

Nonlinear transversely vibrating beams by the homotopy perturbation method with an auxiliary term

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Abstract

This paper presents the high order frequency-amplitude relationship for nonlinear transversely vibrating beams with odd and even nonlinearities, using Homotopy Perturbation Method with an auxiliary term (HPMAT). The governing equations of vibrating buckled beam, beam carrying an intermediate lumped mass, and quintic nonlinear beam are investigated to exhibit the reliability and ability of the proposed asymptotic approach. It is demonstrated that two terms in series expansions are sufficient to obtain a highly accurate periodic solutions. The integrity of the analytical solutions is verified by numerical results.

Keywords: Homotopy Perturbation Method with an Auxiliary Term, Non-linear vibrating beams, Frequency-amplitude relationship.

1. Introduction

Mathematical modeling of many engineering systems such as beam structures often leads to nonlinear ordinary or partial differential equations. Structures like helicopter rotor blades, space craft antennae, flexible satellites, airplane wings, robot arms, high-rise buildings, long-span bridges and drill strings can be modeled as a beam-like member. An exact formulation of the beam problem was first investigated in terms of general elasticity equations by Love [1]. The problem of the vibrating beams was formulated in terms of the partial differential equation of motion by many researchers [2-19] with different boundary conditions. Sedighi et al. [2] have presented the advantages of some effective analytical approaches such as Min-Max Approach, Parameter Expansion Method, Hamiltonian Approach, Variational Iteration Method and Energy Balance Method on the asymptotic solutions of governing equation of transversely vibrating cantilever beams. The analytical expression for geometrically non-linear vibration of clamped-clamped Euler-Bernoulli beams including non-linear strain-displacement relationship has been obtained by Barari et al. [3]. The application of new equivalent functions for dead-zone and preload nonlinearities on the dynamical behavior of beam vibration using Parameter Expansion Method (PEM) have been investigated by Sedighi et al. [4-9]. Nikkiah Bahrami et al. [10] used Modified wave approach to calculate the natural frequencies and mode shapes of arbitrary non-uniform beams. Non-linear modal analysis of rotating beams studied by [11, 12]. Zohoor and Kakavand [12] obtained the equations of motion, for three-dimensional rotating Euler-Bernoulli and Timoshenko beams on flying supports using finite element method. When the vibration amplitudes are moderate or large, the geometric nonlinearity must be included. Numerical solutions such as finite element and boundary element methods have no capability of giving parametric solutions. Therefore, they cannot be used to investigate the global and qualitative behavior of the system. Some approximate solutions such as perturbation methods can overcome

deficiencies of exact and numerical methods.

In recent years, substantial progress has been made in analytical solutions for nonlinear equations without small parameters. Such methods have also been used to find approximate solutions to nonlinear oscillators (REF). There have been several classical approaches employed to solve the governing nonlinear differential equations to study the nonlinear vibrations including perturbation methods: He's Max-Min Approach (MMA) [20], Homotopy analysis Method [21, 22], Variational Iteration Method-II [23], Variational Approach [24], Homotopy Perturbation Method (HPM) [25], Adomian decomposition [26], Energy Balance Method [27, 28], Multistage Adomian Decomposition Method [29], Variational Iteration Method [30, 31], Monotone Iteration Schemes [32], Hamiltonian Approach [33], Navier and Levy-type solution [34, 35] and Parameter Expansion Method (PEM) [36].

In order to exhibit the accuracy of HPMAT to approximate the solution of nonlinear oscillators, different nonlinear ordinary differential equations of vibrating beams are considered in this paper. The second-order frequency-amplitude relation is achieved using a novel method called HPMAT. The results presented in this paper demonstrate that Homotopy Perturbation Method with an auxiliary term is very effective and convenient for nonlinear beam vibrations in which the highly nonlinear governing equations exist.

2. Homotopy Perturbation Method with an Auxiliary Term (HPMAT)

Consider a general nonlinear governing equation

$$L(q) + N(q) = 0 \quad (1)$$

where L and N are the linear and nonlinear operator, respectively. Recently, He [37] proposed the homotopy equation with an auxiliary term as follows

$$\tilde{L}(q) + p(L(q) - \tilde{L}(q) + N(q)) + \alpha p(1-p)q = 0 \quad (2)$$

where α is an auxiliary parameter, p is an embedding parameter $\tilde{L}$ is a linear operator with a possible unknown constant and $\tilde{L}(q) = 0$ can approximately describe the solution property. In the nonlinear vibrating problems the linear operator $\tilde{L}$ is chosen in the form [37]:

$$\tilde{L}(q) = \ddot{q} + \omega^2 q \quad (3)$$

where ω is the frequency which must be determined. The solution of the problem is then expressed in a power series in p . Using the standard perturbation method, the nonlinear governing equation is divided into several consequent linear equations which can be used in order to find the accurate solution of the system (REF).

3. Applications

The applicability of HPMAT approach, as described above, is verified through the solution of different nonlinear governing equations of vibrating beams. To check the validity of the proposed method, the results of the present study are also compared with those obtained from the numerical solutions.

