Journal of Applied and Computational Mechanics, Vol. 1, No. 2, (2015), 59-66
DOI: 10.22055/jacm.2014.10756

Shahid Chamran University

Journal of
Applied and Computational Mechanics

Solution of strongly nonlinear oscillator problem arising in Plasma Physics with Newton Harmonic Balance Method

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Received June 21 2014; revised July 27 2014; accepted for publication August 13 2014.
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Abstract

In this paper, Newton Harmonic Balance Method (NHBM) is applied to obtain the analytical solution for an electron beam injected into a plasma tube where the magnetic field is cylindrical and increases towards the axis in inverse proportion to the radius. Periodic solution is analytically verified and consequently the relation between the Natural Frequency and the amplitude is obtained in an analytical form. A comparison of the period of the oscillation and obtained solution with the exact result illustrates that the NHBM is a powerful and efficient tool for solving nonlinear vibration equations.

Keywords: Electron beam, Frequency–Amplitude Relation, Plasma Physics, Newton Harmonic Balance Method

1. Introduction

Oscillation systems have been widely used in many areas of physics and engineering [1]. These systems have significant importance in engineering, particularly in mechanical and structural dynamics because there are many practical engineering components consisting of vibrational elements that can be modeled using oscillatory systems such as elastic beams or mass-on-moving belt or nonlinear pendulum and vibration of a milling machine [2-4].

Mainly, nonlinear vibrations of oscillatory systems are modeled by nonlinear differential equations. Obtaining exact solution for these nonlinear problems is difficult and time-consuming, thus researchers tried to find new approaches to overcome it. Recently, many authors used various analytical methods to solve nonlinear equations in mechanical systems. Some kind of these methods like Homotopy Perturbation Method (HPM), Homotopy Analysis Method (HAM), Parameter Expanding Method (PEM) and Variational Iteration Method (VIM) are powerful methods and can used for almost all types of nonlinear equations [5-16]. Some other methods like Frequency Amplitude Formulation (FAF), Max-Min Approach (MMA), Energy Balance Method (EBM), Harmonic Balance Method (HBM) and Newton Harmonic Balance Method (NHBM) were introduced for oscillatory systems [17-26]. In these types of methods, the results are achieved by obtaining the motion frequency and having the initial conditions.

NBHM incorporates both Newton's Method and Harmonic Balance Method. Wu et al. [24] introduced this method and applied it to three examples. Lai et al. [25] analyzed first, second and third order analytical approximation for a second order differential equation with cubic quantic nonlinearities. Also, they compared various orders of obtained frequency with exact one.

In the current research, NHBM is applied to obtain analytical approximate solution for an electron beam into a plasma tube in which the restoring force is inversely proportional to the dependent variable. This singular nonlinear oscillator has been recently studied by some researchers using a modified generalized method; the parameter expanding method [14], the homotopy perturbation method [16], rational harmonic balance method [19], the standard harmonic balance method [19,23] and coupled method [27].

2. Governing Equation

Now, we consider a problem of some importance in plasma physics concerning an electron beam injected into a plasma tube where the magnetic field is cylindrical and increases towards the axis in inverse proportion to the radius. The beam is injected parallel to the axis, but the magnetic field bends the path towards the axis. The governing equation for the path $u(x)$ of the electrons is as follows:

$$\ddot{u}+\frac{1}{u}=0, \tag{1}$$

With initial conditions:

$$u(0)=A \qquad \dot{u}(0)=0 \tag{2}$$

3. Basic idea of Newton Harmonic Balancing Method

The Newton Harmonic Balancing Method (NHBM) is composed of the Newton and Harmonic Balance method. For the first time, this method was presented by Wu [23] in 2006.

The equation of motion of vibrating systems can be defined in the following form:

$$\frac{d^2u}{dt^2}+f\left(u,\frac{du}{dt},\frac{d^2u}{dt^2}\right)=0 \tag{3}$$

The initial conditions are as follows:

$$u(0)=A\ ,\ \frac{du}{dt}(0)=0 \tag{4}$$

Assumed that Eq. (3) is an odd function. It means that $f\left(-u,-\frac{du}{dt},-\frac{d^2u}{dt^2}\right)=-f\left(u,\frac{du}{dt},\frac{d^2u}{dt^2}\right)$.

By substitution $\tau=\omega t$, Eqs. (3) and (4) can be rewritten as follows:

$$\begin{aligned}&\omega^2u''+f(u,\omega u',\omega^2u'')=0\\&u(0)=A\ ,\ u'(0)=0\end{aligned} \tag{5}$$

Where prime denotes the derivation with respect to τ.

