, a numerical method for the fractional cable equation which involves two Riemann-Liouville fractional derivatives is proposed. Our computational technique is based on collocation idea where a combination of Bernoulli polynomials and Sinc functions are used to approximate the solution to this problem. The constructed approximation by our method convert the fractional cable equation into a set of algebraic equations. Also, we provide two numerical examples to confirm the accuracy and effectiveness of the present method.

Keywords: Fractional cable equation, Bernoulli polynomials, Riemann-Liouville fractional derivative, Sinc function, Numerical solution.

1. Introduction

During the last few decades, a lot of attention has been paid to the fractional differential equations (FDEs) because of their extensive engineering applications. For example, these equations frequently appear in modeling many areas of fluid mechanics, viscoelasticity, biology, pharmacy, and control systems [16, 24]. Today, different numerical methods have been used to solve FDEs (see for example [2, 3, 4, 5, 13, 24, 26, 27, 28, 32-34] and references therein).

The standard model that explains electrodiffusion of ions in nerve cells is given by the following Nernst–Planck equation [17],

$$\frac{\partial C_k}{\partial t} = D_k \frac{\partial^2 C_k}{\partial x^2} + \frac{F z_k}{RT} \frac{\partial V}{\partial x} - \frac{4}{D z_k} \frac{\partial F z_k}{\partial x}$$

where $C_k$ is the concentration of ionic species with diffusivity $D_k$ and charge $z_k$. Also, $V$ is the membrane voltage. We refer to [17] for more details. If we consider slowly varying ionic concentrations along the axial direction, $\partial C_k / \partial t \approx 0$, then the standard cable equation (SCE) is obtained. To cover the anomalous diffusion in the movement of the ions, SCE needs to be modified. In recent years, to model anomalous electrodiffusion of ions in spiny dendrites, the fractional cable equation (FCE) from the fractional Nernst-Planck equation is derived (see [17, 56] and references therein).

In this study, we consider the following FCE [35]:

$$\frac{\partial u(x,t)}{\partial t} = D_1^{-\gamma} \left( \frac{\partial^2 u(x,t)}{\partial x^2} \right)^{-\nu} D_1^{-\gamma} u(x,t) + g(x,t), \quad (x,t) \in \Omega \times [0,T),$$

with initial and boundary conditions

$$u(x,0) = \theta(x), \quad x \in \Omega,$$

$$u(x,t) = 0, \quad (x,t) \in \partial \Omega \times [0,T].$$

Here $\Omega = (0,L); 0 < \gamma_1, \gamma_2 < 1; \nu > 0$ and $\mu^2$ are constants. Also, $g(x,t)$ is a given function and $u(x,t)$ is an unknown function. In Equation (1) the fractional derivatives are Riemann-Liouville type and are defined by:

Definition 1.1. Suppose that $u(t)$ be in $C[a,b]$ and $n - 1 < \gamma < n$, then the Riemann-Liouville fractional derivative of order $\gamma$ is
defined by [23]
\[ D^\gamma_t u(t) = \frac{1}{\Gamma(n-\gamma)} \frac{d^{n-\gamma}}{dt^{n-\gamma}} \int_0^t (t-s)^{\gamma-1} u(s) ds, \quad t \in [0, T], \]
where, \( \Gamma \) denoting the Gamma function and we have [16, 23]
\[ D^\gamma_t (\tau_1 u(t) + \tau_2 u(t)) = \tau_1 D^\gamma_t u_1(t) + \tau_2 D^\gamma_t u_2(t), \quad (\tau_1 \text{ and } \tau_2 \text{ are constants}), \]
(4)
\[ D^\gamma_t t^n = \frac{\Gamma(n+1)}{\Gamma(n+1-\gamma)} t^{n-\gamma}, \quad n > -1. \] (5)

In recent years, a lot of numerical methods to solve problem (1)-(3) have been devised such as homotopy analysis method [12], implicit numerical methods [19], finite element method [35], implicit compact difference scheme [11], Galerkin finite element method [36], finite difference/ Legendre spectral schema [18], meshless method [8], explicit numerical method [25], and discontinuous Galerkin finite element method [33]. We also refer the interested reader to [21, 31].

