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Research Article

Ritz Method Application to Bending, Buckling and Vibration Analyses of Timoshenko Beams via Nonlocal Elasticity

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Abstract. Bending, buckling and vibration behaviors of nonlocal Timoshenko beams are investigated in this research using a variational approach. At first, the governing equations of the nonlocal Timoshenko beams are obtained, and then the weak form of these equations is outlined in this paper. The Ritz technique is selected to investigate the behavior of nonlocal beams with arbitrary boundary conditions along them. To find the equilibrium equations of bending, buckling, and vibration of these structures, an analytical procedure is followed. In order to verify the proposed formulation, the results for the nonlocal Timoshenko beams with four classical boundary conditions are computed and compared wherever possible. Since the Ritz technique can efficiently model the nano-sized structures with arbitrary boundary conditions, two types of beams with general boundary conditions are selected, and new results are obtained.

Keywords: Ritz method, Weak form, Bending, Buckling, Vibration, Nonlocal Timoshenko beam.

1. Introduction

It is well known that the nano-sized structures such as nano-beams and nano-plates are the main components in NEMS devices, and thus they have received increasing attention in industries due to wonderful mechanical properties [1-8]. There have been a number of experimental and theoretical studies in the literature dealing with the mechanical properties of the nano-sized structures, and therefore, it can be found that the size effect has a major role on the static and dynamic behavior of materials.

Due to the difficulties in experimental specification at the nano-scale and their being time-consuming, numerical simulation and analysis of the nano-structures have been presented extensively by the researchers, and they became interested to develop the size-dependent continuum theories. Therefore, in modeling the small sized structures, various size-dependent continuum theories have received increasing attention. For instance, micro-morphic theory [9], micro-polar theory [9, 10], couple stress theory [11], and nonlocal elasticity theory [12] can be pointed out.

The nonlocal elasticity theory which was formally initiated by the paper of Eringen [12] has been adopted by many researchers, as can be seen in the literature. According to the theory of Eringen, the stress at any point in the body depends not only on the strain at that point, but also on strains at all other points of the body. This definition is based on the atomic theory of lattice dynamics and some experimental observations on phonon dispersion. It is also important to note that the nonlocal elasticity theory was used in two general forms: nonlocal differential elasticity and nonlocal integral elasticity [13], but the former is more popular due to its simplicity.

Many researchers have used the nonlocal elasticity theory to examine the bending, buckling, and vibration analyses of nano-beams depending on the beam model. The study by Peddieson et al. [14] can be considered to be a pioneering work



which first applied the nonlocal elasticity theory of Eringen to the nanotechnology. In this work, a nonlocal version of the Euler-Bernoulli beam theory with the nonlocal differential elasticity approach is formulated, and the flexural behavior of nano-beams is studied. In the field of nonlocal integral elasticity, Polizzotto [15] reported three variational principles including the total potential energy, the complementary energy, and the mixed Hu-Washizu principles.

It is worth noting that most of the attention in the literature has been focused on deriving the governing equations and the corresponding boundary conditions of nano-beams and nano-plates by the well-known technique of variational calculus and with the nonlocal differential elasticity approach. Therefore, the exact solution of most analyses on the nano-beams with simple domains and classical boundary conditions can be found in the literature [1, 2, 16-25]. Such solutions are not generally available for the nano-sized structures with complicated geometries and general boundary conditions. It's quite obvious that the finite element and Ritz methods can efficiently analyze the structures with arbitrary boundary conditions. So, this has made these techniques an efficient alternate to the previous solutions for nano-sized beams depending on the beam model. The latest research activities on nonlocal elastic structures with nonlocal differential elasticity theory and using the finite element and Ritz approaches are the works done by Phadikar et al. [26] and Ghannadpour et al. [27]. In these works, the finite element and Ritz formulations for nonlocal Euler-Bernoulli beams have been derived.

