Journal of Applied and Computational Mechanics, Vol. 2, No. 1, (2016), 42-53
DOI: 10.22055/jacm.2016.12270

Shahid Chamran University

Journal of
Applied and Computational Mechanics

Saint-Venant Torsion of Non-homogeneous Anisotropic Bars

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Received July 8 2016; revised August 16 2016; accepted for publication August 18 2016.
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Abstract

The BEM is applied to the solution of the torsion problem of non-homogeneous anisotropic non-circular prismatic bars. The problem is formulated in terms of the warping function. This formulation leads to a second order partial differential equation with variable coefficients, which is subjected to a generalized Neumann type boundary condition. The problem is solved using the Analog Equation Method (AEM). According to this method, the governing equation is replaced by a Poisson's equation subjected to a fictitious source under the same boundary condition. The fictitious load is established using the Boundary Element Method (BEM) after expanding it to a finite series of radial basis functions. The method has all the advantages of the pure BEM, since the discretization and integration are limited only to the boundary. Numerical examples are presented which illustrate the efficiency and accuracy of the method.

Keywords: Anisotropic materials, Non-homogeneous media, Elasticity, Bars, Torsion.

1. Introduction

Analytical solutions have been presented for homogeneous and isotropic bars [1-4], anisotropic bars [1] and discontinuously non-homogeneous [2] with simple geometry. An approximate solution for homogeneous anisotropic bars was introduced in [5], reducing the problem to the solution of a Fredholm integral equation of the second kind with a regular kernel. Realistic engineering problems are solved by numerical techniques. The Finite Difference Method (FDM) was used along with a relaxation procedure for homogeneous multiply connected bars presented by Shaw [6] and for both homogeneous and discontinuously nonhomogeneous bars with simply and multiply connected boundaries by Ely and Zienkiewicz [7]. The Finite Element Method (FEM) has been developed by Zienkiewicz and Cheng [8] for the solution of homogeneous and discontinuously non-homogeneous bars with isotropic material, using the stress function, while Herrmann [9], and Krahula and Lauterbach [10] used the warping function to solve isotropic and homogeneous bars. Valliappan and Pulmano [11] adopted the same method for discontinuously non-homogeneous anisotropic bars. The Boundary Element Method (BEM) has efficiently been applied to the torsion problem of composite shafts (discontinuously inhomogeneous) consisting of isotropic material [12-15], whereas a Complex Variable Boundary Element Method (CVBEM) has been developed for the solution of the same problem [16-18]. Dumir and Kumar [19] also used CVBEM to solve the torsion problem of homogeneous anisotropic bars. Furthermore, the BEM was used to the solution of non-uniform torsion of homogeneous isotropic bars [20] and of multi-material composite bars [21]. Recently, some analytical solutions to the problem have emerged in the literature for non-homogeneous cylindrical bars [22], for orthotropic bars with inhomogeneous rectangular cross section [23], and for non-homogeneous anisotropic cylindrical bars [24].

In this paper, the torsion problem for non-homogeneous anisotropic bars is formulated in terms of the warping function and also a boundary-only solution is developed to solve the resulting partial differential equation with variable coefficients under Neumann type boundary condition. The method is based on the concept of the analog equation [25], according to which the governing differential equation is replaced by a Poisson's type equation with an unknown fictitious source under the same boundary condition. The fictitious source is established using a procedure based on BEM. The warping function, as well as its derivatives, is computed from their integral representations, which are used as mathematical formulae. Several example problems are presented which illustrate the method and demonstrate its efficiency and accuracy. Moreover, useful conclusions are drawn for the torsional response of nonhomogeneous anisotropic bars.

2. Problem Formulation and Derivation of the Governing Equations

Consider a non-circular prismatic bar consisting of nonhomogeneous anisotropic linearly elastic material. It is assumed that the bar possesses one plane of symmetry, which coincides with its cross section. The cross-section occupies the two-dimensional, in general multiply connected, domain Ω in the xy -plane bounded by the $K+1$ nonintersecting contours $\Gamma_0, \Gamma_1, \ldots \Gamma_K$ (see Fig. 1). The problem is formulated in terms of the warping function using the semiverse method. Due to the inhomogeneity of the material, the elastic constants are position dependent and consequently the coefficients of the governing partial differential equation are variable. The governing equation is derived from three material cases (a) Non-homogeneous anisotropic, (b) Homogeneous anisotropic and (c) Non-homogeneous isotropic.

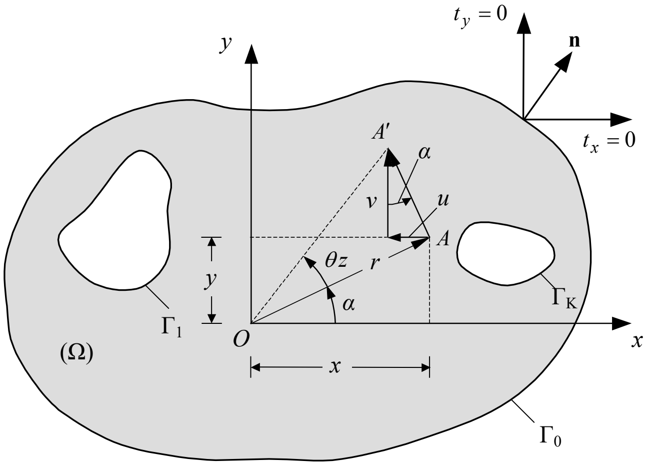

Fig. 1. Domain Ω occupied by the cross-section.

2.1 Non-homogeneous anisotropic bar

We consider a bar of arbitrary cross-section twisted by moments M_t applied at its ends. The cross-section is constant along the length of the bar. According to Saint-Venant's torsion theory, the deformation of the bar consists of (a) rotations of the cross-sections about an axis passing through the *twist center* of the bar, and (b) warping of the cross-sections, which is the same for all sections. Choosing the origin of the coordinate system at the twist center of an end section, the rotation at a distance z is θz, where θ is a constant expressing the rotation of a cross-section per unit length. Assuming that this rotation is small, the three displacements u, v and w of a point $A(x, y, z)$ due to the rotation are determined as (see Fig. 1) [26]

$$u = -\theta z y \tag{1}$$

$$v = \theta z x \tag{2}$$

$$w = \theta \phi(x, y) \tag{3}$$

where $\phi(x, y)$ is the warping function.