Applying Newton's procedure, the displacement and squared angular frequency can be expressed as Eq. (6), in which Δu_1 and $\Delta\omega_1^2$ are small increments of original displacement u_1 and squared angular frequency ω_1^2, respectively.

$$\begin{aligned}u(\tau)&=u_1(\tau)+\Delta u_1(\tau)\\\omega^2&=\omega_1^2+\Delta\omega_1^2\end{aligned} \tag{6}$$

Substituting Eq. (6) into Eq. (5) and linearizing it for the first-order analytical approximation, we set Eq. (7) as bellow:

$$u_1(\tau)=A\cos\tau, \qquad \Delta u_1=\Delta u''=\Delta\omega_1^2=0 \tag{7}$$

For the second analytical approximation, we set Eq. (8) and solve a set of simultaneous equations in terms of c_1 and $\Delta\omega_1^2$.

$$\Delta u_1=c(\cos\tau-\cos3\tau) \tag{8}$$

The corresponding approximate analytical periodic solution, $u(t)$, and the second-order analytical approximate frequency, ω, will be as follows:

$$\omega=\sqrt{\omega_1^2+\Delta\omega_1^2} \tag{9}$$

$$u(t)=(A+c)\cos\omega t-(c)\cos3\omega t \tag{10}$$

4. Application

We rewrite Eq. (1) in the following form:

$$\ddot{u}+\frac{1}{u}=0, \qquad \rightarrow \qquad u^2\ddot{u}+u=0 \tag{11}$$

By substituting $\tau = \omega t$, Eq. (11) is changed to:

$$\omega^2 u^2(\tau)u''(\tau)+u(\tau)=0 \tag{12}$$

Where the initial conditions are:

$$u(0)=A \qquad u'(0)=0 \tag{13}$$

Here A is the maximum amplitude of the system.

Substituting Eq. (6) into Eq. (12), it is obtained:

$$(\omega_1^2+\Delta\omega_1^2)(u_1+\Delta u_1)^2(u_1''+\Delta u_1'')+(u_1+\Delta u_1)=0 \tag{14}$$

Linearization of Eq. (14) with respect to Δu_1 and $\Delta\omega_1^2$ yields:

$$u_1''(u_1^2\omega_1^2+2u_1\omega_1^2\Delta u_1+u_1^2\Delta\omega_1^2)+u_1^2\omega_1^2\Delta u_1''+u_1+\Delta u_1=0 \tag{15}$$

With substituting Eq. (7) into Eq. (15) for first order approximation and avoiding the presence of secular terms, the angular frequency may be written as:

$$-A^2\omega_1^2\cos^3\tau+\cos\tau=0 \quad \Rightarrow \quad \omega_1=\sqrt{\frac{4}{3}}A^{-1} \tag{16}$$

For the second analytical approximation, by substituting Eq. (8) into Eq. (15) and expanding the obtained expression in a trigonometric series and then by putting the coefficients of $\cos\tau$ and $\cos 3\tau$ equal to zero, results achieved in a set of simultaneous equations in terms of $\Delta\omega_1^2$ and c:

$$-3A^3\Delta\omega_1^2+c(4+2A^2\omega_1^2)+4A-3A^3\omega_1^2=0 \tag{17}$$

$$-A^3\Delta\omega_1^2+c(-4+19A^2\omega_1^2)-A^3\omega_1^2=0 \tag{18}$$

Solving Eq. (17) and (18) simultaneously, it is obtained:

$$\Delta\omega_1^2=-\frac{16-92A^2\omega_1^2+55A^4\omega_1^4}{A^2(-16+55A^2\omega_1^2)} \tag{19}$$

$$c=\frac{4A}{-16+55A^2\omega_1^2} \tag{20}$$

The second order analytical approximate frequency and system response may be written as:

$$\omega=\sqrt{\omega_1^2+\Delta\omega_1^2}=\sqrt{\frac{4}{3A^2}-\frac{16-92A^2\omega_1^2+55A^4\omega_1^4}{A^2(-16+55A^2\omega_1^2)}} \tag{21}$$

$$u(t)=(A+c)\cos\omega t-(c)\cos 3\omega t \tag{22}$$

5. Result

Motion equation of electron beam into plasma tube was investigated and solved with NHBM. The obtained frequency by applying the second order approximation of NHBM (Eq. (21)) was compared with exact solution and other analytical solutions in table 1. Also Figs. 1 and 2, show the displacement response for this problem, u(t), and analytical solution of du/dt for A=2, respectively. From this figures and table 1, NHBM has excellent agreement with other analytical or exact results and provides suitable approximation for this type of problems.