To the best knowledge of the author, no work has been reported for the Ritz formulation and the weak form of the governing equations of Timoshenko beams with the nonlocal differential elasticity approach. Thus, the aim of this paper is to obtain analytical expressions for the stiffness, buckling stiffness, and mass matrices of the nonlocal Timoshenko beams in bending, buckling, and vibration analyses. To do this, at first, the governing equations of nonlocal Timoshenko beams are obtained, and then the weak form of these equations is outlined in this paper. The Ritz technique is selected to investigate the behavior of the nonlocal beams with arbitrary boundary conditions along them. To find the equilibrium equations of bending, buckling, and vibration of these structures, an analytical procedure is followed.

2. Governing equations

The governing equations for bending, buckling, and vibration of nonlocal Timoshenko beams are derived in this section. To obtain the governing equations of nonlocal Timoshenko beams, it is assumed that x and z axes are taken along the length and thickness of the beam, respectively. According to the shear deformation beam theory, the strain-displacement relations are given by [28]:

$$\varepsilon_{xx} = z \frac{d\phi}{dx} \quad (1)$$

$$\gamma_{xz} = \phi + \frac{dw}{dx} \quad (2)$$

where z is the coordinate along the thickness of the beam and is measured from the mid-plane of the beam, ϕ is the rotation due to bending, w is the transverse displacement, ε_{xx} is the normal strain, and γ_{xz} is the transverse shear strain.

The virtual strain energy δU , the virtual potential energy δV of an axial load P and the transverse distributed load $q = q(x)$, and the virtual kinetic energy δT of a Timoshenko beam by assuming a free harmonic motion and including the effect of rotary inertia can be written as follows [28]:

$$\delta U = \int_0^L \int_A (\sigma_{xx} \delta \varepsilon_{xx} + \sigma_{xz} \delta \gamma_{xz}) dA dx \quad (3)$$

$$\delta V = - \int_0^L \left(P \frac{dw}{dx} \frac{d\delta w}{dx} + q \delta w \right) dx \quad (4)$$

$$\delta T = \int_0^L \left(\rho A \omega^2 w \delta w + \rho I \omega^2 \phi \delta \phi \right) dx \quad (5)$$

In the above equations, σ_{xx} is the normal stress, σ_{xz} is the transverse shear stress, L is the length of the beam, A is the cross-sectional area of the beam, I is the second moment of area, ω is the circular frequency of vibration, and ρ is the mass density of the beam material. The final form of the virtual strain energy considering the shear correction factor K_s can be expressed as equation (6) by substituting equations (1) and (2) into equation (3).

$$\begin{aligned} \delta U &= \int_0^L \int_A \left(\sigma_{xx} \delta \left(z \frac{d\phi}{dx} \right) + \sigma_{xz} \delta \left(\phi + \frac{dw}{dx} \right) \right) dA dx \\ &= \int_0^L \int_A \left(\sigma_{xx} z \frac{d\delta \phi}{dx} + \sigma_{xz} \left(\delta \phi + \frac{d\delta w}{dx} \right) \right) dA dx \\ &= \int_0^L \left(M \frac{d\delta \phi}{dx} + Q \left(\delta \phi + \frac{d\delta w}{dx} \right) \right) dx \end{aligned} \quad (6)$$

where $M = \int_A \sigma_{xx} z dA$ and $Q = K_s \int_A \sigma_{xz} dA$ are the bending moment and the shear force, respectively. According to the Hamilton principle for the Timoshenko beam theory:

$$\delta T - \delta U - \delta V = 0 = \int_0^L \left(\rho A \omega^2 w \delta w + \rho I \omega^2 \phi \delta \phi - M \frac{d\delta \phi}{dx} - Q \delta \phi - Q \frac{d\delta w}{dx} + P \frac{dw}{dx} \frac{d\delta w}{dx} + q \delta w \right) dx \quad (7)$$