The main purpose of this work is to solve problem (1)-(3) by the Sinc-Bernoulli collocation method. In our method, \( u(x,t) \) is approximated by the Bernoulli polynomials in time direction and the Sinc functions in space direction. Our method has the advantage of converting the solution of FCE given (1)-(3) into the solution of algebraic equations. Therefore, the computation becomes very simple and the corresponding algorithm is computer-oriented. Numerical methods related to the Sinc functions was developed as an efficient method in [20, 30]. These methods are widely used for solving many problems arising in applied sciences. Also, these methods are used as an efficient and effective tool for solving fractional or ordinary differential equations, the main reason being two: first, their desirable behavior towards singularity problems associated with the simple implementation of the method, and second, because of their exponential convergence rate [20, 30]. For some recently published papers on Sinc methods, we refer the interested reader to [1, 6, 9, 22, 23, 27, 28, 29] and references therein.

This paper is organized as follows: The next section is devoted to the some basic definitions and results of Bernoulli polynomials and Sinc functions. In Section 3, we construct the Sinc-Bernoulli collocation method for solving the problem (1)-(3). In Section 4, we obtain the error bound. Numerical simulations are reported in Section 5.

2. Mathematical Preliminaries

2.1 Sinc functions

The Sinc function is defined by [30]
\[ \text{Sinc}(x) = \frac{\sin(\pi x)}{\pi x}, \quad x \in \mathbb{R}. \]
Also, for a given \( h > 0 \) and \( i = 0, \pm 1, \pm 2, \ldots \), the \( i \) th Sinc functions are defined as
\[ S(i,h)(x) = \text{Sinc}\left(\frac{x - ih}{h}\right). \]
Let \( h > 0 \). Then \( g(x) \), which is defined on \( \mathbb{R} \), has the cardinal series representation of the form
\[ C(g,h)(x) = \sum_{i=-\infty}^{\infty} g_i S(i,h)(x), \quad g_i = g(ih), \]
whenever this series converges. \( C(g,h)(x) \) is called the whittaker cardinal expansion of \( g \) and its properties have been broadly studied in [20]. Our aim in this paper, is to construct the approximation over \( \Omega \). For this aim, we use the conformal one-to-one map
\[ \phi(z) = \log\left(\frac{z}{L-z}\right), \]
which maps \( D = \{ z \in \mathbb{C}, \arg(z/L-z) < d \leq \pi/2 \} \), onto the strip
\[ D_\phi = \{ z \in \mathbb{C}, |\phi(z)| < d \}. \]
Note that, \( \phi(\Omega) = \mathbb{R}, \phi(0) = -\infty \) and \( \phi(L) = \infty \). The Sinc basis functions on \( (0,L) \) are defined as
\[ S_L(x) = S(i,h)\phi(x) = \text{Sinc}\left(\frac{\phi(x) - ih}{h}\right). \]
Here \( S(i,h)\phi(x) \) is defined by \( S(i,h)(\phi(x)) \).

Also, for \( h > 0 \), a set of Sinc points \( x_r \) on \( \Omega \) is defined by
\[ x_r = \phi^{-1}(rh) = \frac{L\exp(rh)}{1 + \exp(rh)}, \quad r = 0, \pm 1, \pm 2, \ldots. \] (6)

**Definition 2.1.** ([30]). The space \( L_2(D) \) is the family of all analytic functions \( u \) in \( D \), such that
\[
|u(x)| \leq C_1 \frac{|x|^{\alpha}}{(1 + |x|^{\alpha})^\beta}, \quad x \in D, \ 0 < \alpha \leq 1,
\]

where \( C_1 \) is a constant and \( \beta(x) = e^{\gamma x} \).

The following theorem tells us that Sinc interpolation on \( L(D) \) converge exponentially.