3.1 Uniform beam carrying a lumped mass

Figure 1 shows the schematic representation of the free vibration of a beam carrying a lumped mass along its span. The uniform beam considered in this section has length l and mass m per unit length, hinging on the base to a rotational spring with stiffness K_r , and carries a lumped mass M at an arbitrary intermediate point d along the beam span. By neglecting the rotary inertia and shear deformation, the following non-dimensional parameters are introduced:

$$\zeta = \frac{s}{l}, \eta = \frac{d}{l}, \mu = \frac{M}{ml} \quad (4)$$

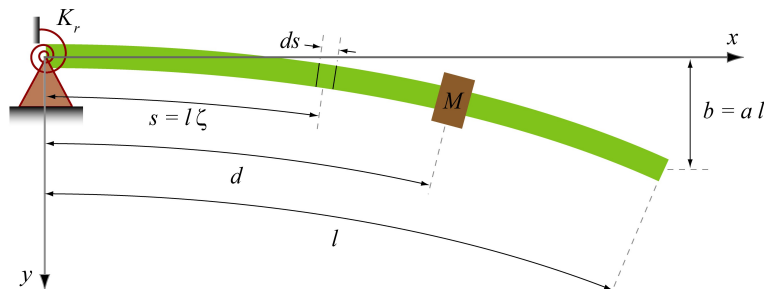


Fig. 1. Configuration of a beam carrying a lumped mass

The kinetic and potential energies of the beam can be expressed as [38]:

$$T = \frac{ml}{2} \int_0^1 [1 + \mu \delta(\zeta - \eta)] (\dot{x}^2 + \dot{y}^2) d\zeta \quad (5)$$

$$U = \frac{EI\lambda^4}{2} \int_0^1 [y''^2 + (\lambda y' y'')^2 + \lambda^4 y'^4 y''^2] d\zeta \quad (6)$$

where $\delta(\zeta - \eta)$ is Dirac's function, $\lambda = 1/l$ and prime represents differentiation with respect to the dimensionless arc length ζ . The solution can be assumed in the form of $y(\zeta, t) = u(t)\phi(\zeta)$, with $\phi(\zeta)$ a normalized eigenfunction of the corresponding linear problem, $u(t)$ the time dependent variable. Using Rayleigh-Ritz procedure with single linear mode, the following Lagrangian function is obtained,

$$L = \frac{ml}{2} \left[\alpha_1 \dot{u}^2 + \alpha_3 \lambda^2 u^2 \dot{u}^2 + \alpha_4 \lambda^2 u^2 \dot{u}^2 + \alpha_5 \lambda^4 u^4 \dot{u}^2 + \alpha_6 \lambda^4 u^4 \dot{u}^2 - \frac{EI\lambda^4}{m} (\alpha_2 u^2 + \alpha_7 \lambda^2 u^4 + \alpha_8 \lambda^4 u^6) \right] \quad (7-a)$$

in which

$$\begin{aligned} \alpha_1 &= \int_0^1 \phi^2 d\zeta + \mu \phi^2(\eta), \quad \alpha_2 = \int_0^1 \phi''^2 d\zeta, \quad \alpha_3 = \int_0^1 \left(\int_0^\zeta \phi'^2 d\zeta \right)^2 d\zeta, \quad \alpha_4 = \mu \left[\left(\int_0^\zeta \phi'^2 d\zeta \right)^2 \right]_{\zeta=\eta} \\ \alpha_5 &= \int_0^1 \left[\left(\int_0^\zeta \phi'^2 d\zeta \right) \left(\int_0^\zeta \phi'^4 d\zeta \right) \right] d\zeta, \quad \alpha_6 = \mu \left[\left(\int_0^\zeta \phi'^2 d\zeta \right) \left(\int_0^\zeta \phi'^4 d\zeta \right) \right]_{\zeta=\eta}, \quad \alpha_7 = \int_0^1 \phi'^2 \phi''^2 d\zeta, \quad \alpha_8 = \int_0^1 \phi'^4 \phi''^2 d\zeta \end{aligned} \quad (7-b)$$

In order to eliminate the variable x in the governing equations, the following constraint equation for the inextensional beams is used [39]:

$$(1 + \lambda x')^2 + (\lambda y')^2 = 1 \quad (8)$$

Applying Euler-Lagrange equation, the following temporal governing equation for a restrained uniform beam carrying an intermediate lumped mass is obtained as (see ref. [38]):

$$\ddot{q} + \lambda q + \varepsilon_1 q^2 \ddot{q} + \varepsilon_1 q \dot{q}^2 + \varepsilon_2 q^4 \ddot{q} + 2\varepsilon_2 q^3 \dot{q}^2 + \varepsilon_3 q^3 + \varepsilon_4 q^5 = 0, \quad q(0) = A, \dot{q}(0) = 0, \quad (9)$$