By performing integration by parts and setting the coefficients of the variation of the displacement components to zero, the following Euler differential equations are obtained:

$$\frac{dM}{dx} = Q - \rho I \omega^2 \phi \quad (8)$$

$$\frac{dQ}{dx} = P \frac{d^2 w}{dx^2} - \rho A \omega^2 w - q \quad (9)$$

The essential and natural boundary conditions of the beam are also obtained as:

$$w = 0 \quad \text{or} \quad V = Q - P \frac{dw}{dx} = 0 \quad (10)$$

$$\phi = 0 \quad \text{or} \quad M = 0 \quad (11)$$

It is seen from the above equations that the obtained governing equations are the same as the local Timoshenko beam theory due to classical elasticity assumptions. As will be shown below, due to the nonlocal constitutive relations, the bending moment and shear force expressions for the nonlocal beam theory are different. It is known that the constitutive equation of classical elasticity is an algebraic relationship between the stress and strain tensors. But, in the nonlocal differential elasticity, the constitutive relation is appeared in the form of differential constitutive equations. In one-dimensional case and by assuming that the constitutive relation for the shear stress and strain remains the same as in the local beam theory, these equations can be simplified as [9,12]

$$\sigma_{xx} - \eta^2 \frac{d^2 \sigma_{xx}}{dx^2} = E \varepsilon_{xx} \quad (12)$$

$$\sigma_{xz} = G \gamma_{xz} \quad (13)$$

where E is Young's modulus, G is the shear modulus, and η is the scale coefficient that incorporates the small scale effect. Multiplying equation (12) by $z dA$ and integrating the result over the area A yields:

$$M - \eta^2 \frac{d^2 M}{dx^2} = EI \frac{d\phi}{dx} \quad (14)$$

The bending moment and shear force expressions for the nonlocal beam theory can be written in the following forms by substituting the governing equations (8) and (9) into equation (14) and also multiplying equation (13) by the shear correction factor K_s and integrating over the area A .

$$M = EI \frac{d\phi}{dx} + \eta^2 \left(P \frac{d^2 w}{dx^2} - \rho A \omega^2 w - \rho I \omega^2 \frac{d\phi}{dx} - q \right) \quad (15)$$

$$Q = K_s GA \left(\phi + \frac{dw}{dx} \right) \quad (16)$$

By substituting the equations (15) and (16) into equations (8) and (9), the final form of the governing equations of the nonlocal Timoshenko beam can be obtained in terms of the displacement components w and ϕ .

$$-P \frac{d^2 w}{dx^2} + K_s GA \left(\frac{d\phi}{dx} + \frac{d^2 w}{dx^2} \right) + \rho A \omega^2 w + q = 0 \quad (17)$$

$$\eta^2 P \frac{d^3 w}{dx^3} + \left(EI - \eta^2 \rho I \omega^2 \right) \frac{d^2 \phi}{dx^2} - \eta^2 \rho A \omega^2 \frac{dw}{dx} - \eta^2 \frac{dq}{dx} - K_s GA \left(\frac{dw}{dx} + \phi \right) + \rho I \omega^2 \phi = 0 \quad (18)$$

The final form of the governing equations may be written as

$$-\Omega k \frac{d^2 \bar{w}}{dx^2} + \left(\frac{d^2 \bar{w}}{dx^2} + \frac{d\bar{\phi}}{dx} \right) + \lambda^2 \Omega \bar{w} + \Omega \bar{q} = 0 \quad (19)$$

$$\alpha^2 k \Omega \frac{d^3 \bar{w}}{d\bar{x}^3} + \Omega \left(1 - \frac{\alpha^2 \lambda^2}{\zeta^2} \right) \frac{d^2 \phi}{d\bar{x}^2} - \alpha^2 \lambda^2 \Omega \frac{d\bar{w}}{d\bar{x}} - \Omega \alpha^2 \frac{d\bar{q}}{d\bar{x}} + \frac{\lambda^2 \Omega}{\zeta^2} \phi - \left(\frac{d\bar{w}}{d\bar{x}} + \phi \right) = 0 \quad (20)$$