**Theorem 2.2.** ([30]) Let \( u(x) \) be in \( L(D) \), \( 0 < \alpha \leq 1 \) and \( d > 0 \), let \( m \) be a positive integer, and let \( h = \sqrt{d/m} \), then there is a constant \( C_1 \), which is independent of \( m \) such that

\[
\sup_{x \in \Omega} |u(x) - \sum_{l=0}^{m} u(x_l)S_l(x)| \leq C_1 \exp(-\sqrt{d/m})
\]

Also, we need the following relationships [30].

\[
\epsilon^{(0)}_{ij} = [S(x)]_{x=x_0} = \begin{cases} 1, & i = r, \\ 0, & i = r, \end{cases}
\]

\[
\epsilon^{(1)}_{ij} = \frac{d}{dx^1} [S(x)]_{x=x_0} = \begin{cases} 0, & i = r, \\ \frac{(1-\gamma)^{r-1}}{r}, & i = r, \end{cases}
\]

\[
\epsilon^{(2)}_{ij} = \frac{d^2}{dx^2} [S(x)]_{x=x_0} = \begin{cases} \frac{(1-\gamma)^{r-1}}{r}, & i = r, \\ \frac{(1-\gamma)^{r-1}}{r}, & i = r, \end{cases}
\]

### 2.2 The shifted Bernoulli polynomials

The Bernoulli basis polynomials of order \( i \) are defined as [15]

\[
\beta_i(x) = \sum_{j=0}^{i} \frac{B_j}{j!} x^{j-1}, \quad x \in [0, 1],
\]

Here, \( B_j = \beta_j(0), \ j = 0, 1, 2, \ldots, i \) are called the Bernoulli numbers. These numbers can be obtained by using the generating function

\[
\frac{x}{e^x - 1} = \sum_{j=0}^{\infty} B_j \frac{x^j}{j!},
\]

where \( B_{2j+1} = 0 \) for \( j \in \mathbb{N} \) (for more details see [15]). As mentioned in [14], the Bernoulli polynomials belong to the space of \( L([0, 1]) \) and \( Y = \text{span} \{ \beta_0(x), \ldots, \beta_i(x) \} \) is a complete subspace of \( L([0, 1]) \). Now suppose that \( f(x) \in L([0, 1]) \), since \( Y \) is a finite subspace of \( L([0, 1]) \), according to the best approximation, there exist the unique coefficients \( \{ c_i \}_{i=0}^{\infty} \) such that [14]

\[
f(x) \approx f_\infty(x) = \sum_{i=0}^{\infty} c_i \beta_i(x).
\]

In fact, \( f_\infty(x) \) is the best approximation to \( f \) out of \( Y \).

In the following theorem, an explicit formula for Bernoulli numbers is given. This formula specifies the relationship between Bernoulli numbers and Striling numbers.

**Theorem 2.3.** ([10]). For \( i \in \mathbb{N} \), the Bernoulli numbers \( B_i \) can be computed by

\[
B_i = 1 + \sum_{j=1}^{i-1} \frac{S(1+2l,j+1)S(2l,2l-j)}{(2j)!} \frac{2j}{(2l+1)} \sum_{k=1}^{\infty} \frac{S(2l,j)S(1+2l,2l+1-j)}{(2k)!} \frac{(1-\gamma)^{r-1}}{(r-1)!(2l+1-j)!},
\]

where

\[
S(r,s) = \frac{1}{s!} \sum_{k=0}^{s-r} (-1)^{s-r-k} \frac{s!}{k!(s-k)!} k^r, \quad 1 \leq s \leq r,
\]

are Striling numbers of the second kind.

To apply the Bernoulli polynomials \( \beta_i(x) \) on \([0, T]\), we use the change of variable \( t = xT \). Then the shifted Bernoulli polynomials \( \tilde{\beta}_i(t) = \beta_i(t/T) \), for \( t \in [0, T] \), are obtained as:

\[
\tilde{\beta}_i(t) = \sum_{j=0}^{i} \frac{B_j}{j!} \left( \frac{1}{T} \right)^{j-i} B_j t^{j-i}.
\]