The non-vanishing strain components corresponding to the assumed displacement field are given as [26]

$$\gamma_{xz} = \theta\left(\frac{\partial \phi}{\partial x} - y\right) \tag{4}$$

$$\gamma_{yz} = \theta\left(\frac{\partial \phi}{\partial y} + x\right) \tag{5}$$

Which, using the stress strain relations for anisotropic bodies [1], are written as

$$\alpha_{55}\tau_{xz} + \alpha_{45}\tau_{yz} = \theta\left(\frac{\partial \phi}{\partial x} - y\right) \tag{6}$$

$$\alpha_{45}\tau_{xz} + \alpha_{44}\tau_{yz} = \theta\left(\frac{\partial \phi}{\partial y} + x\right) \tag{7}$$

where $\alpha_{44} = \alpha_{44}(x, y)$, $\alpha_{45} = \alpha_{45}(x, y)$ and $\alpha_{55} = \alpha_{55}(x, y)$ $(x, y) \in \Omega \cup \Gamma$ are the position dependent material constants of the non-homogeneous anisotropic bar. According to the positive definiteness criterion of the strain energy density, the following relations must be held: $\alpha_{44} > 0$, $\alpha_{55} > 0$, $\alpha_{44}\alpha_{55} - \alpha_{45}^2 > 0$ [1, 24].

Solving Eqs. (6) and (7) for τ_{xz} and τ_{yz}, we have

$$\tau_{xz} = \theta\left[A_{44}\left(\frac{\partial\phi}{\partial x} - y\right) + A_{45}\left(\frac{\partial\phi}{\partial y} + x\right)\right] \tag{8}$$

$$\tau_{yz} = \theta\left[A_{45}\left(\frac{\partial\phi}{\partial x} - y\right) + A_{55}\left(\frac{\partial\phi}{\partial y} + x\right)\right] \tag{9}$$

where

$$A_{44} = \frac{\alpha_{44}}{\alpha_{44}\alpha_{55} - \alpha_{45}^2}, \quad A_{45} = -\frac{\alpha_{45}}{\alpha_{44}\alpha_{55} - \alpha_{45}^2}, \quad A_{55} = \frac{\alpha_{55}}{\alpha_{44}\alpha_{55} - \alpha_{45}^2} \qquad (10),\ (11),\ (12)$$

substituting Eqs. (8) and (9) into the equilibrium equations and the boundary conditions for the 3-D elastic body, we obtain the following boundary value problem for the warping function $\phi = \phi(x, y)$

$$A_{44}\frac{\partial^2\phi}{\partial x^2} + 2A_{45}\frac{\partial^2\phi}{\partial x\partial y} + A_{55}\frac{\partial^2\phi}{\partial y^2} + D\frac{\partial\phi}{\partial x} + E\frac{\partial\phi}{\partial y} = g(x, y) \quad \text{in} \quad \Omega \tag{13}$$

$$\nabla\phi\cdot\mathbf{m} = ym_x - xm_y \quad \text{on} \quad \Gamma \tag{14}$$

where

$$D = \frac{\partial A_{44}}{\partial x} + \frac{\partial A_{45}}{\partial y}, \quad E = \frac{\partial A_{55}}{\partial y} + \frac{\partial A_{45}}{\partial x}, \quad g = yD - xE \qquad (15),\ (16),\ (17)$$

and $\mathbf{m}$ is the vector in the direction of the conormal to Γ, whose components are

$$m_x = A_{44}n_x + A_{45}n_y, \quad m_y = A_{45}n_x + A_{55}n_y \qquad (18),\ (19)$$

In the previous relations, n_x, n_y are the direction cosines of the unit vector normal to the boundary. Note that

$$\frac{\partial\phi}{\partial x} = n_x\frac{\partial\phi}{\partial n} - n_y\frac{\partial\phi}{\partial t}, \quad \frac{\partial\phi}{\partial y} = n_x\frac{\partial\phi}{\partial t} + n_y\frac{\partial\phi}{\partial n} \qquad (20),\ (21)$$

equation (14) may also be written as

$$(m_x n_x + m_y n_y)\frac{\partial\phi}{\partial n} + (m_y n_x - m_x n_y)\frac{\partial\phi}{\partial t} = ym_x - xm_y \tag{22}$$

where $\partial\phi/\partial n$ is the outward normal derivative and $\partial\phi/\partial t$ the tangential one.

The moment resultant on the cross-section $z = 0$ is given as

$$M_t = \int_\Omega (x\tau_{yz} - y\tau_{xz})\,d\Omega = G_0\theta I_t \tag{23}$$

where G_0 is a constant with the dimension of shear modulus and I_t is the torsion constant defined as

$$I_t = \frac{1}{G_0}\int_\Omega\left[A_{55}x^2 - 2A_{45}xy + A_{44}y^2 - (A_{44}y - A_{45}x)\frac{\partial\phi}{\partial x} + (A_{55}x - A_{45}y)\frac{\partial\phi}{\partial y}\right]d\Omega \tag{24}$$

The boundary stress τ_{tz} is expressed

$$\begin{aligned}\tau_{tz} &= -\tau_{xz}\,n_y + \tau_{yz}\,n_x \\ &= \theta\left[-\frac{\partial\phi}{\partial x}(A_{44}n_y - A_{45}n_x) + \frac{\partial\phi}{\partial y}(A_{55}n_x - A_{45}n_y) + (A_{55}x - A_{45}y)n_x + (A_{44}y - A_{45}x)n_y\right]\end{aligned} \tag{25}$$

2.2 Homogeneous anisotropic bar

When the anisotropic body is homogeneous, the quantities A_{44}, A_{45}, A_{55} are constants and Eqs. (13) and (14) are reduced to [26]

$$A_{44}\frac{\partial^2\phi}{\partial x^2} + 2A_{45}\frac{\partial^2\phi}{\partial x\partial y} + A_{55}\frac{\partial^2\phi}{\partial y^2} = 0 \quad \text{in} \quad \Omega \tag{26}$$

$$\nabla\phi\cdot\mathbf{m} = ym_x - xm_y \quad \text{on} \quad \Gamma \tag{27}$$

In this case, the domain integral in Eq. (24) can be transformed into a line integral on boundary appropriately using the Gauss divergence theorem. Thus, we have

$$I_t = \frac{1}{G_0}\int_\Gamma\left\{\left[A_{44}(xy^2 - y\phi) - A_{45}\left(\tfrac{1}{2}x^2y - x\phi\right)\right]n_x + \left[A_{55}(x^2y + x\phi) - A_{45}\left(\tfrac{1}{2}xy^2 + y\phi\right)\right]n_y\right\}ds \tag{28}$$

Equation (28) highly simplifies the evaluation of I_t, since only boundary values of ϕ are involved in it. The boundary stress τ_{tz} is given by Eq. (25).