where

$$\varepsilon_1 = \frac{\alpha_3 \alpha_4}{p^4 \alpha_1}, \quad \varepsilon_2 = \frac{\alpha_5 \alpha_6}{p^4 \alpha_1}, \quad \varepsilon_3 = \frac{2\alpha_7}{\alpha_2}, \quad \varepsilon_4 = \frac{2\alpha_8}{\alpha_2} \quad (10)$$

in which dots are derivatives with respect to the dimensionless time $\tau = (EI\lambda^4/m)^{1/2} (\alpha_2/\alpha_1)^{1/2} t$ and the following non-dimensional parameters are utilized:

$$q = \frac{\bar{p}u}{l}, \quad \bar{p}^4 = \frac{m\bar{\omega}^2 l^4}{EI} \quad (11)$$

where $\bar{\omega}$ is the fundamental frequency of the beam. By comparing equations (9) and (1), one can find that the linear and nonlinear parts of the governing equation are:

$$\begin{aligned} L(q) &= \ddot{q} + \lambda q \\ N(q) &= \varepsilon_1 q^2 \ddot{q} + \varepsilon_1 q \dot{q}^2 + \varepsilon_2 q^4 \ddot{q} + 2\varepsilon_2 q^3 \dot{q}^2 + \varepsilon_3 q^3 + \varepsilon_4 q^5 \end{aligned} \quad (12)$$

Accordingly, using equation (12) the Homotopy equation can be constructed as follows:

$$\ddot{q} + \omega^2 q + p \left[(\lambda - \omega^2) q + \varepsilon_1 q^2 \ddot{q} + \varepsilon_1 q \dot{q}^2 + \varepsilon_2 q^4 \ddot{q} + 2\varepsilon_2 q^3 \dot{q}^2 + \varepsilon_3 q^3 + \varepsilon_4 q^5 \right] + \alpha p(1-p)q = 0 \quad (13)$$

Assume that the solution can be expressed in a power series in p as follows:

$$q = q_0 + p q_1 + p^2 q_2 + \dots \quad (14)$$

Substituting equation (14) into (13), and using the standard perturbation method gives the first differential equation as:

$$\ddot{q}_0 + \omega^2 q_0 = 0, \quad q_0(0) = A, \quad \dot{q}_0(0) = 0 \quad (15)$$

Solving equation (15) yields:

$$q_0 = A \cos(\omega \tau) \quad (16)$$

substituting q_0 into the governing equation results into:

$$\begin{aligned} \ddot{q}_1 + \omega^2 q_1 + \left(\lambda A - \frac{1}{2} \varepsilon_1 A^3 \omega^2 + \frac{3}{4} \varepsilon_3 A^3 - A \omega^2 + \frac{5}{8} \varepsilon_4 A^5 + \alpha A - \frac{3}{8} \varepsilon_2 A^5 \omega^2 \right) \cos(\omega \tau) \\ + \left(-\frac{7}{16} \varepsilon_2 A^5 \omega^2 - \frac{1}{2} \varepsilon_1 A^3 \omega^2 + \frac{5}{16} \varepsilon_4 A^5 + \frac{1}{4} \varepsilon_3 A^3 \right) \cos(3\omega \tau) \\ + \left(\frac{1}{16} \varepsilon_4 A^5 - \frac{3}{16} \varepsilon_2 A^5 \omega^2 \right) \cos(5\omega \tau) = 0, \quad q_1(0) = 0, \quad \dot{q}_1(0) = 0 \end{aligned} \quad (17)$$

Eliminating the secular term leads to:

$$c_1(\omega, \alpha) = \lambda A - \frac{1}{2} \varepsilon_1 A^3 \omega^2 + \frac{3}{4} \varepsilon_3 A^3 - A \omega^2 + \frac{5}{8} \varepsilon_4 A^5 + \alpha A - \frac{3}{8} \varepsilon_2 A^5 \omega^2 = 0 \quad (18)$$

Thereby, the first-order approximate solution with assumption $\alpha = 0$ is achieved as:

$$\omega = \sqrt{\frac{8\lambda + 5\varepsilon_4 A^4 + 6\varepsilon_3 A^2}{8 + 4\varepsilon_1 A^2 + 3\varepsilon_2 A^4}} \quad (19)$$

In order to achieve the high accuracy solution, the third part of the right-hand side of equation (14) must be included. A special solution to the equation (17) can be expressed as:

$$q_1(\tau) = -\frac{1}{128} \frac{A^3 (7\varepsilon_2 A^2 \omega^2 + 8\varepsilon_1 \omega^2 - 5\varepsilon_4 A^2 - 4\varepsilon_3)}{\omega^2} \cos(3\omega\tau) - \frac{A^5 (-\varepsilon_4 + 3\varepsilon_2 \omega^2)}{384\omega^2} \cos(5\omega\tau) \quad (20)$$

Using the perturbation method and substituting equation (20) leads to the third differential equation for $q_2(\tau)$.