Now, we obtain the Riemann-Liouville fractional derivative of \( \tilde{\beta}_i(t) \).
Theorem 2.4. Assume that $\tilde{\beta}_i(t)$ is a shifted Bernoulli polynomial of degree $i$ as (11) and also let $\gamma > 0$, then

$$D^{-\gamma}_{t} \tilde{\beta}_i(t) = \sum_{l=0}^{i} Z^{(l)}_\gamma t^{-l-\gamma}, \quad i = 0, 1, \ldots,$$

where

$$Z^{(l)}_\gamma = \frac{\tilde{\beta}_l(t)}{\Gamma(1+1-l-\gamma)} \left(\frac{1}{T}\right)^{1-l} B_{l},$$

Proof. Using Equations (4), (5) and (11), the proof is straightforward. It is worth to mention here that, the ordinary first-order derivative of $\tilde{\beta}_i(t)$ can be determined as

$$\frac{d \tilde{\beta}_i(t)}{dt} = \sum_{l=1}^{i} Z^{(l)}_\gamma t^{-l-1}, \quad i = 1, 2, \ldots.$$

3. The Sinc-Bernoulli Collocation Method

To solve the FCE as given in (1)-(3), we use $(n+1)$ terms of shifted Bernoulli polynomials, in time, and $(2m+1)$ terms of Sinc functions, in space, to approximate $u(x,t)$. Thus, we have

$$u(x,t) \approx u_{m,n}(x,t) = \sum_{j=0}^{n} \sum_{i=0}^{m} u_{j}(x) \tilde{\beta}_i(t),$$

where $\{u_{j}\}$ are $(n+1)(2m+1)$ unknown coefficients. Note that $u_{m,n}(x,t)$ satisfies the boundary conditions in (3), since when $x$ tend to 0 and $t$ then $\tilde{\beta}_i(t)$ tend to zero.

Theorem 3.1. Let $u(x,t)$ is approximated by $u_{m,n}(x,t)$ as (13). Also, let $0 < \gamma_1, \gamma_2 < 1$ and $x_r$ be Sinc points. Then we have

$$D^{-\gamma_1}_{t} u_{m,n}(x,t) = \sum_{j=0}^{n} \sum_{i=0}^{m} u_{j}(x) Z^{(i)}_{\gamma_1} t^{-i-\gamma_1},$$

$$D^{-\gamma_2}_{x} u_{m,n}(x,t) = \sum_{j=0}^{n} \sum_{i=0}^{m} u_{j}(x) Z^{(i)}_{\gamma_2} t^{-i-\gamma_2},$$

where $E_x = \tilde{\phi} (x) \tilde{\phi}_{1}^{(1)} + \tilde{\phi} (x) \tilde{\phi}_{1}^{(2)}$.

Proof. Using Equations (7), (12) and (13), we obtain

$$\frac{\partial u_{m,n}(x,t)}{\partial t} = \sum_{j=0}^{n} \sum_{i=0}^{m} u_{j}(x) Z_{\gamma_1}^{(i)} t^{-i-1},$$

$$\frac{\partial^2 u_{m,n}(x,t)}{\partial x^2} = \sum_{j=0}^{n} \sum_{i=0}^{m} u_{j}(x) Z_{\gamma_2}^{(i)} t^{-i-1}.$$

Similarly, by using Theorem 2.4 and Equation (7), we have

$$D^{-\gamma_2}_{t} u_{m,n}(x,t) = \sum_{j=0}^{n} \sum_{i=0}^{m} u_{j}(x) \tilde{\beta}_i(t),$$

$$D^{-\gamma_1}_{x} u_{m,n}(x,t) = \sum_{j=0}^{n} \sum_{i=0}^{m} u_{j}(x) \tilde{\beta}_i(t).$$

Finally, employing Equations (8), (9) and Theorem 2.4 we obtain

$$D^{-\gamma_1}_{t} \left( \frac{\partial^2 u_{m,n}(x,t)}{\partial x^2} \right) = \sum_{j=0}^{n} \sum_{i=0}^{m} u_{j}(x) \left( \frac{d S(x)}{dx} \right)_{x=x_r} D^{-\gamma_2}_{x} \tilde{\beta}_i(t),$$

$$= \sum_{j=0}^{n} \sum_{i=0}^{m} u_{j}(x) \left( \tilde{\phi}(x) \tilde{\phi}_{1}^{(1)} + \tilde{\phi}(x) \tilde{\phi}_{1}^{(2)} \right) D^{-\gamma_2}_{x} \tilde{\beta}_i(t),$$