where the non-dimensional terms are defined as follows:

$$\begin{aligned} \bar{x} &= \frac{x}{L}; \bar{w} = \frac{w}{L} \\ k &= \frac{PL^2}{EI} && \text{buckling load parameter} \\ \bar{q} &= \frac{qL}{K_s GA} && \text{distributed load parameter} \\ \lambda^2 &= \omega^2 \frac{\rho AL^4}{EI} && \text{frequency parameter} \\ \Omega &= \frac{EI}{K_s GAL^2} && \text{shear deformation parameter} \\ \alpha &= \frac{\eta}{L} && \text{scaling effect parameter} \\ \zeta &= \frac{L\sqrt{A}}{\sqrt{I}} && \text{slenderness ratio} \end{aligned} \quad (21)$$

and the non-dimensional bending moment and non-dimensional shear force expressions can be similarly obtained as

$$\bar{M} = \frac{d\phi}{d\bar{x}} + \alpha^2 \left(k \frac{d^2 \bar{w}}{d\bar{x}^2} - \lambda^2 \bar{w} - \frac{\lambda^2}{\zeta^2} \frac{d\phi}{d\bar{x}} - \bar{q} \right) \quad (22)$$

$$\bar{Q} = \phi + \frac{d\bar{w}}{d\bar{x}} \quad (23)$$

where $\bar{M} = ML / EI$ and $\bar{Q} = Q / K_s GA$. It is noted that the governing equations of the local Timoshenko beam can be retrieved by setting $\alpha = 0$ in the above equations.

3. Weak form derivation

It is known that the weak form of a differential equation is a weighted-integral statement that is equivalent to both the governing differential equation and the associated natural boundary conditions. Derivation of the weak form of the governing equations for the nonlocal Timoshenko beams is outlined [28] in this section. The weak form of the governing equations (19) and (20) can be obtained by:

$$\int_0^1 \psi_1 \left(-\Omega k \frac{d^2 \bar{w}}{d\bar{x}^2} + \left(\frac{d^2 \bar{w}}{d\bar{x}^2} + \frac{d\phi}{d\bar{x}} \right) + \lambda^2 \Omega \bar{w} + \Omega \bar{q} \right) d\bar{x} = 0 \quad (24)$$

$$\int_0^1 \psi_2 \left(\alpha^2 k \Omega \frac{d^3 \bar{w}}{d\bar{x}^3} + \Omega \left(1 - \frac{\alpha^2 \lambda^2}{\zeta^2} \right) \frac{d^2 \phi}{d\bar{x}^2} - \alpha^2 \lambda^2 \Omega \frac{d\bar{w}}{d\bar{x}} - \alpha^2 \Omega \frac{d\bar{q}}{d\bar{x}} + \frac{\lambda^2 \Omega}{\zeta^2} \phi - \left(\frac{d\bar{w}}{d\bar{x}} + \phi \right) \right) d\bar{x} = 0 \quad (25)$$

where ψ_1 and ψ_2 are weight functions which satisfy the homogeneous essential boundary conditions on $\bar{w}$ and ϕ , respectively. The weak statements can be written in the following final form after performing integration once by parts on the first two terms of equation (24) and on the first four terms of equation (25) and using equations (10), (11), (22), and (23):

$$\int_0^1 \left(-\Omega k \frac{d\bar{w}}{d\bar{x}} \frac{d\psi_1}{d\bar{x}} + \left(\frac{d\bar{w}}{d\bar{x}} + \phi \right) \frac{d\psi_1}{d\bar{x}} - \lambda^2 \Omega \bar{w} \psi_1 - \Omega \bar{q} \psi_1 \right) d\bar{x} - [\psi_1 \bar{V}]_0^1 = 0 \quad (26)$$