Solution of the third equation should not contain the so-called secular term $\cos(\omega\tau)$. To ensure so, the right-hand side of this equation should not contain the terms $\cos$, that is, the coefficients of $\cos$ must be zero:

$$c_2(\omega, \alpha) = -\frac{3}{32} A^5 \varepsilon_1 \omega^2 \varepsilon_3 - \frac{45}{256} A^9 \varepsilon_4 \omega^2 \varepsilon_2 - \frac{5}{32} A^7 \varepsilon_4 \omega^2 \varepsilon_1 - \frac{7}{64} A^7 \varepsilon_2 \omega^2 \varepsilon_3 + \frac{7}{32} A^7 \varepsilon_1 \omega^4 \varepsilon_2 + \frac{65}{512} A^9 \varepsilon_2^2 \omega^4 \\ + \frac{5}{64} A^7 \varepsilon_3 \varepsilon_4 + \frac{3}{32} A^5 \varepsilon_1^2 \omega^4 - A \alpha \omega^2 + \frac{95}{1536} A^9 \varepsilon_2^2 + \frac{3}{128} A^5 \varepsilon_3^2 = 0 \quad (21)$$

Solving equations (18) and (21) simultaneously leads to:

$$\omega = \frac{1}{\sqrt{112\varepsilon_2 \varepsilon_1 A^6 + 65\varepsilon_2^2 A^8 + 48\varepsilon_1^2 A^4 - 512 - 192A^4 \varepsilon_2 - 256A^2 \varepsilon_1}} \\ \times \left(24\varepsilon_3 \varepsilon_1 A^4 + 28\varepsilon_3 \varepsilon_2 A^6 - 160\varepsilon_4 A^4 - 192\varepsilon_3 A^2 + 40\varepsilon_1 \varepsilon_4 A^6 - 256\lambda + 45\varepsilon_2 \varepsilon_4 A^8 \right. \\ \left. - \frac{2}{3} [184320\varepsilon_4 A^6 \varepsilon_3 + 94080A^8 \varepsilon_3^2 + 96768\varepsilon_3^2 A^4 - 180\varepsilon_3 \varepsilon_1 A^{12} \varepsilon_2 \varepsilon_4 - 180\varepsilon_2^2 \varepsilon_4 A^{14} \varepsilon_3 - 32256\lambda \varepsilon_3 \varepsilon_2 A^6 + 147456\lambda^2 \right. \\ \left. - 13824\varepsilon_3^2 \varepsilon_1 A^6 + 221184\varepsilon_3 \lambda A^2 - 19008\varepsilon_3^2 \varepsilon_2 A^8 - 10560\varepsilon_4^2 \varepsilon_1 A^{10} + 184320\lambda A^4 - 18720\varepsilon_2^2 \varepsilon_2 A^{12} + 180\varepsilon_1^2 \varepsilon_4^2 A^{12} - 46080\varepsilon_1 \varepsilon_4 \lambda A^6 \right. \\ \left. - 75\varepsilon_2^2 \varepsilon_4^2 A^{16} + 9\varepsilon_3^2 \varepsilon_2^2 A^{12} - 51840\lambda \varepsilon_2 \varepsilon_4 A^8 + 120\varepsilon_1 \varepsilon_4^2 \varepsilon_2 A^{14} - 28800\varepsilon_1 \varepsilon_4 A^8 - 41760\varepsilon_2 \varepsilon_3 \varepsilon_4 A^{10} - 27648\varepsilon_1 \varepsilon_3 \lambda A^4]^{1/2} \right)^{1/2} \quad (22)$$

In order to verify the effectiveness of the proposed approach to approximate the analytical solution, the analytical solutions and numerical results have been plotted simultaneously. As can be seen in Fig. 2, the second order approximation for $q(\tau)$ for different system parameters exhibits an excellent agreement with numerical results.

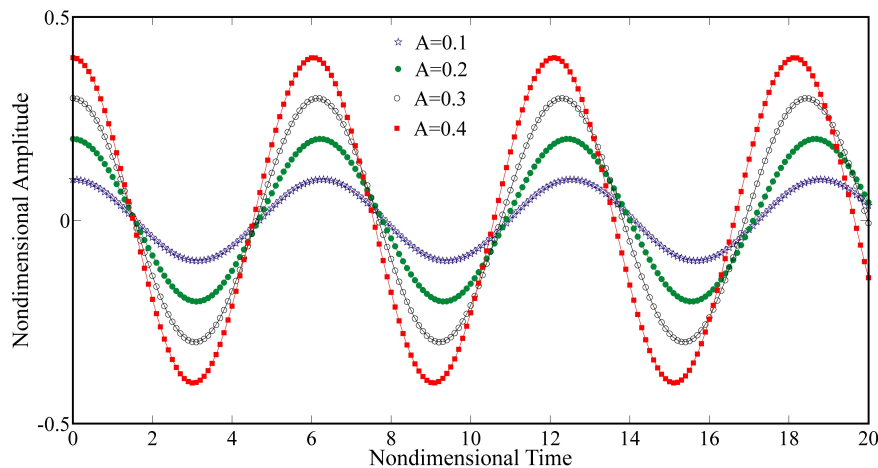


Fig. 2. Comparison between numerical results (solid lines) and analytical solutions (symbols) for different initial conditions

