Vibration analysis of a rotating closed section composite Timoshenko beam by using differential transform method

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Abstract

This study introduces the Differential Transform Method (DTM) in the analysis of the free vibration response of a rotating closed section composite, Timoshenko beam, which features material coupling between flapwise bending and torsional vibrations due to ply orientation. The governing differential equations of motion are derived using Hamilton's principle and solved by applying DTM. The natural frequencies are calculated and the effects of the bending-torsion coupling, the slenderness ratio and several other parameters on the natural frequencies are investigated using the computer package, Mathematica. Wherever possible, comparisons are made with the studies in open literature.

1. Introduction

Solving the problem of determining vibration characteristics of rotating beams is a requirement in various branches of engineering. The determination of the structural response and modal frequencies is essential in designing rotating structural elements such as helicopter blades, airplane propellers and turbo machinery blades [1, 2]. The common models for beams are Euler-Bernoulli beams and Timoshenko beams. In the latter (i.e., the classical model), the ratio of the thickness of the beam to the length is considered to be relatively small and thus, the effect of transverse shear and the rotary inertia is negligible. The former, on the other hand, can account for appreciable thickness, so the transverse shear and rotary inertia are not neglected.

Zhou [3] was the first to adopt the differential transformation method (DTM) in engineering applications. The first application of DTM was in solving the initial boundary value problems in electrical circuit analysis. Ozdemir and Kaya [4] applied DTM to calculate the natural frequency of non-uniform beams. In another study, Mei [5] employed DTM for rotating beams.

This article examines the transverse vibration of the rotating closed section composite beam with constant cross-section. The effects of the rotational speed on natural frequency and shape modes are also investigated.

2. Formulation

2.1 Composite model

A straight composite beam with length L, height h and breadth b is shown in Fig. 1. In the right-handed Cartesian coordinate system, the x-axis is the centroidal axis of the beam. The flapwise displacement, $w(x,t)$ and the torsional rotation, $\psi(x,t)$ occur about the y-axis and the x-axis, respectively. Here, x and t, denote the spanwise coordinate and the time, respectively. The beam rotates with a constant angular velocity, Ω. Since the cross-sections of the

beam have symmetry in both planes, the x-axis is also the locus of the geometric shear centers of the beam cross-sections. Therefore, the beam features material coupling between flapwise bending and torsional vibrations only due to ply orientation.

2.2 Potential and Kinetic Energy Expressions

The total potential energy expression, U is given by Ref. [6] and the derivation of the kinetic energy expression, $\wp$ is made in this study. As a result, the following expressions are obtained for a rotating, composite Timoshenko beam:

$$U = \frac{1}{2}\int_0^L \left\{ EI_y \left(\theta'\right)^2 + T\left[\left(w'\right)^2 + \left(I_\alpha/\rho A\right)\left(\psi'\right)^2\right] + 2K\,\theta'\psi' + kAG\left(w'-\theta\right)^2 + GJ\left(\psi'\right)^2 \right\}dx \tag{1}$$

$$\wp = \frac{1}{2}\int_0^L \left\{ \rho A\dot{w}^2 + I_\alpha\dot{\psi}^2 + 2\rho\Omega I_z\left(\psi\dot{\theta}+\theta\dot{\psi}\right) + \rho I_y\left[\Omega^2\left(\theta^2+\psi^2\right)+2\Omega\left(\theta\dot{\psi}-\psi\dot{\theta}\right)+\dot{\theta}^2\right] \right\}dx \tag{2}$$

where primes and dots denote differentiation with respect to spanwise coordinate x and time t, respectively. Here ρ is the material density; A is the cross sectional area; ρA is the mass per unit length, I_y and I_z are the second moments of inertia of the beam cross-section about the y and z axes, respectively; I_α is the polar mass moment of inertia per unit length about the x axis; EI_y, GJ, K and kGA are the flapwise bending rigidity, torsional rigidity, bending–torsion coupling rigidity and shear rigidity of the composite beam, respectively.

3. Governing Equations of Motion

The governing differential equations of motion and the associated boundary conditions are obtained by applying the Hamilton's principle to the energy expressions given by Eqs. (1) and (2).