Table 1. Comparison of the frequency obtained via 2nd order NHBM approximation with HBM, CHV2 and the exact solution

A	HBM[20]	Present study Eq.(21)	CHV2[27]	Exact solution [20]
0.1	12.7279	12.1999	12.4953	12.5331
1	1.2728	1.2200	1.2495	1.2533
10	0.1273	0.1220	0.1249	0.1253
50	0.0254	0.0244	0.0245	0.0250
100	0.0127	0.0122	0.0124	0.0125

Comparison between frequencies obtained by NHBM and other analytical methods is illustrated in figure 3. The effect of A which has been studied in the phase plane figures shows in Figure. 4 for four different initial amplitude. In addition, Fig. 5 shows the displacement behavior of the system versus time and the initial amplitude. Also, time histories of the error percentage for four analytical methods are illustrated in Fig. 6. In the other word, Fig. 6 shows errors with respect to time marching solution for $A=1$. The ruptures that happened in this figure are the points where the equation has root in them. So, we can neglect these critical points. Hence, as shown in Fig. 6, the NHBM error percentage decreased during the time.

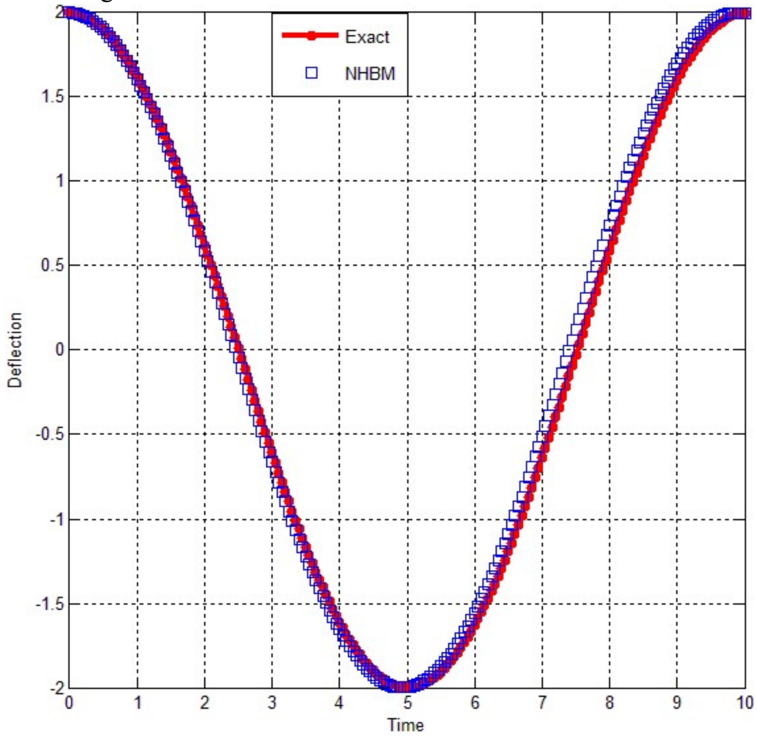

Fig. 1. Comparison of analytical solution of u(t) based on time with the exact solution for A=2