3.2 Transversely vibrating quintic nonlinear beam

In this example, the governing equation of quintic nonlinear beam, shown in Fig. 3, is considered. The governing equation of the hinged–hinged vibrating beam without damping effects subjected to constant axial force P_0 is expressed as [22]:

$$mw_{tt} + EI \left[\frac{27}{2} w_x^2 w_{xx}^3 - 3w_{xx}^3 - 3w_x^2 w_{xxxx} + \frac{9}{4} w_x^4 w_{xxxx} \right] + EI w_{xxxx} + P_0 w_{xx} + \frac{3}{2} P_0 w_x^2 w_{xx} = 0 \quad (23)$$

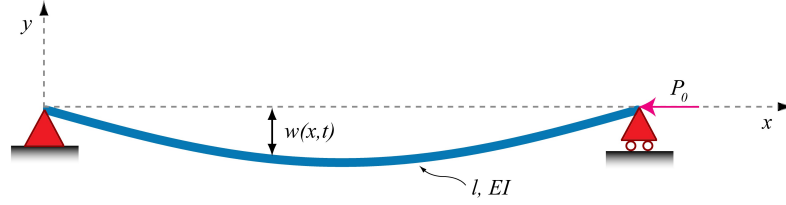


Fig. 3. Schematic representation of transversely vibrating beam

where the symbol w denotes the displacement of the flexible beam which may be represented as $w(x, t) = q(t) \sin \pi x / l$. By introducing the non-dimensional variables,

$$\tau = \sqrt{\frac{EI}{ml^4}}, \quad q^* = \frac{lq}{r^2} \quad (24)$$

where r is radius of gyration of beam cross-section and applying Bubnov-Galerkin decomposition method, the non-linear governing equation of transverse vibrating beams can be extracted as:

$$\frac{d^2 q(\tau)}{d\tau^2} + \omega_n^2 q(\tau) + \beta (q(\tau))^3 + \delta (q(\tau))^5 = 0 \quad (25)$$

where

$$\omega_n^2 = \pi^2 (\pi^2 - \bar{\varepsilon}_0), \quad \beta = -\left(\bar{\varepsilon}_0 + \frac{7\pi^2}{8}\right) \left(\frac{r\pi}{l}\right)^4, \quad \delta = \frac{27}{20} \left(\frac{r\pi}{l}\right)^8, \quad \bar{\varepsilon}_0 = \frac{P_0 l^2}{2EI}, \quad \rho = \frac{r\pi}{l} \quad (26)$$

The homotopy equation for the quintic nonlinear beam vibration can be expressed as:

$$\ddot{q} + \omega^2 q + p \left[(\omega_n^2 - \omega^2) q + \beta q^3 + \delta q^5 \right] + \alpha p (1-p) q = 0 \quad (27)$$

Substituting equation (14) into (27) yields:

$$\ddot{q}_0 + \omega^2 q_0 = 0, \quad q_0(0) = A, \quad \dot{q}_0(0) = 0 \quad (28)$$

$$\ddot{q}_1 + \omega^2 q_1 + \omega_n^2 q_0 - \omega^2 q_0 + \alpha q_0 + \beta q_0^3 + \delta q_0^5 = 0, \quad q_1(0) = 0, \quad \dot{q}_1(0) = 0 \quad (29)$$

Solving equation (28) gives the following initial solution for q_0 as:

$$q_0 = A \cos(\omega \tau) \quad (30)$$

Substituting equation (30) into (29) results in,

$$\begin{aligned} \ddot{q}_1 + \omega^2 q_1 + \left(\frac{5}{8} \delta A^5 + \frac{3}{4} \beta A^3 - A \omega^2 + \alpha A + \omega_n^2 A \right) \cos(\omega \tau) \\ + \left(\frac{1}{4} \beta A^3 + \frac{5}{16} \delta A^5 \right) \cos(3\omega \tau) + \frac{1}{16} \delta A^5 \cos(5\omega \tau) = 0, \quad q_1(0) = 0, \quad \dot{q}_1(0) = 0 \end{aligned} \quad (31)$$

Eliminating the secular term lead to:

$$c_1(\omega, \alpha) = \frac{5}{8} \delta A^5 + \frac{3}{4} \beta A^3 - A \omega^2 + \alpha A + \omega_n^2 A = 0 \quad (32)$$

The first-order approximate solution with assumption $\alpha = 0$ is achieved as:

$$\omega = \sqrt{\omega_n^2 + \frac{3}{4} \beta A^2 + \frac{5}{8} \delta A^4} \quad (33)$$

To achieve the high accuracy solution, the third part of the right-hand side of equation (27) must be included. A particular solution of equation (31) can be expressed as:

$$q_1(\tau) = \frac{1}{128} \frac{A^3 (5\delta A^2 + 4\beta)}{\omega^2} \cos(3\omega \tau) + \frac{\delta A^5}{384\omega^2} \cos(5\omega \tau) \quad (34)$$