The proof is complete. We now describe the numerical collocation scheme to problem (1)-(3). By substituting $u_{m,n}(x,t)$ into (1) we have
Next, we collocate (14) at the Sinc points \( \{x_j\}_{m-r}^m \) and the shifted Legendre roots \( \{\xi_j\}_{l=0}^n \) of \( L_{n+1}(2(t/t) - 1) \). Here \( L_n(t) \), \(-1 \leq t \leq 1\) is the well-known Legendre polynomial of order \( n \) (see [4]). Then by applying Theorem 3.1, for \( r = -m, \ldots, m \) and \( l = 1, \ldots, n \), we obtain

\[
\sum_{j=1}^{n} \sum_{k=1}^{n} u_j \mathcal{D}_j^{(r)} \mathcal{D}_k^{(s)} u_j \mathcal{D}_k^{(s)} + \sum_{j=1}^{n} \sum_{k=1}^{n} u_j \mathcal{D}_j^{(r)} \mathcal{D}_k^{(s)} u_j \mathcal{D}_k^{(s)} + \mu^2 \sum_{j=1}^{n} \sum_{k=1}^{n} u_j \mathcal{D}_j^{(r)} \mathcal{D}_k^{(s)} u_j \mathcal{D}_k^{(s)} = g(x, t). \tag{15}
\]

Moreover, by substituting \( u_{m,n}(x, t) \) into (2) we have

\[
\sum_{j=1}^{m} \sum_{k=1}^{n} u_j \mathcal{S}_j(x) \mathcal{J}_k(t) = \theta(x). \tag{16}
\]

Finally, the FCE (1)-(3) is reduced to linear algebraic equations (15) and (16) which can be solved for the unknown coefficients \( \{u_j\} \). So by using (15), \( u_{m,n}(x, t) \) can be found. The results of this section can be summarized in the following algorithm.

BEGIN
Input \( m \) and \( n \).

Step 1: Choose the roots of order \( n + 1 \) of shifted Legendre polynomials as \( n + 1 \) collocation points: \( \{\xi_j\}_{l=0}^n \).

Step 2: Select the \( 2m + 1 \) Sinc collocation points \( \{x_j\}_{m-r}^m \) by (6).

Step 3: Create the \( (2m + 1) \times n \) equations by (15).

Step 4: Create the \( 2m + 1 \) equations by (16).

Step 5: Solve the \( (2m + 1) \times (n + 1) \) linear algebraic equations given in step 3 and step 4 for the unknown coefficients \( \{u_j\} \).

Step 6: Find the numerical solution as \( u_{m,n}(x, t) = \sum_{j=1}^{m} \sum_{k=1}^{n} u_j \mathcal{S}_j(x) \mathcal{J}_k(t) \).

END.

4. Error Bounds

In this section, we will find an upper bound for the truncated Sinc-Bernoulli series of \( u(x, t) \). For simplicity, we set \( T = 1 \).

**Lemma 4.1.** ([7]). Suppose \( \psi \in C^{(n+1)}[0, 1] \) and \( v_j(t) \) is the approximation of \( \psi(t) \) by using Bernoulli polynomials as given in (10), then, we have

\[
\|v_j(t) - \psi(t)\| \leq \frac{\Lambda}{(n+1)! \sqrt{3} + 2n},
\]

where \( \Lambda = \max_{t \in [0, 1]} |\psi^{(n+1)}(t)| \).

In order to estimate \( \|u(x, t) - u_{m,n}(x, t)\| \) we need to introduce the space

\[
H_f(\alpha) = \{u(x, t) : u(x, t) = \sum_{j=1}^{m} \psi_j(x) \psi_j(t), \text{ such that } \psi_j(x) \in L_1(D) \text{ and } \psi_j(t) \in C[0, 1] \}
\]

where \( \alpha \in (0, 1) \), \( r \in \mathbb{N} \), and \( J = [0, 1] \times [0, 1] \).