$$\int_0^1 \left(\alpha^2 k \Omega \frac{d^2 \bar{w}}{d\bar{x}^2} \frac{d\psi_2}{d\bar{x}} + \Omega \left(1 - \frac{\alpha^2 \lambda^2}{\zeta^2} \right) \frac{d\phi}{d\bar{x}} \frac{d\psi_2}{d\bar{x}} - \alpha^2 \lambda^2 \Omega \bar{w} \frac{d\psi_2}{d\bar{x}} - \alpha^2 \Omega \bar{q} \frac{d\psi_2}{d\bar{x}} - \frac{\lambda^2 \Omega}{\zeta^2} \phi \psi_2 + \left(\frac{d\bar{w}}{d\bar{x}} + \phi \right) \psi_2 \right) d\bar{x} - \Omega [\psi_2 \bar{M}]_0^1 = 0 \quad (27)$$

where $\bar{V} = V / K_s GA$. The coefficients of the weight functions in the boundary integrals are called secondary variables, and their specifications constitute the natural boundary conditions.

As can be seen in the above equations, all the terms are either bilinear or linear. The bilinear and linear functional associated with the problem are given by adding the expressions in equations (26) and (27) and collecting the coefficients.

$$\int_0^1 \left(\alpha^2 k \Omega \frac{d^2 \bar{w}}{dx^2} \frac{d\bar{\psi}_2}{dx} - \alpha^2 \lambda^2 \Omega \bar{w} \frac{d\bar{\psi}_2}{dx} - \Omega k \frac{d\bar{w}}{dx} \frac{d\bar{\psi}_1}{dx} - \lambda^2 \Omega \bar{w} \bar{\psi}_1 - \frac{\lambda^2 \Omega}{\zeta^2} \phi \bar{\psi}_2 + \Omega \left(1 - \frac{\alpha^2 \lambda^2}{\zeta^2} \right) \frac{d\phi}{dx} \frac{d\bar{\psi}_2}{dx} \right. \\ \left. + \left(\frac{d\bar{w}}{dx} + \phi \right) \left(\frac{d\bar{\psi}_1}{dx} + \bar{\psi}_2 \right) - \Omega \bar{q} \left(\bar{\psi}_1 + \alpha^2 \frac{d\bar{\psi}_2}{dx} \right) \right) dx - [\bar{\psi}_1 \bar{V}]_0^1 - \Omega [\bar{\psi}_2 \bar{M}]_0^1 = 0 \quad (28)$$

It can be seen that the first two terms in equation (28) are non-symmetric bilinear. Therefore, it is impossible to construct the associated quadratic functional form. In the rest of the article, the Ritz solution is developed.

4. Ritz solution

The Ritz technique requires the expansion of unknown functions in infinite series, and by assuming sufficient number of functions, it is possible to obtain a very high accuracy solution to the problem considered. In this study, the displacement functions $\bar{w}$ and ϕ are written in the form of a series of Chebyshev polynomials multiplied by the boundary functions $f(\bar{x})$ and $g(\bar{x})$, which ensure that the displacement components satisfy the essential boundary conditions of the beam, as follows:

$$\bar{w}(\bar{x}) = \underbrace{\prod_{m=1}^{n_s} (\bar{x}_m - \bar{x})^{n_m}}_{f(\bar{x})} \sum_{j=1}^{n_t} a_j P_{j-1}(\bar{x}) \quad (29)$$

$$\phi(\bar{x}) = \underbrace{\prod_{m=1}^{n_s} (\bar{x}_m - \bar{x})^{n_m-1}}_{g(\bar{x})} \sum_{j=1}^{n_t} b_j P_{j-1}(\bar{x}) \quad (30)$$

