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Time integration of rectangular membrane free vibration using spline-based differential quadrature

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Abstract

In this paper, numerical spline-based differential quadrature is presented for solving the boundary and initial value problems, and its application is used to solve the fixed rectangular membrane vibration equation. For the time integration of the problem, the Runge–Kutta and spline-based differential quadrature methods have been applied. The Runge–Kutta method was unstable for solving the problem, with large errors in its results, but the spline-based differential quadrature method obtained results that agree with the exact solution. The relative errors were calculated and investigated for different values of time and spatial nodes of discretisation. It seems that the spline-based differential quadrature method is proper for the full simulation of membrane vibration in both spatial and temporal solutions. For the time solving of the membrane vibration, conventional methods, such as the Runge–Kutta method, are not useful even if the time steps are considered too small.

Keywords: Runge–Kutta method, spline-based differential quadrature method, membrane vibration, time integration.

1. Introduction

The differential quadrature method (DQM) has attracted appreciable attention over the past two decades. This method, which was first introduced by Belman and Casti [1], can be used for calculating the natural frequencies of carbon nanotubes [2], functionally graded Euler–Bernoulli beams [3], rotating beams [4], plates and shells of variable thickness [5], nanotube-laminated composites [6], and doubly-curved shells [7]. The DQM can also be used in a vibration analysis of nanobeams [8] and nanoplates [9,14] based on nonlocal theories.

In the DQM, any derivation can be approximated by weighting some coefficients, and based on how these coefficients are calculated, there are different types of DQM. In generalized [4–9, 12] and harmonic [2] differential quadrature methods, the coefficients are calculated by using Lagrange interpolation polynomial and harmonic functions, respectively. In the modified cubic B-spline differential quadrature method (SDQM), a modification of cubic B-spline functions is used to find the weighting coefficients [10].

The DQM can also be applied in time integration. Tanaka and Chen [13] presented a combined application of the dual reciprocity boundary element method and the differential quadrature method to time-dependent diffusion problems. They found that the DQM is numerically accurate, stable and computationally efficient in computing dynamic problems.

In this paper, the rectangular membrane equation was solved by discretising the spatial derivatives by the SDQM. As the initial value problem, the obtained equation was solved numerically by the Runge–Kutta method, but its solution was very unstable even when the time step was considered too small. Therefore, the SDQM was used by discretising the temporal derivatives. The relative errors of the computed middle displacement of the membrane are listed in Table 1, and these results show good agreement with the exact solution. It seems that conventional time integration methods, such as Runge–Kutta, cannot be applied to initial value equations obtained by using the SDQM to discretise the spatial derivatives. It is noted that the Newmark method failed; instead, the membrane vibration equation can be solved numerically with good accuracy by using the SDQM.

2. Weighting coefficient for SDQM

The approximation of a function derivative with respect to space variable at a node in SDQM is given by [11]:

$$D_n\{f(x)\}_i = \sum_{j=0}^{N} C_{i,j}^{(n)} f(x_j) \tag{1}$$

where $i, j = 0, 1/2, 1, 3/2, 2, ..., N-2, N-3/2, N-1, N-1/2, N$ and D_n is a differential operator of order n. The subscript i indicates the value of $D_n\{f(x)\}$ at node x_i in which $x_i = ih$ and for $0 \le x \le 1$, h is equal to $\frac{1}{N}$. In the SDQM, eq. (1) should be satisfied when function $f(x)$ takes the cardinal spline basis functions $\Omega_j(x)$; therefore, weighting coefficients are given in explicit forms:

$$C_{i,j}^{(n)} = \Omega_j^{(n)}(x_i) \tag{2}$$

The cardinal cubic B-spline interpolation function is written as follows:

$$\begin{aligned}
&\Omega_0(x) = 277\Phi_{-2}(x) + 21\Phi_{-1}(x) + \Phi_0(x)\\
&\Omega_{\frac{1}{2}}(x) = 288\Phi_{-2}(x)\\
&\Omega_1(x) = -3608\Phi_{-2}(x) - 164\Phi_{-1}(x) + \Phi_1(x)\\
&\Omega_{\frac{3}{2}}(x) = 6048\Phi_{-2}(x) + 288\Phi_{-1}(x)\\
&\Omega_2(x) = -3423\Phi_{-2}(x) - 164\Phi_{-1}(x) + \Phi_2(x)\\
&\Omega_3(x) = 440\Phi_{-2}(x) + 21\Phi_{-1}(x) + \Phi_3(x)\\
&\Omega_4(x) = -21\Phi_{-2}(x) - \Phi_{-1}(x) + \Phi_4(x)\\
&\Omega_i(x) = \Phi_i(x), \quad 5 \le i \le N-5\\
&\Omega_{N-4}(x) = -21\Phi_{N+2}(x) - \Phi_{N+1}(x) + \Phi_{N-4}(x)\\
&\Omega_{N-3}(x) = 440\Phi_{N+2}(x) + 21\Phi_{N+1}(x) + \Phi_{N-3}(x)\\
&\Omega_{N-2}(x) = -3423\Phi_{N+2}(x) - 164\Phi_{N+1}(x) + \Phi_{N-2}(x)\\
&\Omega_{N-\frac{3}{2}}(x) = 6048\Phi_{N+2}(x) + 288\Phi_{N+1}(x)\\
&\Omega_{N-1}(x) = -3608\Phi_{N+2}(x) - 164\Phi_{N+1}(x) + \Phi_{N-1}(x)\\
&\Omega_{N-\frac{1}{2}}(x) = 288\Phi_{N+2}(x)\\
&\Omega_N(x) = 277\Phi_{N+2}(x) + 21\Phi_{N+1}(x) + \Phi_N(x)
\end{aligned} \tag{3}$$

where

$$\begin{aligned}
\Phi_j(x) = &\frac{10}{3}\varphi_3(x-x_j) - \frac{4}{3}\left[\varphi_3(x-x_j+\frac{h}{2}) + \varphi_3(x-x_j-\frac{h}{2})\right]\\
&+\frac{1}{6}\left[\varphi_3(x-x_j+h) + \varphi_3(x-x_j-h)\right]
\end{aligned} \tag{4}$$

in which

$$\varphi_3(x)=\frac{1}{6h^3}\begin{cases}0 & x\le -2h\\ (x+2h)^3 & -2h\le x\le -h\\ (x+2h)^3-4(x+h)^3 & -h\le x\le 0\\ (2h-x)^3-4(h-x)^3 & 0\le x\le h\\ (2h-x)^3 & h\le x\le 2h\\ 0 & x\ge 2h\end{cases} \tag{5}$$

For solving the initial problem using the SDQM, the following discretisation can be applied for the first-order and second-order derivatives, respectively. [11]

$$D_1\{f(x)\}_i=C_{i0}^{(1)}f(x_0)+\sum_{j=\frac{1}{2}}^{M}C_{ij}^{(1)}f(x_j) \tag{6}$$

$$D_2\{f(x)\}_i=C_{i0}^{(1)}f'(x_0)+\sum_{j=\frac{1}{2}}^{M}C_{ij}^{(1)}C_{j0}^{(1)}f(x_0)+\sum_{j=\frac{1}{2}}^{M}C_{ij}^{(1)}\sum_{k=\frac{1}{2}}^{M}C_{jk}^{(1)}f(x_k) \tag{7}$$

3. Spatial SDQM discretisation

In this article, the free vibration of a fixed-edge rectangular membrane with a and b lengths is studied.