**Theorem 4.2.** Suppose \( u(x, t) \in H_f(\alpha) \) and \( u_{m,n}(x, t) \) is the approximation of \( u(x, t) \) as in Equation (13). Also, suppose \( h = \sqrt{\epsilon \alpha m} \), then

\[
\|u(x, t) - u_{m,n}(x, t)\| \leq K \sqrt{m} \exp(-\frac{(2m + 1)K}{(n+1)! \sqrt{2m + 3}})
\]

where \( K \) is a constant.

**Proof.** Employing triangle inequality, we get

\[
\|u(x, t) - u_{m,n}(x, t)\| \leq \|u(x, t) - \sum_{j=1}^{m} \sum_{k=1}^{n} u_j \mathcal{S}_j(x) \mathcal{J}_k(t)\| + \|\sum_{j=1}^{m} \sum_{k=1}^{n} u_j \mathcal{S}_j(x) \mathcal{J}_k(t)\|.
\]
Since \( u(x,t) \in H^r(t) \), we obtain
\[
\left\| u(x,t) - \sum_{l=-m}^{m} u(x,t)S_l(x) \right\| \leq \sum_{l=-m}^{m} \| w_l(t) \| \left\| v_l(x) - \sum_{m=-l}^{l} v_m(x)S_l(x) \right\|
\]
\[
\leq \sum_{l=1}^{r} \left\| \| w_l(t) \| \right\| \left\| v_l(x) - \sum_{m=-l}^{l} v_m(x)S_l(x) \right\|
\]
(18)

According to Theorem 2.2, one obtains
\[
\zeta \pi \alpha = - \left( \sum_{l=1}^{r} \right) \leq - \left( \sum_{l=1}^{r} \right) \leq - \left( \sum_{l=1}^{r} \right)
\]
\[
\zeta \pi \alpha = - \left( \sum_{l=1}^{r} \right) \leq - \left( \sum_{l=1}^{r} \right) \leq - \left( \sum_{l=1}^{r} \right)
\]
(19)

where \( \zeta, I = 1, \ldots, r \), are constant. Also, since \( w_l(t) \in C([0,1]) \), there exist real numbers \( \{ W_l \}_{l=1}^{r} \) such that \( |w_l(t)| \leq W_l \). Thus, from (18), we conclude that
\[
\left\| u(x,t) - \sum_{m=-l}^{l} u(x,t)S_l(x) \right\| \leq rW \sqrt{m} \exp(-\sqrt{\pi}d\alpha),
\]
(20)

where \( W = \max_{l \in \mathbb{N}} \{ W_l \} \) and \( \zeta = \max_{l \in \mathbb{N}} \{ \zeta_l \} \). Now, since \( |S_l(x)| \leq 1 \), we get
\[
\left\| \sum_{m=-l}^{l} u(x,t)S_l(x) - \sum_{m=-l}^{l} u_mS_l(x) \right\| \leq \sum_{m=-l}^{l} \left\| u_m \right\|
\]
(21)

Also, using Lemma 4.1 we have
\[
\left\| u(x,t) - \sum_{j=0}^{n} u_j \right\| \leq \frac{\eta_i}{(n+1) \sqrt{n+\frac{3}{2}}}, 
\]
(22)

where \( \eta_i = -m, \ldots, m \), are constant. Finally, by using (20) and (21), we have
\[
\left\| \sum_{m=-l}^{l} u(x,t)S_l(x) - \sum_{m=-l}^{l} u_mS_l(x) \right\| \leq \frac{(2m+1) \eta_i}{(n+1) \sqrt{n+\frac{3}{2}}}
\]
(23)

where \( \eta = \max \{ \eta_i \}_{i=-m}^{m} \). Let \( K = \max \{ rW, \eta \} \). The desired result can be achieved by inserting (19) and (22) into (17).