where a_j and b_j are the unknown undetermined coefficients, n_s is the number of supports along the beam, n_t is the number of terms needed in the series, $\bar{x}_m$ is the non-dimensional distance of m^{th} support from the origin, n_m takes values of 1 & 2 corresponding to the simply-supported and clamped conditions at m^{th} support, respectively, and $P_j(\bar{x})$ is the j^{th} Chebyshev polynomial. By substituting the displacement fields into the weak form equations (26) and (27) and assuming the constant non-dimensional transverse distributed load $\bar{q} = \bar{q}_0$, the following equations are obtained:

$$\sum_{j=1}^{n_t} \left(K_{ij}^{11} - \beta B_{ij}^{11} - \lambda^2 M_{ij}^{11} \right) a_j + \sum_{j=1}^{n_t} K_{ij}^{12} b_j - Q_i^1 = 0; (i = 1, 2, \dots, n_t) \quad (31)$$

$$\sum_{j=1}^{n_t} \left(K_{ij}^{21} - \beta B_{ij}^{21} - \lambda^2 M_{ij}^{21} \right) a_j + \sum_{j=1}^{n_t} \left(K_{ij}^{22} - \lambda^2 (M_{ij}^{22} + M_{ij}^{'22}) \right) b_j - Q_i^2 = 0; (i = 1, 2, \dots, n_t) \quad (32)$$

where the terms $K_{ij}^{11}, B_{ij}^{11}, M_{ij}^{11}, K_{ij}^{12}, K_{ij}^{21}, B_{ij}^{21}, M_{ij}^{21}, K_{ij}^{22}, M_{ij}^{22}, M_{ij}^{'22}, Q_i^1$, and Q_i^2 are given in details in the appendix, and the eigenvalue $\beta = k / \bar{k}$ represents the ratio of the actual buckling load parameter and the applied load parameter due to the applied in-plane load $\bar{P}(\bar{k} = \bar{P} L^2 / EI)$. The equations (31) and (32) can be rewritten in the matrix form as

$$\left(\begin{bmatrix} K^{11} & K^{12} \\ K^{21} & K^{22} \end{bmatrix} - \beta \begin{bmatrix} B^{11} & 0 \\ B^{21} & 0 \end{bmatrix} - \lambda^2 \begin{bmatrix} M^{11} & 0 \\ M^{21} & M^{22} + M^{'22} \end{bmatrix} \right) \begin{Bmatrix} a \\ b \end{Bmatrix} = \begin{Bmatrix} Q^1 \\ Q^2 \end{Bmatrix} \quad (33)$$

or

$$([K] - \beta[B] - \lambda^2[M])\{U\} = \{Q\} \quad (34)$$

It is emphasized that the corresponding equations of the local Timoshenko beam can be retrieved by setting $[B^{21}] = [M^{21}] = [M^{'22}] = 0$ in equation (34) where the sub-matrices $[B^{21}]$ and $[M^{21}]$ have non-symmetric effects on the equations of the nonlocal Timoshenko beams.

5. Results

Before presentation of the new results and discussion on the capability of the Ritz method, it is necessary to perform a convergence study and validate the developed method by applying it to some classical problems presented in the literature. To Journal of Applied and Computational Mechanics, Vol. 4, No. 1, (2018), 16-26

do this, the convergence studies are carried out for the non-dimensional maximum deflection $\bar{w}_{\max}$, the buckling load parameter k , and the natural frequency parameter λ of a clamped-simply supported nano-beam. The following data are adopted in generating the bending, buckling, and vibration results, and a summary of these results is presented in Table 1.

nano-tube with diameter $d=0.678$ nm,
 Young's modulus $E=5.5$ TPa,
 Poisson's ratio $\nu=0.19$,
 shear correction factor $K_s=0.563$,
 effective tube thickness $t=0.066$ nm,
 scaling effect parameter $\alpha=0.5$,
 applied load parameter $\bar{k} = 1$, and distributed load parameter $\bar{q} = 1$.