Equations of motion:

$$\rho I_y\ddot{\theta} - \rho I_y\Omega^2\theta - EI_y\theta'' - kAG\left(w'-\theta\right) - K\psi'' - 2\rho I_y\Omega\dot{\psi} = 0 \tag{3}$$

$$\rho A\ddot{w} - Tw'' - kAG\left(w''-\theta'\right) = 0 \tag{4}$$

$$I_\alpha\ddot{\psi} - \rho I_y\Omega^2\psi - GJ\psi'' - \left(I_\alpha/\rho A\right)\left(T\psi'\right)' - K\theta'' + 2\rho I_y\Omega\dot{\theta} = 0 \tag{5}$$

where T is the centrifugal force that is given as follows:

$$T(x) = \int_x^L \rho A\Omega^2(R+x)dx \tag{6}$$

Boundary Conditions:

• The geometric boundary conditions at the cantilever end, x=0, of the composite beam,

$$w(0,t) = \theta(0,t) = \psi(0,t) = 0 \tag{7}$$

• The natural boundary conditions at the free end, x=L, of the composite beam,

$$EI_y\theta' + K\psi' = 0 \tag{8}$$

$$Tw' + kAG\left(w'-\theta\right) = 0 \tag{9}$$

$$\left(TI_\alpha/\rho A\right)\psi' + K\theta' + GJ\psi' = 0 \tag{10}$$

3.1 Exponential Solution and Dimensionless Equations of Motion

A sinusoidal variation of $w(x,t)$, $\psi(x,t)$ and $\theta(x,t)$ with a circular natural frequency ω is assumed and the functions are approximated as exponential solutions.

$$w(x,t) = \bar{w}(x)e^{i\omega t} \tag{11-a}$$

$$\psi(x,t) = \bar{\psi}(x)e^{i\omega t} \tag{11-b}$$

$$\theta(x,t) = \bar{\theta}(x)e^{i\omega t} \tag{11-c}$$

Additionally, the following dimensionless parameters are used to enable comparisons with the studies in open literature:

$$\bar{x} = \frac{x}{L}, \quad \tilde{w} = \frac{\bar{w}}{L}, \quad \delta = \frac{R}{L}, \quad r^2 = \frac{I}{AL^2} \tag{12}$$

If Eqs. (11-a,11-b,11-c) are substituted into Eqs. (3)-(5) and the dimensionless parameters are used, equations of motion can be rewritten as follows:

$$A_1 \frac{d^2\bar{\theta}}{d\bar{x}^2} + A_2\bar{\theta} + A_3 \frac{d^2\bar{\psi}}{d\bar{x}^2} + A_4\bar{\psi} + A_5 \frac{d\tilde{w}}{d\bar{x}} = 0 \tag{13}$$

$$\left(B_1 - \bar{x}\delta - \frac{\bar{x}^2}{2}\right)\frac{d^2\tilde{w}}{d\bar{x}^2} + \left(B_2 - \bar{x}\right)\frac{d\tilde{w}}{d\bar{x}} + B_3\tilde{w} + B_4 \frac{d\bar{\theta}}{d\bar{x}} = 0 \tag{14}$$

$$\left(C_1 - \bar{x}\delta - \frac{\bar{x}^2}{2}\right)\frac{d^2\bar{\psi}}{d\bar{x}^2} + \left(C_2 - \bar{x}\right)\frac{d\bar{\psi}}{d\bar{x}} + C_3\bar{\psi} + C_4 \frac{d^2\bar{\theta}}{d\bar{x}^2} + C_5\bar{\theta} = 0 \tag{15}$$

where the dimensionless coefficients are:

$$A_1 = 1 \quad , \quad A_2 = \frac{\rho AL^4 r^2}{EI_y}\left(\omega^2 + \Omega^2\right) - \frac{kAGL^2}{EI_y} \quad , \quad A_3 = \frac{K}{EI_y} \quad , \quad A_2 = 2i\omega\Omega\frac{\rho AL^4 r^2}{EI_y} \quad , \quad A_5 = \frac{kAGL^2}{EI_y} \quad ,$$

$$B_1 = \delta + \frac{1}{2} + \frac{kAG}{\rho AL^2\Omega^2} \quad , \quad B_2 = -\delta \quad , \quad B_3 = \left(\frac{\omega}{\Omega}\right)^2 \quad , \quad B_4 = \frac{kAG}{\rho AL^2\Omega^2} \quad , \quad C_1 = \delta + \frac{1}{2} + \frac{GJ}{I_s L^2\Omega^2} \quad , \quad C_2 = B_2 \quad , \quad C_3 = \left(\frac{\omega}{\Omega}\right)^2 + \frac{\rho AL^2 r^2}{I_s} \quad ,$$

$$C_4 = \frac{K}{I_s L^2\Omega^2} \quad , \quad C_4 = \frac{K}{I_s L^2\Omega^2} \quad , \quad C_5 = -2i\frac{\omega}{\Omega}\frac{\rho AL^2 r^2}{I_s} \quad .$$