2.3 Non-homogeneous isotropic bar

For a nonhomogeneous but isotropic bar, Eqs. (13) and (14) are reduced to

$$G\nabla^2\phi+\frac{\partial G}{\partial x}\frac{\partial \phi}{\partial x}+\frac{\partial G}{\partial y}\frac{\partial \phi}{\partial y}=y\frac{\partial G}{\partial x}-x\frac{\partial G}{\partial y} \quad \text{in} \quad \Omega \tag{29}$$

$$\nabla\phi\cdot\mathbf{n}=yn_x-xn_y \quad \text{on} \quad \Gamma \tag{30}$$

where $G=G(x,y)$ is the variable shear modulus of the material and is given as

$$G=1/\alpha_{44}=1/\alpha_{55} \tag{31}$$

Finally, Eqs. (24) and (25) are reduced to

$$I_t=\frac{1}{G_0}\int_\Omega G\left(x\frac{\partial \phi}{\partial y}-y\frac{\partial \phi}{\partial x}+x^2+y^2\right)d\Omega \tag{32}$$

$$\tau_{tz}=G\theta\left(\frac{\partial \phi}{\partial t}+yn_y+xn_x\right) \tag{33}$$

2.4 Existence of the solution

The Green's reciprocal identity for the second order partial differential operator L defined by Eq. (13) is written as [26]

$$\int_\Omega\left[vL(u)-uL^*(v)\right]d\Omega=\int_\Gamma\left(Xn_x+Yn_y\right)ds \tag{34}$$

where L^* is the adjoint operator to L defined as

$$L^*(v)=\frac{\partial^2(A_{44}v)}{\partial x^2}+2\frac{\partial^2(A_{45}v)}{\partial x\partial y}+\frac{\partial^2(A_{55}v)}{\partial y^2}-\frac{\partial(Dv)}{\partial x}-\frac{\partial(Ev)}{\partial y} \tag{35}$$

and

$$X=A_{44}\left(v\frac{\partial u}{\partial x}-u\frac{\partial v}{\partial x}\right)+A_{45}\left(v\frac{\partial u}{\partial y}-u\frac{\partial v}{\partial y}\right) \tag{36}$$

$$Y=A_{45}\left(v\frac{\partial u}{\partial x}-u\frac{\partial v}{\partial x}\right)+A_{55}\left(v\frac{\partial u}{\partial y}-u\frac{\partial v}{\partial y}\right) \tag{37}$$

Applying Eq. (34) for the function $u=\phi$ satisfying Eq. (13) and $v=1$, we have

$$\int_\Omega\left(yD-xE\right)d\Omega=\int_\Gamma\left[\left(A_{44}\frac{\partial \phi}{\partial x}+A_{45}\frac{\partial \phi}{\partial y}\right)n_x+\left(A_{45}\frac{\partial \phi}{\partial x}+A_{55}\frac{\partial \phi}{\partial y}\right)n_y\right]ds \tag{38}$$

Equation (38) represents the necessary condition so that the Neumann-type boundary value problem of Eqs. (13) and (14) has a solution. We can prove that this condition is always valid.

Using Gauss divergence theorem, the domain integral in Eq. (38) is transformed into a line integral as

$$\int_\Omega\left(yD-xE\right)d\Omega=\int_\Gamma\left[\left(yA_{44}-xA_{45}\right)n_x+\left(yA_{45}-xA_{55}\right)n_y\right]ds \tag{39}$$

Equating Eqs. (38) and (39), and moving all terms on the left hand side yields

$$\int_\Gamma\left\{\left[A_{44}\left(\frac{\partial \phi}{\partial x}-y\right)+A_{45}\left(\frac{\partial \phi}{\partial y}+x\right)\right]n_x+\left[A_{45}\left(\frac{\partial \phi}{\partial x}-y\right)+A_{55}\left(\frac{\partial \phi}{\partial y}+x\right)\right]n_y\right\}ds=0 \tag{40}$$

which, taking into account Eqs. (8) and (9), can be written as

$$\int_\Gamma\left(\tau_{xz}n_x+\tau_{yz}n_y\right)ds=0 \tag{41}$$

or

$$\int_\Gamma\tau_{nz}ds=0 \tag{42}$$

Equation (42) is true and therefore the condition (38) is always satisfied for the torsion problem.

3. The Analog Equation Method

The previous three Neumann's type boundary value problems described by Eqs. (13) and (14), Eqs. (26) and (27), Eqs. (29) and (30) are solved using the Analog Equation Method (AEM) [25] as developed in [27].

We present the solution procedure for the following boundary value problem

$$A\phi_{,xx}+2B\phi_{,xy}+C\phi_{,yy}+D\phi_{,x}+E\phi_{,y}+F\phi=g(x,y) \quad \text{in} \quad \Omega \tag{43}$$

$$\beta_1\phi+\beta_2\phi_{,n}=\beta_3 \quad \text{on} \quad \Gamma \tag{44}$$

where $A=A(x,y)$, $B=B(x,y)$,... $F=F(x,y)$ are functions defined in Ω and $\beta_i=\beta_i(s)$ are functions specified on Γ. The boundary value problem described by Eqs. (43) and (44) is solved using the concept of the analog equation [25], which is applied for the problem at hand as follows.

Let $\phi=\phi(x,y)=\phi(\mathbf{y})$ ($\mathbf{y}\in\Omega$) be the sought solution of Eqs. (43) and (44). This function is two times continuously differentiable in Ω. Thus, if the Laplacian operator is applied to it, we have

$$\nabla^2 \phi = b(\mathbf{y}) \tag{45}$$

Equation (45), which henceforth will be referred to as the analog equation, indicates that the solution of Eq. (43) can be established by solving this Poisson equation under the boundary condition (44), if the fictitious source $b = b(\mathbf{y})$ is known. Its establishment is accomplished following the procedure below.

We write the solution of Eq. (45) in the integral form. Thus, we have

$$\varepsilon\phi(\mathbf{x}) = \int_{\Omega} \phi^*(\mathbf{x},\mathbf{y}) b(\mathbf{y}) d\Omega_{\mathbf{y}} - \int_{\Gamma} [\phi^*(\mathbf{x},\xi)\phi,_n(\xi) - \phi_i(\xi)\phi,_n^*(\mathbf{x},\xi)] ds_{\xi} \tag{46}$$

in which $\mathbf{x} \in \Omega \cup \Gamma$, $\mathbf{y} \in \Omega$ and $\xi \in \Gamma$; $\phi^* = \ell n(r)/2\pi$ is the fundamental solution of the Laplace equation and $\phi,_n^*$ is its derivative normal to the boundary with $r = \|\mathbf{y} - \mathbf{x}\|$ or $r = \|\xi - \mathbf{x}\|$ being the distance between the points $\mathbf{x},\mathbf{y}$ or $\mathbf{x},\xi$; $\varepsilon = 1, \alpha/2\pi, 0$ depending on whether $\mathbf{x} \in \Omega$, $\mathbf{x} \in \Gamma$, $\mathbf{x} \notin \Omega \cup \Gamma$, respectively; α is the angle between the tangents to the boundary at point $\mathbf{x}$. For points where the boundary is smooth, it is $\varepsilon = 1/2$.