Table 1. Non-dimensional maximum deflections, critical buckling load parameters, and natural frequency parameters for a clamped-simply supported nano-beam.

n_t	$\bar{w}_{\max}$	k	λ
2	0.0132	3.4628	9.2848
4	0.0154	3.2768	7.5660
6	0.0155	3.2746	7.5494
8	0.0155	3.2746	7.5484
10	0.0155	3.2746	7.5478
12	0.0155	3.2745	7.5474
14	0.0155	3.2745	7.5472
16	0.0155	3.2745	7.5471
18	0.0155	3.2745	7.5470
20	0.0155	3.2745	7.5469

The convergence study shows that about 12 terms are sufficient to obtain reasonably accurate results. However, to ensure the results, every term was calculated using 20 terms. In the next step, to compare the results obtained by the proposed method, the non-dimensional maximum deflection $\bar{w}_{\max}$, the buckling load parameter k , and the natural frequency parameter λ are compared with those reported in the literature [1, 2, &17]. For this purpose, the following data are adopted in generating the bending, buckling, and vibration results.

For vibration analysis:

nano-tube with diameter $d=0.678$ nm,
 Young's modulus $E=5.5$ TPa,
 Poisson's ratio $\nu=0.19$,
 shear correction factor $K_s=0.563$, and effective tube thickness $t=0.066$ nm.

For bending and buckling analyzes:

nano-rod with diameter $d=1$ nm,
 Young's modulus $E=1$ TPa,
 Poisson's ratio $\nu=0.19$,
 shear correction factor $K_s=0.9$
 applied load parameter $\bar{k} = 1$, and distributed load parameter $\bar{q} = 1$.

In Table 2, the results of the nonlocal Timoshenko beams with various boundary conditions at two ends are presented. The values of α are selected in such a way that makes the comparison with other references possible. It is emphasized that the results reported by C.M. Wang et al. [1, 21 & 17] were obtained by exact solution of the governing equations. As can be seen, there is excellent agreement between the obtained results.

As it was noted in [27] for the Euler nonlocal beams, in the vibration analysis of the nonlocal Timoshenko beams and for some values of the scaling effect parameter, no real natural frequencies exist. It is also noted that, in the bending analysis and in the clamped nonlocal beam (CL-CL), the deflection is not affected by the small scale effect.

As mentioned before, the main purpose of this study is to provide some new results presenting the capability of the Ritz method to model the nonlocal Timoshenko beams with arbitrary boundary conditions along them. Therefore, two different nonlocal beams are selected here as Ref. [27]. These beams are different from one another as far as the boundary conditions along them are concerned; one of them is a nonlocal beam with one clamped end and one simple support along it as shown in

Figure 1 which is referred to as beam (A), the other is a beam with four simple supports with various intervals over it as shown in Figure 2 which is referred to as beam (B).

Table 2. Non-dimensional maximum deflections, critical buckling load parameters, and natural frequency parameters for nonlocal beams with various boundary conditions and various scaling effect parameter

		Boundary Conditions			
α	Ref.	SS-SS	CL-SS	CL-CL	CL-FR
Vibration analysis, λ					
0	Ritz solution	9.5660	14.3271	19.8092	3.4636
	[17]	9.5660	14.3224	19.7945	3.4633
0.5	Ritz solution	5.2290	7.5469	10.5111	4.0098
	[17]	5.2290	7.5466	10.5106	4.0096
0.7	Ritz solution	4.0425	5.7884	8.0559	--
	[17]	4.0425	5.7884	8.0559	--
Buckling analysis, k					
0	Ritz solution	9.7112	19.4766	37.0603	2.4574
	[1]	9.7113	19.4765	37.0604	2.4575
0.2	Ritz solution	6.9943	10.9477	14.9291	2.2374
	[1]	6.9943	10.9564	14.9291	2.2374
1	Ritz solution	0.9066	0.9512	0.9737	0.7108
Bending Analysis, $\overline{w}_{\max}$					
0	Ritz solution	0.0132	0.0057	0.0028	0.1258
	[2]	0.0132	--	0.0028	0.1258
0.2	Ritz solution	0.0182	0.0071	0.0028	0.1058
	[2]	0.0182	--	0.0028	0.1058
1	Ritz solution	0.1382	0.0428	0.0028	0.3742
	[2]	0.1382	--	0.0028	0.3742