4. Application of the Differential Transform Method

The Differential Transform Method is a transformation technique based on the Taylor series expansion and is a useful tool to obtain analytical solutions to differential equations. In this method, certain transformation rules are applied and the governing differential equations and the boundary conditions of the system are transformed into a set of algebraic equations in terms of the differential transforms of the original functions and the solution of these algebraic equations gives the desired solution of the problem with considerable accuracy. The application procedure of this method can be found in Ref. [4]. After the Differential Transform Method is applied to Eqs. (13) - (15), the following transformed equations of motion are obtained.

$$A_1(k+2)(k+1)\theta[k+2] + A_2\theta[k] + A_3(k+2)(k+1)\psi[k+2] + A_4\psi[k] \\ + A_5(k+1)W[k+1] = 0 \tag{16}$$

$$B_1(k+2)(k+1)W[k+2] + (B_2 - \delta k)(k+1)W[k+1] + \left[B_3 - \frac{k(k+1)}{2}\right]W[k] \\ + B_4(k+1)\theta[k+1] = 0 \tag{17}$$

$$C_1(k+2)(k+1)\psi[k+2] + (C_2 - \delta k)(k+1)\psi[k+1] + \left[C_3 - \frac{k(k+1)}{2}\right]\psi[k] \\ + C_4(k+1)(k+2)\theta[k+2] + C_5\theta[k] = 0 \tag{18}$$

where $\theta[k]$, $W[k]$ and $\psi[k]$ are the transformed functions of $\bar{\theta}$, $\tilde{w}$, $\bar{\psi}$, respectively. Additionally, transformed boundary conditions are obtained as follows by applying differential transform method to Eqs. (7)-(10).

$$\theta[0] = W[0] = \psi[0] = 0 \qquad , at \ x = 0 \tag{19}$$

$$A_4(k+1)\psi[k+1] + (k+1)\theta[k+1] = 0 \quad , at \ x = L \tag{20}$$

$$B_1(k+1)w[k+1] - \theta[k] = 0 \qquad , at\ x = L \qquad (21)$$

$$C_1(k+1)\psi[k+1] + C_3(k+1)\theta[k+1] = 0, at\ x = L \qquad (22)$$

5. Result and discussion

The computer package Mathematica is used to write a program for the expressions given by Eqs. (16)-(22). In order to validate the computed results, an illustrative example that studies a nonrotating composite Timoshenko beam is taken from Ref. [7] which is solved by using the formulas given above. When the results are compared with the ones given in Ref. [7], it is seen that there is excellent agreement among the results. Additionally, effect of the rotation speed on the natural frequencies is examined and the related graphic is plotted.

The beam model that is studied in Ref. [7] is a uniform nonrotating cantilever glass-epoxy composite beam with a rectangular cross section with a width of 12.7 mm and a thickness of 3.18 mm. Unidirectional plies, each with fiber angles of $+15°$, are used in the analysis. The data used for the analysis are as follows:

$EI = 0.2865\ N.m^2$, $GJ = 0.1891\ N.m^2$, $kGA = 6343.3\ N$

$\mu = 0.0544\ kg/m$, $I_s = 0.777 \times 10^{-6}\ kg.m$, $K = 0.1143\ N.m^2$

$L = 0.1905\ m$, $r^2 = 0.00002322$, $\Omega = 0\ rad/sec$.

In Table 1, the calculated results are compared with the ones given by Ref. [4] and very good agreement is observed. In Table 2, the effect of the rotational speed on the natural frequencies is given. In Figure 2, mode shapes of a rotating composite Timoshenko beam are plotted.
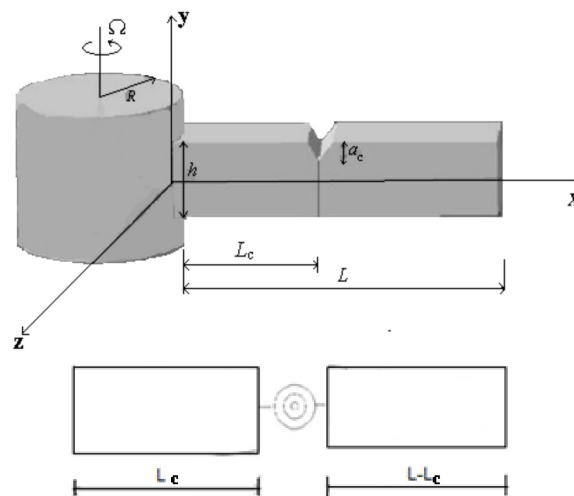

Fig. 1. Configuration of a uniform   rotating composite Timoshenko beam.