Using the perturbation method and substituting equation (34) leads to the third differential equation for $q_2(\tau)$. The solution to this equation should not contain the so-called secular term $\cos(\omega \tau)$; therefore, the coefficients of $\cos$ must be zero:

$$c_2(\omega, \alpha) = -A\alpha\omega^2 + \frac{95}{1536} A^9 \delta^2 + \frac{5}{64} A^7 \delta \beta + \frac{3}{128} \beta^2 A^5 = 0 \quad (35)$$

Solving equations (32) and (35) simultaneously leads to:

$$\omega = \frac{1}{\sqrt{2}} \sqrt{\omega_n^2 + \frac{3}{4} \beta A^2 + \frac{5}{8} \delta A^4 + \frac{1}{96} \sqrt{2304\omega_n^2 + 1512\beta^2 A^4 + 1470\delta^2 A^8 + 3456\omega_n^2 \beta A^2 + 2880\beta \delta A^6 + 2880\omega_n^2 \delta A^4}} \quad (36)$$

In order to demonstrate the integrity of solutions by analytical approach, the analytical results together with the

corresponding numerical results are presented graphically. The second-order approximation of $q(\tau)$, using HPMAT approach, exhibits a good agreement with numerical results from fourth-order Runge–Kutta method, as depicted in Fig. 4.

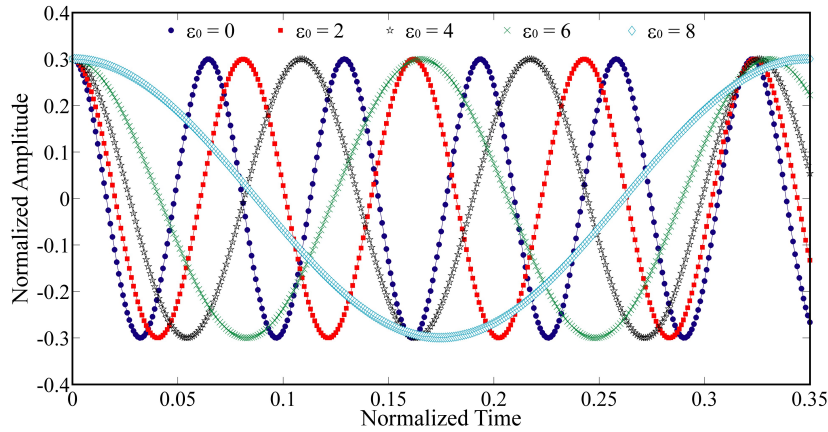


Fig. 4. Comparison between numerical results (solid lines) and analytical solutions (symbols) for different values of $\bar{\varepsilon}_0$

3.3 Buckled beam nonlinear vibration

Assume that the buckled beam, shown in Fig. 5, is the Euler–Bernoulli beam subjected to axial force $\bar{P}_0 + \bar{P}(t)$. The symbol v denotes the dynamic displacement of a point in the middle plane of the flexible beam in z direction. The governing equation of motion of an initially straight buckled beam is given by [40]:

$$m \frac{\partial^2 \bar{v}}{\partial t^2} + EI \frac{\partial^4 \bar{v}}{\partial x^4} + \frac{EA}{l} \frac{\partial^2 \bar{v}}{\partial x^2} \left[\frac{l}{EA} (\bar{P}_0 + \bar{P}(t)) - \frac{1}{2} \int_0^l \left(\frac{\partial \bar{\psi}}{\partial \bar{x}} \right)^2 d\bar{x} \right] - \frac{EA}{l} \frac{\partial^2 \bar{\psi}}{\partial x^2} \int_0^l \frac{\partial \bar{\psi}}{\partial \bar{x}} \frac{\partial \bar{v}}{\partial \bar{x}} d\bar{x} - \frac{EA}{l} \frac{\partial^2 \bar{v}}{\partial x^2} \int_0^l \frac{\partial \bar{\psi}}{\partial \bar{x}} \frac{\partial \bar{v}}{\partial \bar{x}} d\bar{x} - \frac{EA}{2l} \frac{\partial^2 \bar{\psi}}{\partial x^2} \int_0^l \left(\frac{\partial \bar{v}}{\partial \bar{x}} \right)^2 d\bar{x} - \frac{EA}{2l} \frac{\partial^2 \bar{v}}{\partial x^2} \int_0^l \left(\frac{\partial \bar{v}}{\partial \bar{x}} \right)^2 d\bar{x} = -\bar{P}(t) \frac{\partial^2 \bar{\psi}}{\partial x^2} \quad (37)$$

By introducing the spatial and temporal variables as:

$$x = \frac{\hat{x}}{l}, \quad v = \frac{\hat{v}}{r}, \quad t = \hat{t} \sqrt{\frac{EI}{mL^4}}, \quad \rho = \frac{l}{r}, \quad P_0 = \frac{P_0}{EA} \rho^2, \quad P(t) = \frac{\hat{P}(\hat{t})}{EA} \rho^2 \quad (38)$$