$$\frac{\partial^2 u}{\partial t^2}=c^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \tag{8}$$

$$u(0,y,t)=u(a,y,t)=u(x,0,t)=u(x,b,t)=0 \tag{9}$$

where $u(x,y,t)$ is the displacement, x and y are the coordinates and $c^2=\frac{T}{\rho}$ in which T and ρ are tension and density. The initial velocity is zero, and the initial displacement is:

$$u(x,y,0)=0.1(4x-x^2)(2y-y^2) \tag{10}$$

For $c^2=5$, $a=4$ and $b=2$, the exact solution of eq. (8) is:

$$u(x,y,t)=0.426050\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(2m-1)^3(2n-1)^3}\cos\left(\frac{\pi\sqrt{5(m^2+4n^2)}}{4}t\right)\sin\frac{m\pi x}{4}\sin\frac{n\pi y}{2} \tag{11}$$

The SDQM discretisation of the eq. (8) is written as below:

$$\ddot{u}_{i,j}=c^2\left[\frac{1}{a^2}\sum_{m=0}^{N}C_{i,m}^{(2)}u_{m,j}+\frac{1}{b^2}\sum_{m=0}^{N}C_{j,m}^{(2)}u_{i,m}\right] \tag{12}$$

By considering the boundary equations and relations (5), the above equation can be rewritten as follows:

$$\ddot{u}_{i,j}=c^2\left[\frac{1}{a^2}\sum_{m=\frac{1}{2}}^{N-1}C_{i,m}^{(2)}u_{m,j}+\frac{1}{b^2}\sum_{m=\frac{1}{2}}^{N-1}C_{j,m}^{(2)}u_{i,m}\right] \tag{13}$$

4. Runge–Kutta method

For solving eq. (13) as the initial value problem, it should be written in state space.

$$\{\dot{X}\}=[K]\{X\} \tag{14}$$

where

$$\{X\}=\{[1]_{1\times(N-1)}\quad u_{1,1}\quad u_{2,1}\quad \ldots\quad u_{N-1,1}\quad u_{1,2}\quad \ldots\quad u_{N-1,2}\quad \ldots\quad u_{1,N-1}\quad \ldots\quad u_{N-1,N-1}\}^T \tag{15}$$

and $[K]_{2(N-1)\times 2(N-1)}$ contains the corresponding coefficient of each X_i.

By using the Runge–Kutta method, eq. (14) is solved by applying the initial conditions. The result is depicted in Fig. 1. for the middle point of the membrane, $N=15$ and time step 1×10^{-12} by comparing with the exact solution. As shown in the figure, the Runge–Kutta solution diverged at $t=0.14(s)$. In addition, the solution becomes unstable for the other values of N. Also the Newmark method fails for solving eq. (18), and the corresponding solution also becomes unstable. It seems that common time integration methods, such as the Runge–Kutta and Newmark methods, should not be used for problems discretised based on the SDQM. Therefore, the

SDQM method should be applied for the time integration of such problems to achieve stable and more accurate results.

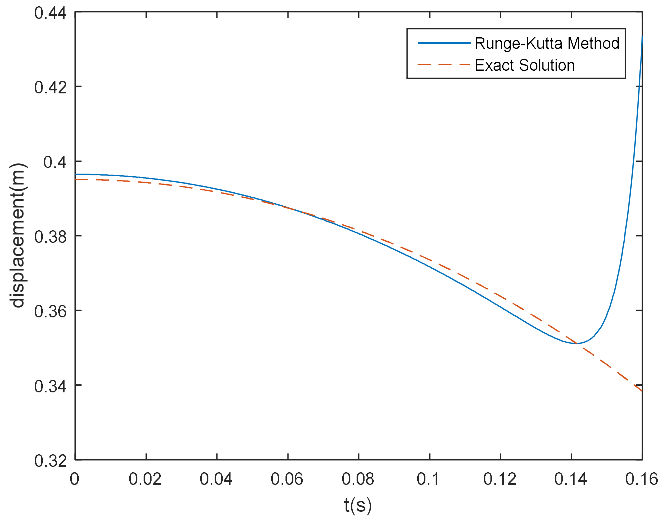

Fig. 1. Comparison of the Runge–Kutta method and the exact solution

5. Numerical solution based on SDQM

Both the time and spatial discretisation of eq. (8) based on SDQM by considering the boundary conditions (9) is presented below:

$$T_{k0}^{(1)}\left(\frac{\partial u}{\partial t}\right)_{i,j,0}+\sum_{m=\frac{1}{2}}^{M}T_{km}^{(1)}T_{m0}^{(1)}u_{i,j,0}+\sum_{m=\frac{1}{2}}^{M}T_{km}^{(1)}\sum_{p=\frac{1}{2}}^{M}T_{mp}^{(1)}u_{i,j,p}=c^2\left[\frac{1}{a^2}\sum_{m=\frac{1}{2}}^{N-1}C_{i,m}^{(2)}u_{m,j,k}+\frac{1}{b^2}\sum_{m=\frac{1}{2}}^{N-1}C_{j,m}^{(2)}u_{i,m,k}\right] \quad (16)$$

where

$$T_{ij}^{(1)}=\frac{C_{ij}^{(1)}}{t_f} \qquad i,j=0,1/2,1,3/2,2,\ldots,M-2,M-3/2,M-1,M-1/2,M \quad (17)$$

in which t_f is the end time of the simulation. By considering zero initial velocity, eq. (16) can be rewritten as follows:

$$\sum_{m=\frac{1}{2}}^{M}T_{km}^{(1)}T_{m0}^{(1)}u_{i,j,0}+\sum_{m=\frac{1}{2}}^{M}T_{km}^{(1)}\sum_{p=\frac{1}{2}}^{M}T_{mp}^{(1)}u_{i,j,p}=c^2\left[\frac{1}{a^2}\sum_{m=\frac{1}{2}}^{N-1}C_{i,m}^{(2)}u_{m,j,k}+\frac{1}{b^2}\sum_{m=\frac{1}{2}}^{N-1}C_{j,m}^{(2)}u_{i,m,k}\right] \quad (18)$$

The above equation can be written in matrix form:

$$[A][U]+[U][B]+[C]=0 \quad (19)$$

where

$$[U]=\begin{bmatrix} u_{1,1,1} & \cdots & u_{1,1,M} \\ \vdots & \cdots & \vdots \\ u_{N-1,1,1} & \cdots & u_{N-1,1,M} \\ u_{1,2,1} & \cdots & u_{1,2,M} \\ \vdots & \cdots & \vdots \\ u_{N-1,2,1} & \cdots & u_{N-1,2,M} \\ \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots \\ u_{1,N-1,1} & \cdots & u_{1,N-1,M} \\ \vdots & \cdots & \vdots \\ u_{N-1,N-1,1} & \cdots & u_{N-1,N-1,M} \end{bmatrix} \quad (20)$$

and $[A]_{M\times(N+3)^2}$ is the coefficient matrix of the right side of eq. (16), $[B]_{(N+3)^2\times M}$ is the coefficient matrix of the second term in the left side of eq. (16) and $[C]_{M\times M}$ is related to the first term of the left side of eq. (16) representing the initial displacement. Eq. (19), which is called a Sylvester equation, is solved by the “lyap” function in MATLAB software, and the displacement of the rectangular membrane is obtained.

The relative error values of the SDQM in calculating the middle point of the rectangular membrane in $t=1(s)$ for $t_f=2(s)$ are listed in Table 1 for M and N from 30 to 50 nodes. In many cases, the M and N values have satisfactory results (i.e., error ≤ 1%), but in general, the increasing or decreasing of the relative error has no regular, specified manner. For example, when M is equal to 48 nodes or 50 nodes, the relative errors are less than 0.3% for any values of N, but when M is equal to 49 nodes, the relative errors are less than 8%. For both the temporal and spatial solutions of the membrane vibration, the SDQM achieved good agreement with the exact solution. For example, the displacement of the membrane is shown in Fig. 2. for four periods of time.