Table 1. Validation of the Calculated Results

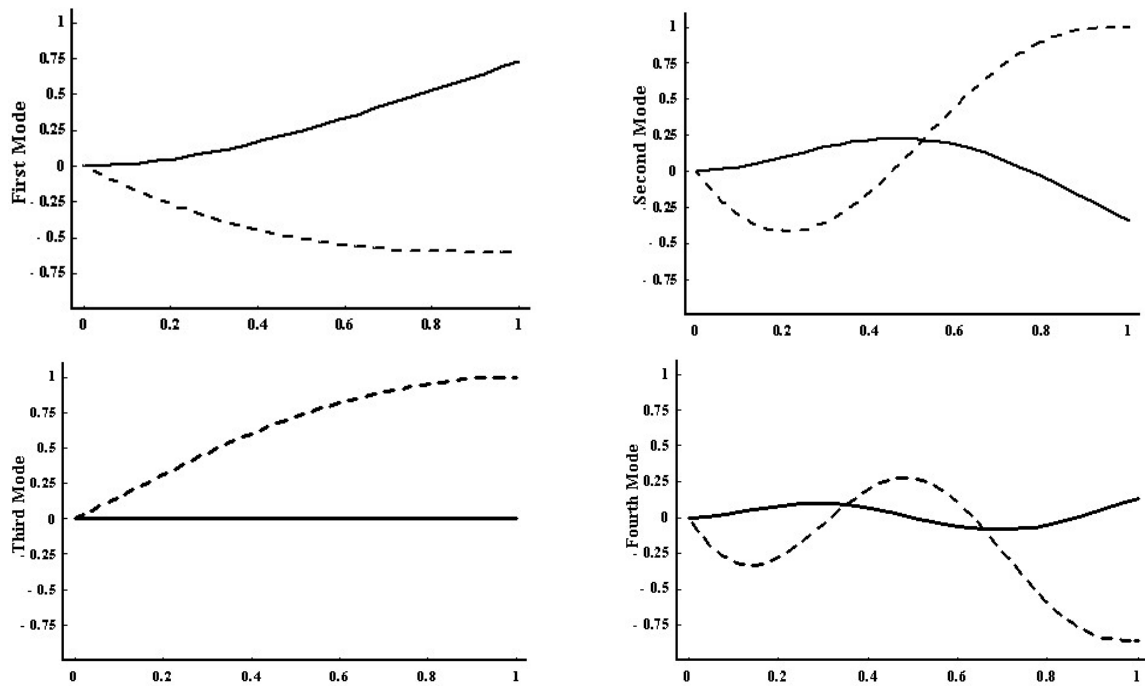| Natural Fequencies	Methods	
	DTM	Ref.[7]
Mode 1	30.747	30.747
Mode 2	189.779	189.779
Mode 3	518.791	189.779
Mode 4	648.169	648.169
Mode 5	986.199	986.199
Mode 6	1564.75	1564.751

Fig. 2. Mode shapes of a rotating composit Timoshenko beam

Table 2. Effect of the rotational speed on the natural frequencies

			Rotatinal Speed Ω (rad / sec)				
	0	**50**	**100**	**150**	**200**	**250**	**300**
	30.7469	32.3631	36.8079	43.2728	51.0809	59.8268	69.3128
	189.7788	189.7328	189.5966	189.3757	189.0808	188.7328	188.3675
	518.7907	518.4607	517.4587	515.7426	513.2417	509.8463	505.4071
Natural Frequencies	648.1686	648.2308	648.1159	647.9228	647.6509	647.2977	646.8609
	986.1988	985.7556	984.4146	982.1438	978.8860	974.5609	969.0584
	1564.751	1564.237	1562.684	1560.066	1556.335	1551.425	1545.248
	1944.308	1944.231	1943.996	1943.603	1943.050	1942.333	1941.446
	2241.306	2240.888	2239.556	2237.103	2233.288	2228.003	2221.297

6. Conclusions

The DTM for vibration analysis of cracked rotating uniform beams is formulated. First, the equation of the beam is converted to a dimensionless one by using DTM and deriving the transformed equation. Besides, comparisons of the results for different configurations of non-cracked beams calculated by the proposed method with those in the literature are made and a good agreement between the results is seen. Then, vibration analysis of cracked rotating tapered beams is carried out. The effects of the crack location and size, rotation speed and hub radius on vibration characteristics of a cracked rotating tapered beam are investigated. It is seen that for a given crack size, the fundamental frequency decreases monotonically when the crack location varies from the free end to the fixed root.
Also, the mode shapes of the cracked rotating tapered beam are obtained. It is seen that with the increase of the crack size, the mode shapes of cracked rotating beam change significantly and a local perturbation occurs at the crack location. The present work provides helpful information for structural design and diagnosis and can be extended to studying the vibration of arbitrarily cracked rotating beams.

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