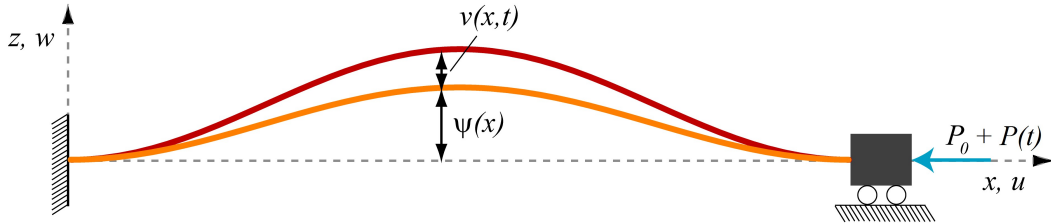


Fig. 5. Configuration of buckled beam subject to axial force

where $r = \sqrt{I/A}$ denotes the radius of gyration of the cross-section, ρ is a non-dimensional parameter which measures the slenderness of the beam and l is the beam length. Thereby, the non-dimensional form of governing equation can be obtained as:

$$\frac{\partial^2 v}{\partial t^2} + \frac{\partial^4 v}{\partial x^4} + \frac{\partial^2 v}{\partial x^2} \left[P_0 + P(t) - \frac{1}{2} \int_0^1 \left(\frac{\partial \psi}{\partial x} \right)^2 dx \right] - \frac{\partial^2 \psi}{\partial x^2} \int_0^1 \frac{\partial v}{\partial x} \frac{\partial \psi}{\partial x} dx - \frac{\partial^2 v}{\partial x^2} \int_0^1 \frac{\partial v}{\partial x} \frac{\partial \psi}{\partial x} dx - \frac{1}{2} \frac{\partial^2 \psi}{\partial x^2} \int_0^1 \left(\frac{\partial v}{\partial x} \right)^2 dx - \frac{1}{2} \frac{\partial^2 v}{\partial x^2} \int_0^1 \left(\frac{\partial v}{\partial x} \right)^2 dx = -P(t) \frac{\partial^2 \psi}{\partial x^2} \quad (39)$$

It is convenient to express the post-buckling deflection as:

$$\psi = b \varphi(x, \lambda) \quad (40)$$

where φ is the buckling mode corresponding to the eigenvalue λ then we have:

$$b^2 = 4(P_0 - P_c) / \pi^2 \quad (41)$$

In which P_c represents the first buckling load for a clamped-clamped boundary conditions. Finally, the nonlinear governing equation of vibrating initially straight buckled beam can be obtained as [40]:

$$\frac{\partial^2 v}{\partial t^2} + \frac{\partial^4 v}{\partial x^4} + P_c \frac{\partial^2 v}{\partial x^2} - b^2 \frac{\partial^2 \varphi}{\partial x^2} \int_0^1 \frac{\partial \varphi}{\partial x} \frac{\partial v}{\partial x} dx = \frac{1}{2} b \frac{\partial^2 \varphi}{\partial x^2} \int_0^1 \left(\frac{\partial v}{\partial x} \right)^2 dx + b \frac{\partial^2 v}{\partial x^2} \int_0^1 \frac{\partial \varphi}{\partial x} \frac{\partial v}{\partial x} dx + \frac{1}{2} \frac{\partial^2 v}{\partial x^2} \int_0^1 \left(\frac{\partial v}{\partial x} \right)^2 dx \quad (42)$$

For a clamped-clamped straight beam, the lowest eigenvalue, the corresponding buckling mode and critical load are:

$$\lambda = 2\pi, \varphi(x) = \frac{1}{2}(1 - \cos 2\pi x), P_c = 4\pi^2 \quad (43)$$

Applying the weighted residual Bubnov-Galerkin method, the governing equation for cantilever beam vibration can be expressed as:

$$\frac{d^2 q}{dt^2} + \beta_1 q(\tau) + \beta_2 (q(\tau))^2 + \beta_3 (q(\tau))^3 = 0 \quad (44)$$

where

$$\begin{aligned} \beta_1 &= \omega_n^2 \\ \beta_2 &= \pi b \left(\int_0^1 \phi' \sin 2\pi x dx \right) \left(\int_0^1 \phi'' \phi dx \right) + \pi^2 b \left(\int_0^1 \phi \cos 2\pi x dx \right) \left(\int_0^1 (\phi')^2 dx \right) \\ \beta_3 &= -\frac{1}{2} \left(\int_0^1 (\phi')^2 dx \right)^2 \end{aligned} \quad (45)$$

in which ω_n represents the linear natural frequency and $\phi(x)$ is the first eigenmode of clamped-clamped beam. To solve nonlinear ordinary equation (44) analytically, the homotopy equation can be written as:

$$\ddot{q} + \omega^2 q + p \left[(\beta_1 - \omega^2) q + \beta_2 q^2 + \beta_3 q^3 \right] + \alpha p(1-p)q = 0 \quad (46)$$