Eq. (46), when applied to boundary points, yields the boundary integral equation

$$\frac{a}{2\pi}\phi(\mathbf{x}) = \int_{\Omega} \phi^*(\mathbf{x},\mathbf{y}) b(\mathbf{y}) d\Omega_{\mathbf{y}} - \int_{\Gamma} [\phi^*(\mathbf{x},\xi)\phi,_n(\xi) - \phi(\xi)\phi,_n^*(\mathbf{x},\xi)] ds_{\xi} \tag{47}$$

which is a domain-boundary integral equation and could be solved using domain discretization to approximate the domain integrals. This, however, would spoil the advantages of BEM over the domain methods. We can maintain the pure boundary character of the method by converting the domain integrals to boundary line integrals using the following procedure. The unknown fictitious source $b(\mathbf{y})$ is approximated by the series

$$b(\mathbf{y}) = \sum_{j=1}^{M} a_j f_j \tag{48}$$

where $f_j = f_j(r_j)$ is a set of M radial basis approximation functions and a_j are M coefficients to be determined; $r_j = \|\mathbf{y} - \mathbf{x}_j\|$ with $\mathbf{x}_j$ being the M collocation points located in the domain Ω (see Fig. 2).

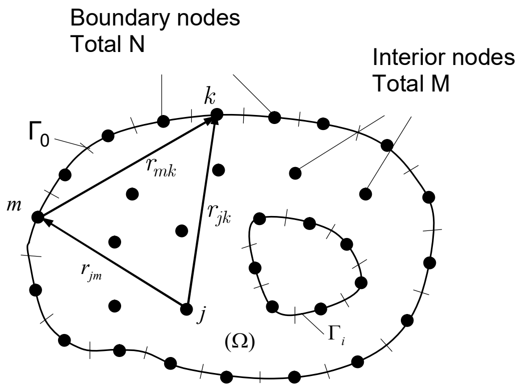

Fig. 2. Boundary discretization and domain nodal points.

Using the Green's reciprocal identity [26], the domain integral in Eq. (47) becomes

$$\begin{aligned}\int_{\Omega} \phi^*(\mathbf{x},\mathbf{y})\phi(\mathbf{y}) d\Omega_{\mathbf{y}} &= \sum_{j=1}^{M} a_j \int_{\Omega} \phi^*(\mathbf{x},\mathbf{y}) f_j(\mathbf{y}) d\Omega_{\mathbf{y}} \\ &= \sum_{j=1}^{M} a_j \left\{ \varepsilon \hat{u}_j(\mathbf{x}) + \int_{\Gamma} \left[\phi^*(\mathbf{x},\xi)\hat{u}_j,_n(\xi) - \hat{u}_j(\xi)\phi,_n^*(\mathbf{x},\xi) \right] ds_{\xi} \right\}\end{aligned} \tag{49}$$

in which $\hat{u}_j(r_j)$ is a particular solution of the equation

$$\nabla^2 \hat{u}_j = f_j \tag{50}$$

and $\hat{u}_j,_n$ its derivative normal to the boundary. Note that $\hat{u}_j$ can be always established when f_j is specified. Taking into consideration Eq. (49), the Eq. (47) is written as

$$\begin{aligned}\frac{a}{2\pi}\phi(\mathbf{x}) = &\sum_{j=1}^{M} a_j \left\{ \varepsilon \hat{u}_j(\mathbf{x}) + \int_{\Gamma} \left[\phi^*(\mathbf{x},\xi)\hat{u}_j,_n(\xi) - \hat{u}_j(\xi)\phi,_n^*(\mathbf{x},\xi) \right] ds_{\xi} \right\} \\ &- \int_{\Gamma} [\phi^*(\mathbf{x},\xi)\phi,_n(\xi) - \phi(\xi)\phi,_n^*(\mathbf{x},\xi)] ds_{\xi}\end{aligned} \tag{51}$$

For points $\mathbf{x} \in \Omega$ $(\varepsilon = 1)$, the warping function $\phi(\mathbf{x})$ can be evaluated from Eq. (46), which by virtue of Eq. (49) becomes

$$\begin{aligned}\phi(\mathbf{x}) = \sum_{j=1}^{M} a_j \left\{ \hat{u}_j(\mathbf{x}) \right. &+ \left. \int_\Gamma \left[\phi^*(\mathbf{x},\xi)\hat{u}_{j,n}(\xi) - \hat{u}_j(\xi)\phi^*_{,n}(\mathbf{x},\xi) \right] ds_\xi \right\} \\ &- \int_\Gamma [\phi^*(\mathbf{x},\xi)\phi_{,n}(\xi) - \phi(\xi)\phi^*_{,n}(\mathbf{x},\xi)] ds_\xi \end{aligned} \tag{52}$$

It is apparent that the first and second derivatives of the warping function $\phi(\mathbf{x})$ can be obtained by direct differentiation, namely

$$\begin{aligned}\phi_{,kl}(\mathbf{x}) = \sum_{j=1}^{M} a_j \left\{ \hat{u}_{j,kl}(\mathbf{x}) \right. &+ \left. \int_\Gamma \left[\phi^*_{,kl}(\mathbf{x},\xi)\hat{u}_{j,n}(\xi) - \hat{u}_j(\xi)\phi^*_{,nkl}(\mathbf{x},\xi) \right] ds_\xi \right\} \\ &- \int_\Gamma [\phi^*_{,kl}(\mathbf{x},\xi)\phi_{,n}(\xi) - \phi(\xi)\phi^*_{,nkl}(\mathbf{x},\xi)] ds_\xi \end{aligned} \tag{53}$$

where $k,l = 0,x,y$. Note that $\phi_{,00} \equiv \phi$. The expressions of the derivatives of the kernel functions are given in the Appendix A.