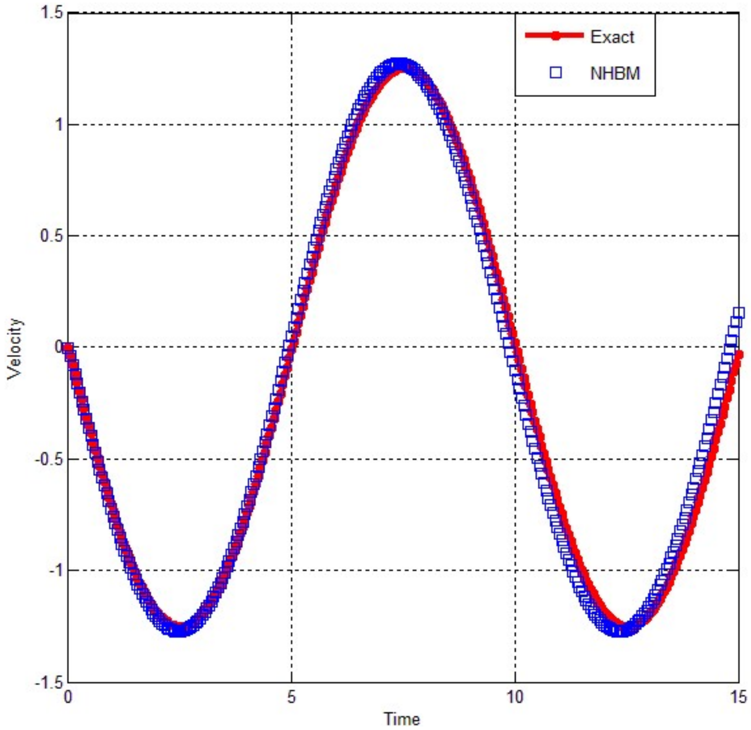

Fig. 2. Comparison of analytical solution of du/dt based on time with the exact solution for A=2

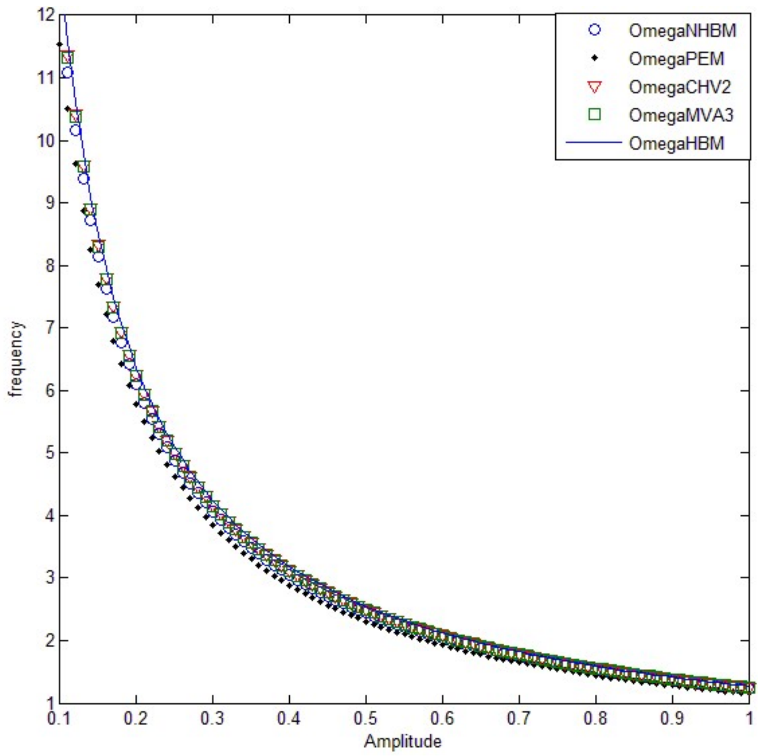

Fig. 3. The results of NHBM and other analytical methods for frequency versus amplitude

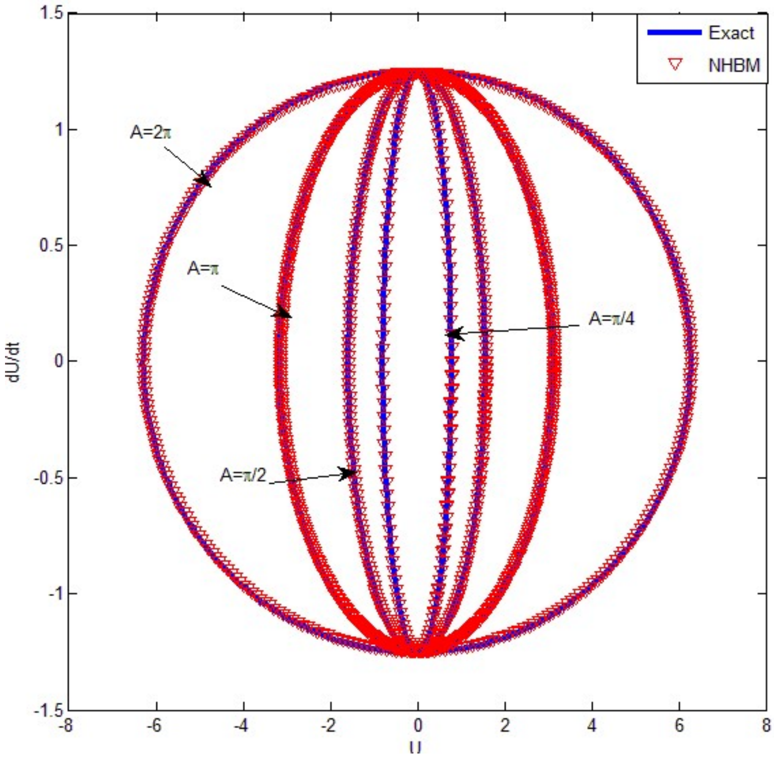

Fig. 4. Comparison of the analytical solution of du/dt based on u(t) with the exact solution