Substituting equation (14) into (44) yields:

$$\ddot{q}_0 + \omega^2 q_0 = 0, \quad q_0(0) = A, \quad \dot{q}_0(0) = 0 \quad (47)$$

$$\ddot{q}_1 + \omega^2 q_1 + \beta_1 q_0 - \omega^2 q_0 + \alpha q_0 + \beta_2 q_0^2 + \beta_3 q_0^3 = 0, \quad q_1(0) = 0, \quad \dot{q}_1(0) = 0 \quad (48)$$

Substituting the initial solution into (48) results into:

$$\begin{aligned} \ddot{q}_1 + \omega^2 q_1 + \left(-A\omega^2 + \alpha A + \frac{3}{4} \beta_3 A^3 + \beta_1 A \right) \cos(\omega\tau) + \frac{1}{2} \beta_2 A^2 (\cos(2\omega\tau) + 1) \\ + \frac{1}{4} \beta_3 A^3 \cos(3\omega\tau) = 0, \quad q_1(0) = 0, \quad \dot{q}_1(0) = 0 \end{aligned} \quad (49)$$

Eliminating the secular term leads to:

$$c_1(\omega, \alpha) = -A\omega^2 + \alpha A + \frac{3}{4} \beta_3 A^3 + \beta_1 A = 0 \quad (50)$$

and the approximate frequency with assumption $\alpha = 0$ is achieved as:

$$\omega = \sqrt{\beta_1 + \frac{3}{4} \beta_3 A^2} \quad (51)$$

a particular solution to equation (49) can be written as:

$$q_1(\tau) = \frac{1}{6} \frac{\beta_2 A^2}{\omega^2} (\cos(2\omega\tau) - 3) + \frac{\beta_3 A^3}{32\omega^2} \cos(3\omega\tau) \quad (52)$$

Substituting equation (52) leads to the third governing equation for $q_2(\tau)$ which should not contain the so-called secular term $\cos(\omega\tau)$, therefore, the coefficients of $\cos$ must be zero:

$$c_2(\omega, \alpha) = -A\alpha\omega^2 + \frac{1}{6} A^3 \beta_2^2 + \frac{3}{128} A^5 \beta_3^2 = 0 \quad (53)$$

Solving equations (50) and (53) simultaneously leads to:

$$\omega = \frac{1}{\sqrt{2}} \sqrt{\beta_1 + \frac{3}{4} \beta_3 A^2 + \frac{1}{24} \sqrt{576\beta_1^2 + 378\beta_3^2 A^4 + 864\beta_3\beta_1 A^2 + 384\beta_2^2 A^2}} \quad (54)$$

To illustrate of validity of the proposed approach for this example, the results are compared with the numerical solution using fourth-order Runge–Kutta method. As can be seen in Fig. 6, the second approximations of $q(\tau)$ using analytical method for different values of parameters, shows an excellent agreement with numerical results.

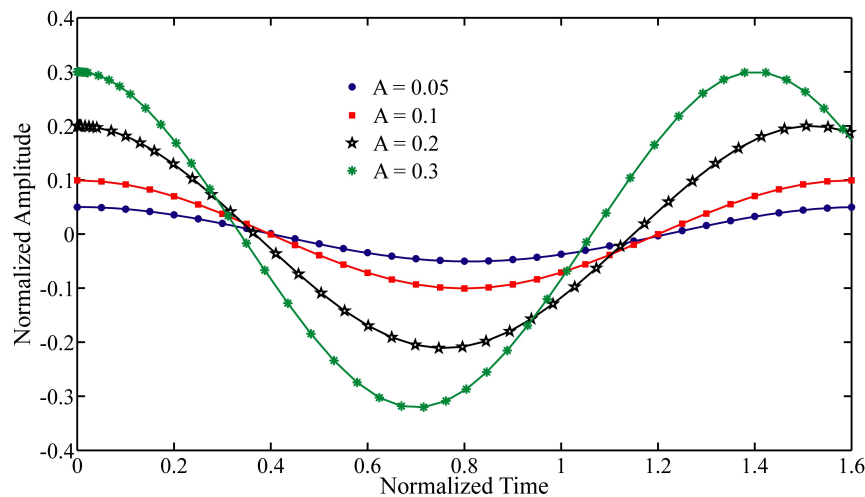


Fig. 6. Comparison between numerical results (solid lines) and analytical solutions (symbols) for different initial conditions

4. Conclusions

In this research, Homotopy perturbation method with an auxiliary term was employed to solve different governing equations of nonlinear vibrating beams. In order to exhibit the accuracy of this approximate approach, different governing equations of nonlinear oscillators were studied. By employing this modern approach, high order frequency-amplitude relation as a function of system parameters was obtained. It is clearly demonstrated that this asymptotic approach is very effective and straightforward to study the nonlinear oscillator systems includes odd and even nonlinear terms. The integrity of the obtained analytical solutions is verified by numerical methods. The accuracy of the results demonstrates that the method can be potentiality used for the analysis of strongly nonlinear oscillation problems.

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