The final step of AEM is to apply Eq. (43) to the M collocation points inside Ω. Thus, we obtain a set of M equations of the form

$$A^j\phi^j_{,xx} + 2B\phi^j_{,xy} + C\phi^j_{,yy} + D\phi^j_{,x} + E\phi^j_{,y} + F\phi^j = g(x^j, y^j) \tag{54}$$

in which $j = 1,2,...M$. Substitution of Eqs. (53) into Eqs. (54) yields a set of M linear algebraic equations which can be used to evaluate the coefficients a_j. This can be implemented only by numerically using the procedure as presented below.

The BEM with constant elements is used to approximate the boundary integrals in Eqs. (51), (52) and (53). In this case, $a = \pi$, hence $\varepsilon = 1/2$. If N is the number of the boundary nodal points (see Fig. 2), then for the m nodal point Eq. (51) is written as

$$\frac{1}{2}\phi^m = \sum_{j=1}^{M} F_{mj}a_j + \sum_{k=1}^{N} \tilde{H}_{mk}\phi^k - \sum_{k=1}^{N} G_{mk}\phi^k_{,n} \tag{55}$$

where

$$\tilde{H}_{mk} = \int_k \phi^*_{,n}(r_{mk})ds, \qquad G_{mk} = \int_k \phi^*(r_{mk})ds \tag{56), (57}$$

$$F_{mj} = \varepsilon\hat{u}^m_j - \sum_{k=1}^{N} \tilde{H}_{mk}\hat{u}^k_j + \sum_{k=1}^{N} G_{mk}(\hat{u}_{,n})^k_j \tag{58}$$

in which $m = 1,2,...,N$ and $j = 1,2,...,M$.

Applying Eq. (55) to all boundary nodal points and using matrix notation yields

$$\mathbf{H}\boldsymbol{\phi} - \mathbf{G}\boldsymbol{\phi}_{,n} + \mathbf{F}\mathbf{a} = \mathbf{0} \tag{59}$$

where $\mathbf{a} = \{a_1, a_2, ..., a_M\}^T$; $\boldsymbol{\phi}$, $\boldsymbol{\phi}_{,n}$ are the vectors of the N boundary nodal values of the warping function and its normal derivative, respectively, and

$$\mathbf{H} = \tilde{\mathbf{H}} - \tfrac{1}{2}\mathbf{I} \tag{60}$$

where $\mathbf{I}$ is the unit matrix. The boundary condition Eq. (44) applied to all boundary points yields

$$\boldsymbol{\beta}_1\boldsymbol{\phi} + \boldsymbol{\beta}_2\boldsymbol{\phi}_{,n} = \boldsymbol{\beta}_3 \tag{61}$$

in which $\boldsymbol{\beta}_1$, $\boldsymbol{\beta}_2$ are known $N \times N$ diagonal matrices, and $\boldsymbol{\beta}_3$ a known $N \times 1$ vector. Now, Eqs. (59) and (61) can be combined to express $\boldsymbol{\phi}$ and $\boldsymbol{\phi}_{,n}$ in terms of $\mathbf{a}$

$$\boldsymbol{\phi} = \mathbf{S}_u\mathbf{a} + \mathbf{s}_u, \qquad \boldsymbol{\phi}_{,n} = \mathbf{S}_b\mathbf{a} + \mathbf{s}_b \tag{62), (63}$$

in which $\mathbf{S}_u$, $\mathbf{S}_b$ are known $N \times M$ rectangular matrices and $\mathbf{d}_u$, $\mathbf{d}_b$ are known vectors.

Moreover, using the same discretization in Eqs. (52), (53) and applying them to the M collocation points we obtain

$$\tilde{\boldsymbol{\phi}} = \mathbf{F}\mathbf{a} + \mathbf{H}\boldsymbol{\phi} - \mathbf{G}\boldsymbol{\phi}_{,n} \tag{64}$$

$$\tilde{\boldsymbol{\phi}}_{,kl} = \mathbf{F}_{kl}\mathbf{a} + \mathbf{H}_{kl}\boldsymbol{\phi} - \mathbf{G}_{kl}\boldsymbol{\phi}_{,n} \tag{65}$$

where $k,l = 0,x,y$. In the previous expression, $\mathbf{H}$, $\mathbf{H}_{kl}$, $\mathbf{G}$, $\mathbf{G}_{kl}$ are known $M \times N$ matrices originating from the integration of the kernels on the boundary elements of Eqs. (52), (53) and $\tilde{\boldsymbol{\phi}}$, $\tilde{\boldsymbol{\phi}}_{,kl}$ are $M \times 1$ vectors including the values of ϕ and its derivatives at the interior collocation points. Note that since the distance r_{jm} in the kernels does not vanish, the resulting line integrals are regular.

Subsequently, substituting Eqs. (65) for Eqs. (54) and using Eqs. (61) and (62) to eliminate $\boldsymbol{\phi}$ and $\boldsymbol{\phi}_{,n}$ yields the following linear system of equations for $\mathbf{a}$

$$\mathbf{K}\mathbf{a} = \mathbf{g} \tag{66}$$

Finally, the coefficient $\mathbf{a}$ is employed in Eqs. (62) and (63) to evaluate $\boldsymbol{\phi}$ and $\boldsymbol{\phi}_{,n}$ while the warping function and its derivatives at the M nodal points are computed from Eqs. (64) and (65). Since points $P \in \Omega$ do not coincide with the nodal points these quantities are evaluated from the discretized counterparts of Eqs. (52) and (53).

4. Numerical Examples

On the basis of the numerical procedure presented in the previous section, a FORTRAN code has been written and numerical results for certain materials have been obtained, which illustrate the applicability, effectiveness and accuracy of the method. The employed approximation functions f_j are the multiquadrics, which are defined as

$$f_j = \sqrt{r^2 + c^2}\,, \quad r = \sqrt{\left(x - x_j\right)^2 + \left(y - y_j\right)^2}\,, \quad j = 1, 2, \ldots, M \tag{67}$$

with x_j, y_j being the collocation nodal points inside Ω and c is the shape parameter [28]. Using these radial basis functions, the particular solution of Eq. (50) is obtained as

$$\hat{u}_j = -\frac{c^3}{3}\ell n\left(c\sqrt{r^2 + c^2} + c^2\right) + \frac{1}{9}\left(r^2 + 4c^2\right)\sqrt{r^2 + c^2} \tag{68}$$

The derivatives of $\hat{u}_j$ are given in the Appendix A.