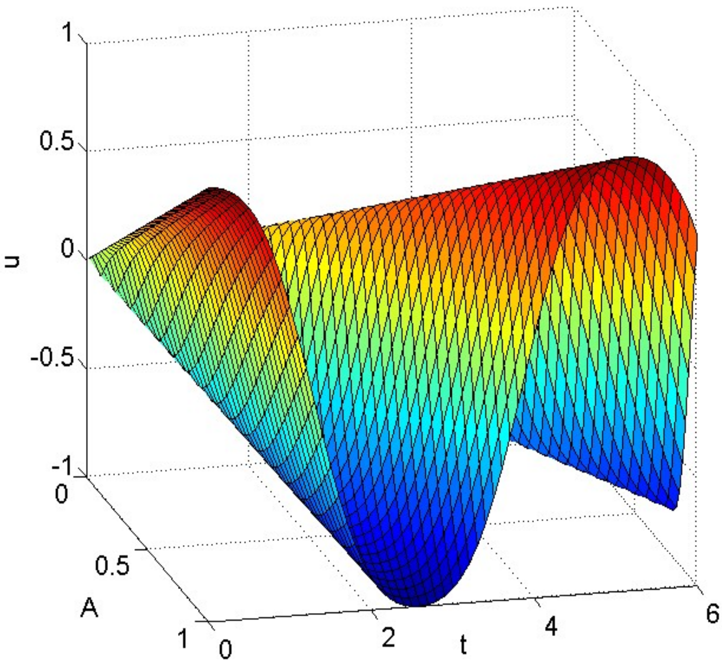

Fig. 5. Influence of the initial amplitude on time histories of NHBM response

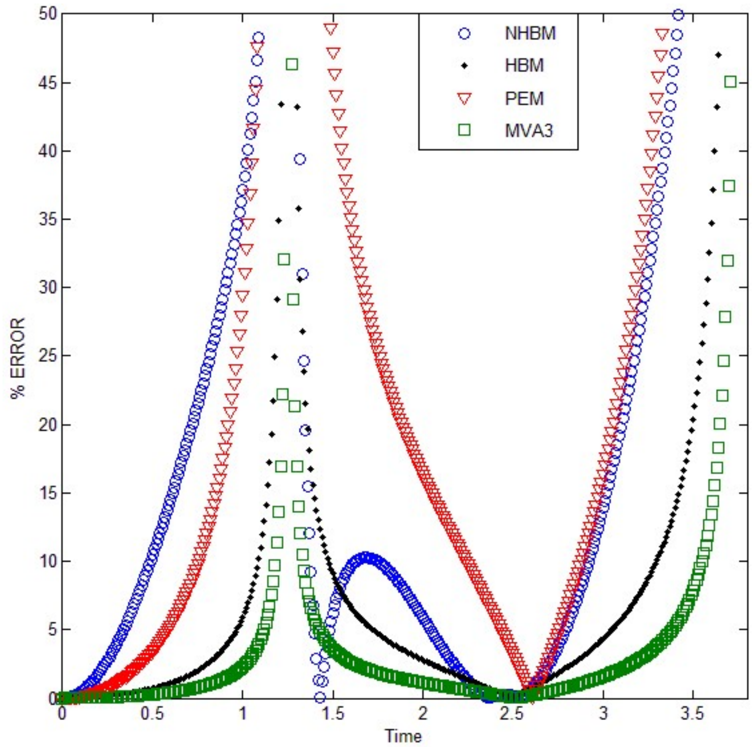

Fig. 6. Error percentage for NHBM and other analytical solutions

6. Conclusion

In this paper, we applied a new method, called Newton Harmonic Balance Method (NHBM) to solve the governing equation of nonlinear oscillation of an electron beam into a plasma tube. The results of NHBM have excellent agreement with the results obtained by the exact solution. This method is easy and uses only one or two iteration leading us to the acceptable results. The error percentage achieved by NHBM decreased during the time against the other analytical solutions. However, the obtained frequency and system response of NHBM is better than four other analytical solutions applied in other works. This method is a powerful and efficient tool for solving nonlinear vibration equations.

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