5. Numerical Examples

In this section, we show the effectiveness of our method with two numerical examples. We report the results based on the following discrete \( l^\infty \) and \( l^2 \) errors, which are defined respectively as:
\[
F^\infty \text{ - error} = \max \left\{ |u_{\text{exact}}(x,t) - u_{\text{approx}}(x,t)| \right\}, 
\]
\[
F^2 \text{ - error} = \left\{ \sum_{i=-m}^{m} \left( |u_{\text{exact}}(x,t) - u_{\text{approx}}(x,t)| \right)^2 \right\}^{\frac{1}{2}}.
\]

Here, \( \{ x_i, t_j \} \) are the collocation points as given in Section 3. Also, in all examples we choose \( h = \pi / \sqrt{m} \). It is worth mentioning that all the numerical calculations of this paper are done by Maple software.

**Example 5.1.** Consider the following FCE [12, 8, 35]
\[
\frac{\partial u(x,t)}{\partial t} - D_+^{\alpha} \left( \frac{\partial^2 u(x,t)}{\partial x^2} \right) + D_+^{\alpha} u(x,t) = 2 \left( t + \frac{\pi t^{\gamma_0}}{\Gamma(2 + \gamma_1)} + \frac{\pi t^{\gamma_1}}{\Gamma(2 + \gamma_2)} \right) \sin(\pi x),
\]
with the initial condition \( u(x,0) = 0 \) and homogeneous conditions. Its exact solution is given as \( u(x,t) = t^\alpha \sin(\pi x) \). This problem is solved with the Sinc-Bernoulli collocation method described in Section 3. Surface of the error function \( u_{\text{approx}}(x,t) - u_{\text{approx}}(x,t) \) with \( n = 2 \) and \( m = 50 \) is shown in Figure 1. Also, Table 2 shows the discrete \( l^\infty \) and \( l^2 \) error for \( n = 2 \) and various \( m, \gamma_0, \gamma_1, \text{ and } \gamma_2 \). It is found that in Table 2, as \( m \) increases, \( l^\infty \) - error and \( l^2 \) - error decreases.
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Fig. 1. Surface of the error function obtained for Example 5.1 with $n = 2$, $m = 50$, $\gamma_1 = 0.2$ and $\gamma_2 = 0.7$.

Fig. 2. Surface of the error function obtained for Example 5.2 with $n = 3$, $m = 50$, $\gamma_1 = \gamma_2 = 0.5$.

Table 1. Values of $f^r − \text{error}$ and $f^r − \text{error}$ with $n = 2$, for Example 1.

<table>
<thead>
<tr>
<th>m</th>
<th>$\gamma_1 = \gamma_2 = 0.5$</th>
<th>$\gamma_1 = \gamma_2 = 0.8$</th>
<th>$\gamma_1 = 0.1, \gamma_2 = 0.9$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>$9.34 \times 10^{-7}$</td>
<td>$2.57 \times 10^{-7}$</td>
<td>$9.30 \times 10^{-7}$</td>
</tr>
<tr>
<td>10</td>
<td>$1.01 \times 10^{-3}$</td>
<td>$2.08 \times 10^{-3}$</td>
<td>$9.94 \times 10^{-4}$</td>
</tr>
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<td>15</td>
<td>$1.17 \times 10^{-4}$</td>
<td>$3.67 \times 10^{-4}$</td>
<td>$1.15 \times 10^{-5}$</td>
</tr>
<tr>
<td>20</td>
<td>$2.00 \times 10^{-5}$</td>
<td>$3.28 \times 10^{-5}$</td>
<td>$2.38 \times 10^{-6}$</td>
</tr>
<tr>
<td>25</td>
<td>$5.32 \times 10^{-6}$</td>
<td>$1.87 \times 10^{-6}$</td>
<td>$5.26 \times 10^{-7}$</td>
</tr>
<tr>
<td>30</td>
<td>$1.45 \times 10^{-6}$</td>
<td>$4.77 \times 10^{-6}$</td>
<td>$1.44 \times 10^{-6}$</td>
</tr>
<tr>
<td>40</td>
<td>$1.39 \times 10^{-7}$</td>
<td>$1.38 \times 10^{-7}$</td>
<td>$5.07 \times 10^{-8}$</td>
</tr>
<tr>
<td>70</td>
<td>$4.86 \times 10^{-10}$</td>
<td>$2.17 \times 10^{-9}$</td>
<td>$4.83 \times 10^{-10}$</td>
</tr>
</tbody>
</table>

Table 2. Values of $f^r − \text{error}$ and $f^r − \text{error}$ with $n = 3$, for Example 2.