4.1 Homogeneous orthotropic and anisotropic bar with elliptical cross-section

As a first example, we treat the torsion problem of the homogeneous orthotropic and anisotropic bar with elliptical cross-section having semi-axes $a = 3$ and $b = 2$ (see Fig. 3). The employed data for the orthotropic material are $a_{44} = 0.5$, $a_{45} = 0$ and $a_{55} = 1.0$, and also the data for the anisotropic material are $a_{44} = 0.5$, $a_{45} = 0.5$ and $a_{55} = 1.0$. The distribution of the $N = 100$ boundary elements and $M = 101$ nodal points is shown in Fig. 4. Numerical results for the torsion constant I_t and the tangential stresses τ_{tz} at points *A* and *B* are given in Table 1. The results are compared with those obtained by the exact solution presented by Lekhnitskii in [1] and are found to be in good agreement. Moreover, the contours of the warping surface of the orthotropic and anisotropic bars are shown in Figs. 5 and 6, respectively.

Table 1. Torsion constant and boundary tangential stresses in Example 4.1

AEM-Orthotropic	AEM-Anisotropic	Exact
	I_t	
61.6961	61.4526	61.6895
	$(\tau_{tz})_A / \theta$	
2.2008	2.2023	2.1819
	$(\tau_{tz})_B / \theta$	
3.2607	3.3296	3.2727

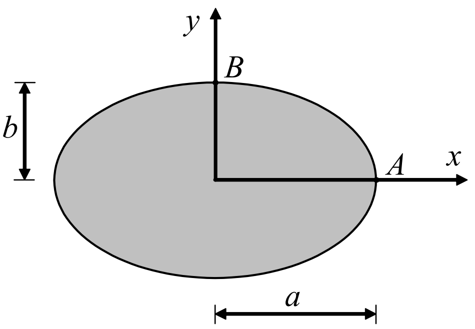

Fig. 3. Elliptical cross-section

4.2 Nonhomogeneous orthotropic bar with elliptical cross-section

As a second example, the orthotropic bar with elliptical cross-section of the previous example has been studied. In this case, the material has position dependent constants which are given as: $a_{44}(x,y) = 0.5 + kx^2y^2$, $a_{45}(x,y) = 0$, and $a_{55}(x,y) = 1 + kx^2y^2$. In this respect, Eqs. (10)-(12) and Eqs. (15)-(17) become

$A_{44} = 1/(1 + kx^2y^2)$, $\quad A_{45} = 0$, $\quad A_{55} = 1/(0.5 + kx^2y^2)$

$D = -2kxy^2/(1 + kx^2y^2)^2$, $\quad E = -2kx^2y/(0.5 + kx^2y^2)^2$

$g = 2kxy\left[x^2/(0.5 + kx^2y^2)^2 - y^2/(1 + kx^2y^2)^2\right]$

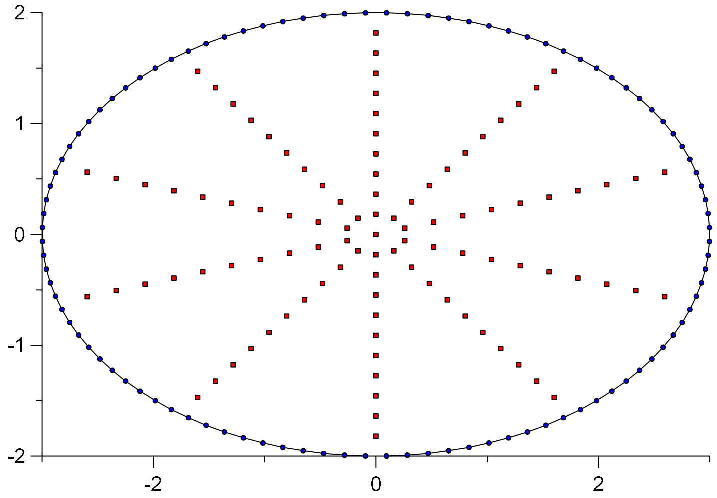

Fig. 4. Boundary and domain nodal points in Example 4.1

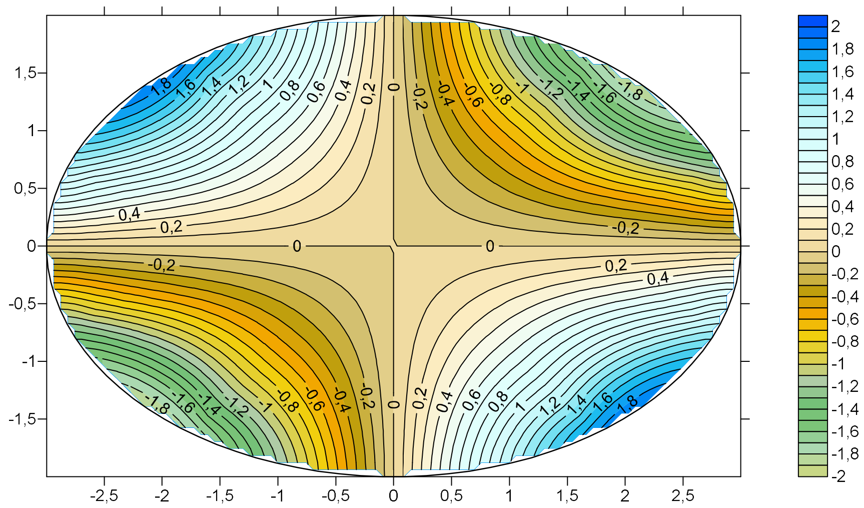

Fig. 5. Contours of the warping surface of the orthotropic bar in Example 4.1

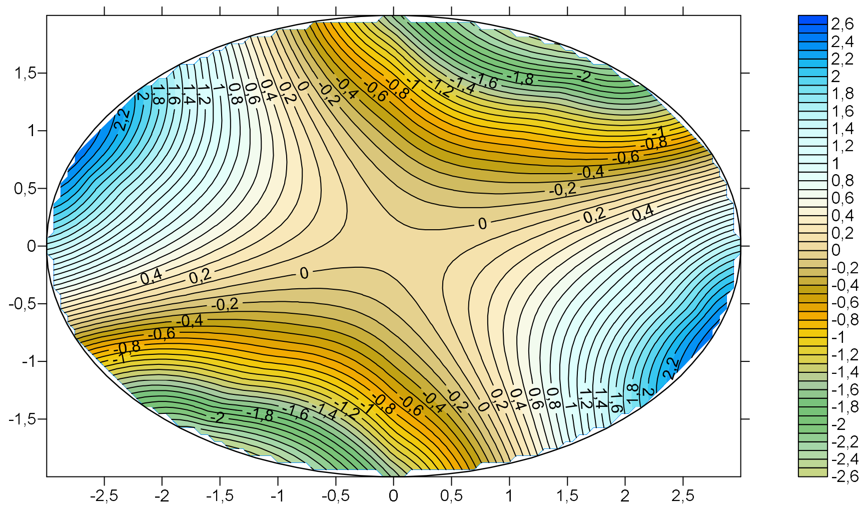

Fig. 6. Contours of the warping surface of the anisotropic bar in Example 4.1

The variation of the torsion constant I_t and the tangential stresses τ_{tz} at points *A* and *B* for various values of the parameter k are presented in Table 2. It is observed that as the parameter k value tends to zero, the solution approaches that of the homogeneous orthotropic material [1]. Moreover, the warping function vector for $k = 0.05$ and $k = -0.05$ are shown in Figs. 7 and 8, respectively.