Table 1. Error values of the $u(\frac{a}{2},\frac{b}{2},1)$

$\left\lvert\frac{exact-SDQM}{exact}\right.$	M																			
	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
31	2.74	0.67	2.06	0.75	0.72	0.39	0.92	0.84	0.85	0.42	0.88	0.44	0.95	0.20	0.35	0.38	0.08	0.05	2.41	0.11
32	2.88	0.67	1.85	0.75	0.78	0.60	1.04	0.95	0.88	0.36	0.96	0.46	0.96	0.26	0.34	0.37	0.06	0.05	5.32	0.04
33	2.85	0.70	1.58	0.71	0.78	0.78	1.04	0.99	0.86	0.27	0.96	0.63	0.87	0.35	0.34	0.32	0.04	0.04	5.65	0.01
34	2.95	0.76	1.41	0.70	0.84	1.04	1.10	1.02	0.90	0.09	1.03	0.40	0.84	0.46	0.36	0.27	0.02	0.02	7.57	0.05
35	2.90	0.77	1.19	0.65	0.84	1.30	1.04	0.94	0.86	0.83	1.00	0.18	0.73	0.52	0.35	0.25	0.07	0.08	3.83	0.12
36	2.96	0.83	1.04	0.62	0.90	1.64	1.05	0.13	0.89	0.24	1.04	0.09	0.69	0.61	0.36	0.27	0.14	0.12	4.67	0.08
37	2.88	0.95	1.00	0.55	0.89	1.93	1.00	0.82	0.86	0.29	1.01	0.01	0.63	0.64	0.34	0.26	0.16	0.15	2.09	0.09
38	2.91	0.97	0.94	0.50	0.94	2.26	1.01	0.69	0.89	0.22	1.04	0.09	0.60	0.71	0.37	0.25	0.25	0.17	1.93	0.14
39	2.80	0.97	0.84	0.44	0.89	2.44	0.97	0.56	0.86	0.13	1.01	0.17	0.54	0.74	0.33	0.28	0.04	0.18	1.73	0.16
40	2.78	1.02	0.75	0.40	0.85	2.59	0.99	0.44	0.74	0.13	1.05	0.25	0.52	0.78	0.32	0.30	0.62	0.22	1.65	0.18
41	2.65	1.04	0.68	0.34	0.65	2.53	0.96	0.41	0.50	0.16	1.06	0.32	0.48	0.76	0.28	0.31	1.35	0.25	1.53	0.17
42	2.57	1.10	0.67	0.26	0.45	2.43	0.98	0.36	0.10	0.24	1.13	0.39	0.47	0.75	0.24	0.33	1.81	0.29	1.51	0.09
43	2.41	1.12	0.79	0.13	0.63	2.18	0.95	0.33	0.30	0.30	1.13	0.44	0.45	0.71	0.20	0.32	1.90	0.33	1.44	0.07
44	2.52	1.18	1.09	0.04	0.94	1.94	0.98	0.31	0.60	0.26	1.17	0.52	0.44	0.65	0.23	0.37	1.90	0.42	1.43	0.30
45	2.29	1.20	1.43	0.30	1.03	1.63	0.86	0.31	0.71	0.09	1.13	0.58	0.43	0.68	0.30	0.39	1.80	0.41	1.49	0.23
46	2.17	1.27	1.72	0.69	1.08	1.36	0.92	0.31	0.77	0.05	1.12	0.67	0.43	0.70	0.37	0.44	1.78	0.37	1.53	0.18
47	2.02	1.29	1.75	1.25	1.06	1.08	0.90	0.26	0.78	0.18	1.03	0.72	0.43	0.65	0.48	0.47	1.70	0.14	1.64	0.17
48	1.90	1.36	1.78	2.16	1.07	0.85	0.90	0.27	0.79	0.28	0.94	0.81	0.43	0.65	0.62	0.56	1.69	0.05	0.06	0.13
49	1.76	1.39	1.68	3.61	1.02	0.64	0.88	0.31	0.78	0.36	0.80	0.85	0.30	0.63	0.78	0.62	1.63	0.00	1.17	0.23
50	1.65	1.46	1.66	6.24	1.02	0.46	0.88	0.41	0.78	0.41	0.67	0.92	0.49	0.64	0.96	0.67	1.63	0.00	1.23	0.35