<table>
<thead>
<tr>
<th>m</th>
<th>$\gamma_1 = \gamma_2 = 0.5$</th>
<th>$\gamma_1 = \gamma_2 = 0.8$</th>
<th>$\gamma_1 = 0.1, \gamma_2 = 0.9$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>$3.02 \times 10^{-4}$</td>
<td>$9.22 \times 10^{-4}$</td>
<td>$3.32 \times 10^{-4}$</td>
</tr>
<tr>
<td>10</td>
<td>$1.09 \times 10^{-4}$</td>
<td>$3.53 \times 10^{-4}$</td>
<td>$1.21 \times 10^{-4}$</td>
</tr>
<tr>
<td>15</td>
<td>$1.50 \times 10^{-5}$</td>
<td>$7.42 \times 10^{-5}$</td>
<td>$1.74 \times 10^{-5}$</td>
</tr>
<tr>
<td>20</td>
<td>$1.32 \times 10^{-5}$</td>
<td>$5.86 \times 10^{-5}$</td>
<td>$1.45 \times 10^{-5}$</td>
</tr>
<tr>
<td>25</td>
<td>$4.36 \times 10^{-6}$</td>
<td>$2.58 \times 10^{-6}$</td>
<td>$4.81 \times 10^{-6}$</td>
</tr>
<tr>
<td>30</td>
<td>$5.69 \times 10^{-7}$</td>
<td>$2.80 \times 10^{-7}$</td>
<td>$6.28 \times 10^{-7}$</td>
</tr>
<tr>
<td>40</td>
<td>$4.21 \times 10^{-8}$</td>
<td>$2.25 \times 10^{-8}$</td>
<td>$4.65 \times 10^{-8}$</td>
</tr>
<tr>
<td>70</td>
<td>$7.54 \times 10^{-12}$</td>
<td>$5.33 \times 10^{-11}$</td>
<td>$8.33 \times 10^{-12}$</td>
</tr>
</tbody>
</table>

Example 5.2. Consider the problem (1)-(3) with $\Omega = (0,1), T = \kappa = \mu = 1$ and

$$g(x,t) = 2t^2 (x^2 - x) + \frac{12(\gamma_1 + 1)}{\Gamma(3 + \gamma_1)} t^{\gamma_1 - 1} + \frac{2(\gamma_2 + 1)}{\Gamma(3 + \gamma_2)} t^{\gamma_2 - 1} + \frac{6t^{\gamma_1 - 1}}{\Gamma(3 + \gamma_1)} (x^2 - x),$$

$$\theta(x) = x^2 - x.$$

This problem has exact solution $u(x,t) = (t^2 + 1)(x^2 - x)$. Surface of the error function $u_{error}(x,t) - u_{ex}(x,t)$ with $n = 3$ and $m = 50$ has been shown in Figure 2. Also, Table 2 represents the $J_e - \text{error}$ and $\bar{J} - \text{error}$ for $n = 3$ and various $m, \gamma_1$, and $\gamma_2$. This table shows that the errors decrease as $m$ increases.

6. Conclusion

In the present work, the combination of shifted Bernoulli polynomials and Sinc functions together with the collocation method were used to reduce the solution of time FCE which involves two fractional derivatives to the solution of a set of algebraic equations. This method was very easy to implement and our numerical results showed that the Sinc-Bernoulli collocation method can solve FCE effectively.

Author Contributions

N. Moshtaghi planned the scheme, initiated the project, and suggested the experiments; A. Saadatmandi conducted the experiments and analyzed the empirical results. The manuscript was written through the contribution of all authors. All authors discussed the results, reviewed, and approved the final version of the manuscript.

Conflict of Interest

The authors declared no potential conflicts of interest with respect to the research, authorship, and publication of this article.

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Numerical Solution of Time Fractional Cable Equation via the Sinc-Bernoulli Collocation Method


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