Table 2. Torsion constant and boundary tangential stresses in Example 4.2

$k=0.05$	$k=0.01$	$k=0.001$	$k=0$	$k=-0.001$	$k=-0.01$	$k=-0.05$
			I_t			
54.7553	60.0935	61.5291	61.6961	61.8637	63.4550	74.2029
			$(\tau_{tz})_A/\theta$			
1.9704	2.1460	2.1951	2.2008	2.2066	2.2613	2.6221
			$(\tau_{tz})_B/\theta$			
3.1486	3.2368	3.2583	3.2607	3.2631	3.2845	3.3135

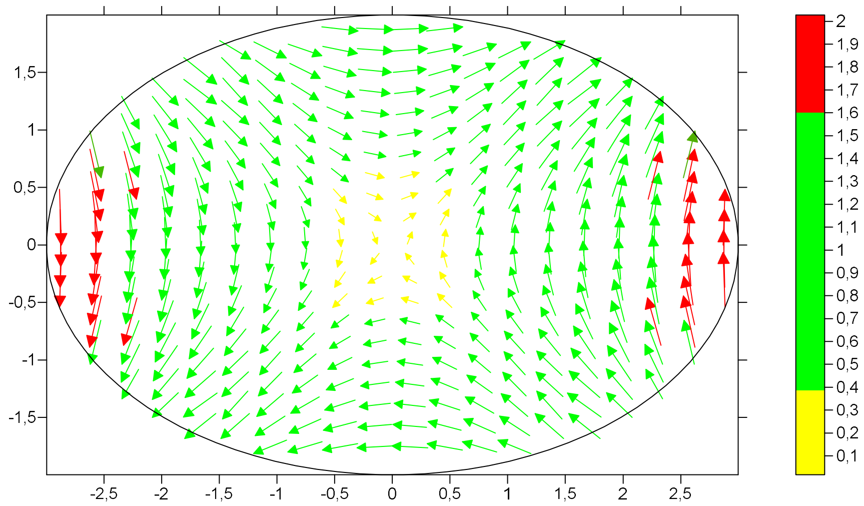

Fig. 7. Warping function vector of the non-homogeneous orthotropic bar ($k=0.05$) in Example 4.2

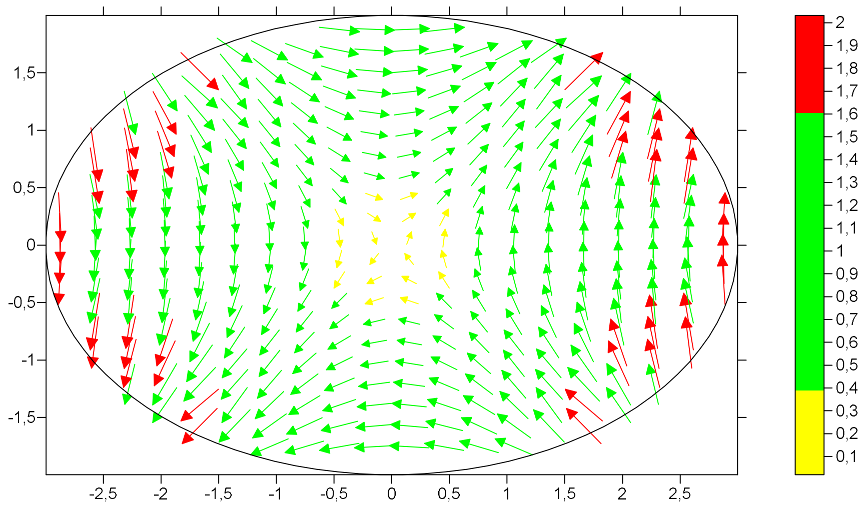

Fig. 8. Warping function vector of the non-homogeneous orthotropic bar ($k=-0.05$) in Example 4.2

4.3 Nonhomogeneous orthotropic bar with elliptical cross-section

In this example, we study the torsion problem of a non-homogeneous isotropic bar with rectangular cross section $a=4.0$ and $b=3.0$ (see Fig. 9). In this case, the material has a variable shear modulus given as: $G=1+kx^2y^2$. The distribution of the $N=144$ boundary elements and $M=181$ nodal points is shown in Fig. 10. The variation of the torsion constant I_t and the tangential stresses τ_{tz} at points A and B for various values of the parameter k are presented in Table 3. It is observed that as the parameter k value tends to zero, the solution approaches that of the homogeneous isotropic material [26]. Moreover, the warping surface of the non-homogeneous isotropic bar for $k=0.5$ is shown in Fig. 11.

Table 3. Torsion constant and boundary tangential stresses in Example 4.3

$k=1$	$k=0.5$	$k=0.1$	$k=0.01$	$k=0.001$	$k=0$
I_t					
33.699	27.901	21.400	19.604	19.414	19.392
$(\tau_{tz})_A/\theta$					
3.0342	2.7321	2.2949	2.1681	2.1545	2.1530
$(\tau_{tz})_B/\theta$					
3.4664	2.9794	2.5385	2.4216	2.4093	2.4079