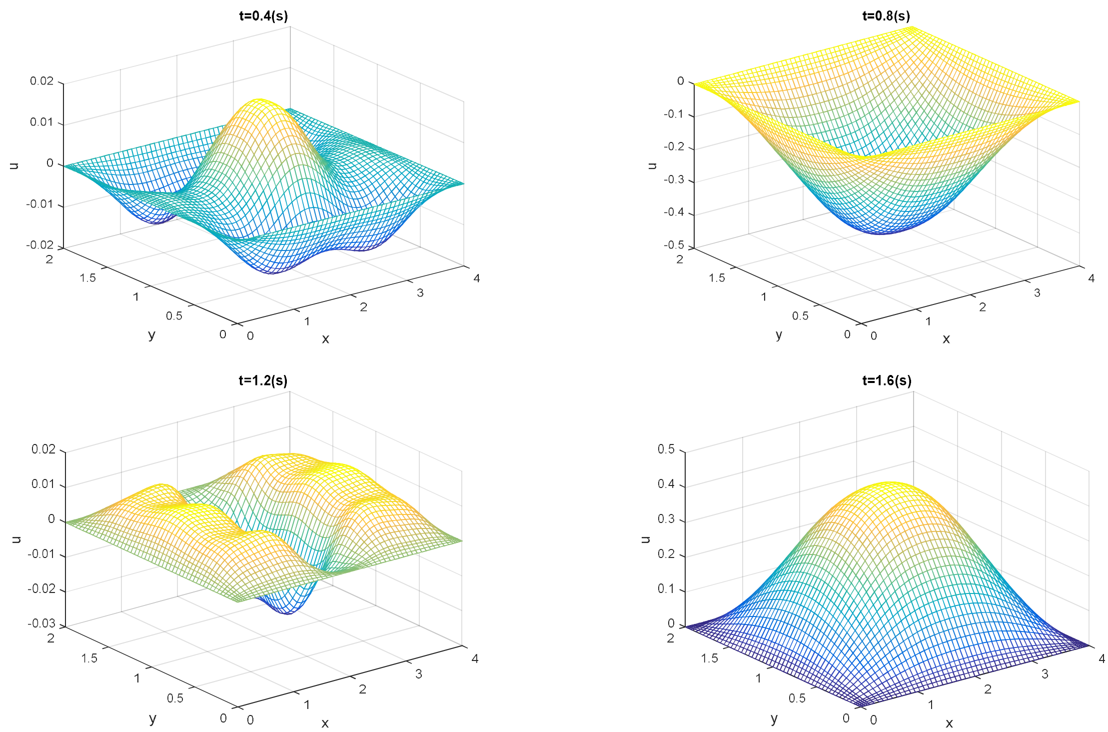

Fig. 2. The displacement of the membrane for four periods of time

6. Conclusion

The full solution of the membrane vibration is divided into two parts: spatial and temporal solutions. The spatial discretisation is applied to the membrane vibration equation easily, and the obtained equation is the initial value problem that could be solved by conventional methods such as the Runge–Kutta and Newmark methods, but the numerical solutions are unstable and have large errors. In this paper, the SDQM is proposed for solving the time solution of the membrane vibration equation instead of conventional time-integration methods. By using the SDQM, the full solution achieves good agreement with exact solution and has good stability.

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