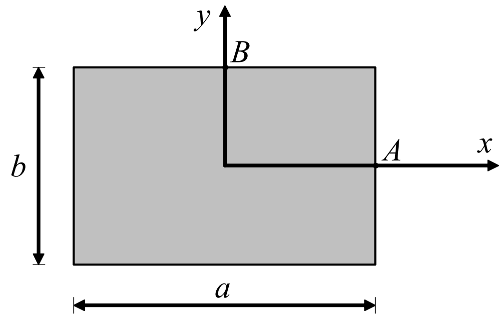

Fig. 9. Rectangular cross-section

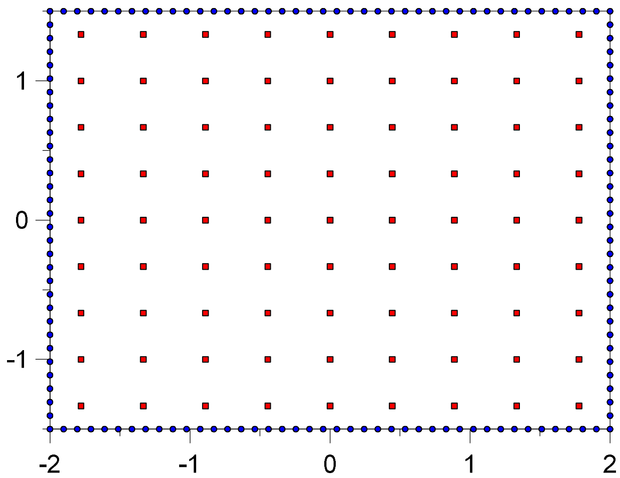

Fig. 10. Boundary and domain nodal points in Example 4.3

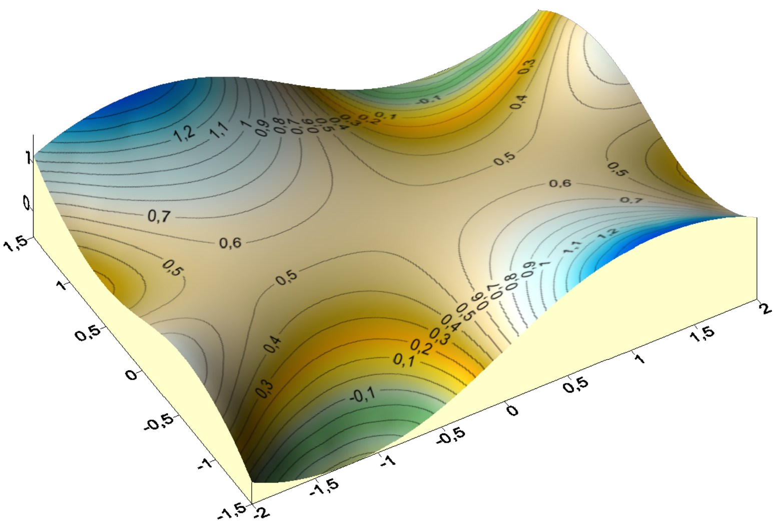

Fig. 11. Warping surface of the non-homogeneous isotropic bar ($k=0.5$) in Example 4.3

5. Conclusions

In this paper, the BEM is developed for the analysis of non-homogeneous anisotropic bars. The developed method is boundary-only in the sense that only the boundary descritization and integration is required. The employed fundamental solution is simple for the Laplace equation. Thus there's no need to establish the fundamental solution of the complete second order partial differential equation with variable coefficients, a problem whose solution is practically impossible or at least computationally complicated. The numerical results demonstrate the efficiency and accuracy of the method.

Appendix A

A.1. Derivatives of the kernel functions

The derivatives of the distance

$$r=\sqrt{(\xi-x)^2+(\eta-y)^2} \qquad \{x,y\}\in\Omega,\ \{\xi,\eta\}\in\Gamma \tag{A.1}$$

are evaluated from the following relations

$$\begin{gathered} r,_x=-r,_\xi=-\frac{\xi-x}{r}, \quad r,_y=-r,_\eta=-\frac{\eta-y}{r} \\ r,_{xx}=\frac{r_y^2}{r}, \quad r,_{yy}=\frac{r,_x^2}{r}, \quad r,_{xy}=-\frac{r,_x r,_y}{r} \\ r,_n=-(r,_x n_x+r,_y n_y), \quad r,_t=-(-r,_x n_y+r,_y n_x) \end{gathered} \tag{A.2}$$

n_ξ, n_η are directional cosines of the outward normal vector to the boundary at point $\{\xi,\eta\}$.

Using Eqs. (A.2) the derivatives of the fundamental solution may be expressed as

$$\begin{gathered} u,_x^*=\frac{1}{2\pi}\frac{r,_x}{r}, \quad u,_y^*=\frac{1}{2\pi}\frac{r,_y}{r}, \quad u,_{xx}^*=\frac{1}{2\pi}\frac{r,_y^2-r,_x^2}{r^2}, \\ u,_{yy}^*=-u,_{xx}^*, \quad u,_{xy}^*=-\frac{1}{\pi}\frac{r,_x r,_y}{r^2}, \quad u,_{nxx}^*=-\frac{1}{\pi}\frac{(r,_y^2-r,_x^2)r,_n+2r,_x r,_y r,_t}{r^3} \\ u,_{nyy}^*=-u,_{nxx}^*, \quad u,_{nxy}^*=-\frac{1}{\pi}\frac{(r,_y^2-r,_x^2)r,_t-2r,_x r,_y r,_n}{r^3} \end{gathered} \tag{A.3}$$

A.2. Derivatives of the function $\hat{u}_j$

Differentiating Eq. (68) gives

$$\begin{gathered} \hat{u}_j,_x=\frac{1}{3\sqrt{r^2+c^2}}(r^2+2c^2-\frac{c^3}{\sqrt{r^2+c^2}+c})(x-x_j) \\ \hat{u}_j,_y=\frac{1}{3\sqrt{r^2+c^2}}(r^2+2c^2-\frac{c^3}{\sqrt{r^2+c^2}+c})(y-y_j) \\ \hat{u}_j,_{xx}=\frac{1}{3(r^2+c^2)^{3/2}}[r^2+c^3\frac{2\sqrt{r^2+c^2}+c}{(\sqrt{r^2+c^2}+c)^2}](x-x_j)+\frac{1}{3\sqrt{r^2+c^2}}(r^2+2c^2-\frac{c^3}{\sqrt{r^2+c^2}+c}) \\ \hat{u}_j,_{yy}=\frac{1}{3(r^2+c^2)^{3/2}}[r^2+c^3\frac{2\sqrt{r^2+c^2}+c}{(\sqrt{r^2+c^2}+c)^2}](y-y_j)+\frac{1}{3\sqrt{r^2+c^2}}(r^2+2c^2-\frac{c^3}{\sqrt{r^2+c^2}+c}) \\ \hat{u}_j,_{xy}=\frac{1}{3(r^2+c^2)^{3/2}}[r^2+c^3\frac{2\sqrt{r^2+c^2}+c}{(\sqrt{r^2+c^2}+c)^2}](x-x_j)(y-y_j) \end{gathered} \tag{A.4}$$

It can be readily proved that

$\lim_{r\to 0}\hat{u}_j,_x=0$, $\lim_{r\to 0}\hat{u}_j,_y=0$

$$\lim_{r\to 0}\hat{u}_j,_{xx}=\frac{c}{2}, \quad \lim_{r\to 0}\hat{u}_j,_{yy}=\frac{c}{2}, \quad \lim_{r\to 0}\hat{u}_j,_{xy}=0 \tag{A.5}$$

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