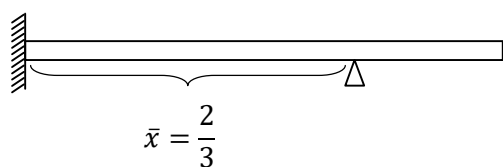


Fig. 1. Nonlocal beam (A)

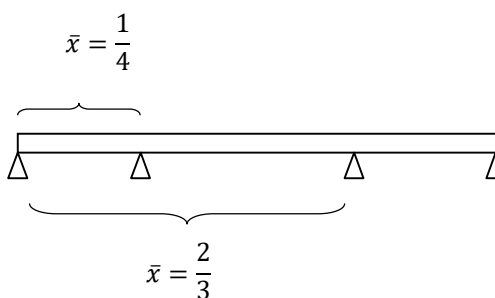


Fig. 2. Nonlocal beam (B)

In order to analyze the bending, buckling, and vibration behaviors of the above mentioned beams (A) and (B), the following data are assumed.

Nano-tube with diameter $d=0.678$ nm,

Young's modulus $E=5.5$ TPa,

Poisson's ratio $\nu=0.19$,

Shear correction factor $K_s=0.563$,

Effective tube thickness $t=0.066$ nm,

Applied load parameter $\bar{k} = 1$, and distributed load parameter $\bar{q} = 100$.

In Table 3, the non-dimensional maximum deflections, the critical buckling load parameters, and the natural frequency parameters are tabulated for the nonlocal beams (A) and (B) and for various scaling effect parameters. The relationship between the buckling ratio (i.e., ratio of the critical buckling load parameter k_{NL} of the nonlocal beam to the corresponding local beam k_L) and the scaling effect parameter, and the relationship between the frequency ratio (i.e., ratio of the critical buckling load parameter λ_{NL} of the nonlocal beam to the corresponding local beam λ_L) and the scaling effect parameter for the first three modes of the beams (A) are also shown in Figures 3 and 4, respectively.

Table 3. Non-dimensional maximum deflections, critical buckling load parameters, and natural frequency parameters for nonlocal beams (A) and (B)

α	Beam (A)				Beam (B)			
	$\bar{w}_{\max}$	Mode No.	K	λ	$\bar{w}_{\max}$	Mode No.	k	λ
0	0.293 ($\bar{x} = 1$)	1	8.5807	15.798	0.0219 ($\bar{x} = 0.45$)	1	57.292	60.953
		2	45.94	37.283		2	81.607	85.254
		3	72.966	84.912		3	102.94	124.97
0.05	0.264 ($\bar{x} = 1$)	1	8.4005	15.773	0.0213 ($\bar{x} = 0.45$)	1	50.115	57.662
		2	41.207	36.622		2	67.782	79.894
		3	61.71	79.44		3	81.876	113.15
0.1	0.176 ($\bar{x} = 1$)	1	7.9026	15.69	0.0198 ($\bar{x} = 0.45$)	1	36.425	50.165
		2	31.479	34.93		2	44.939	68.241
		3	42.185	67.753		3	50.729	90.955

Based on these analyses, it is found that the deflections, buckling loads, and natural frequencies of the considered beams are affected by the small scale effect. The decreasing trend of the buckling loads and natural frequencies of the beams by increasing the small scale effect can be perceived from these figures and also from Table 3. This reduction is more significant for higher buckling and vibration modes. In sum, the nonlocal theory should be used if one needs accurate predictions of high critical buckling loads or natural frequencies of micro and nano-beams which means that the application of the local elasticity models for the nano-sized structures analysis would lead to overprediction of values.

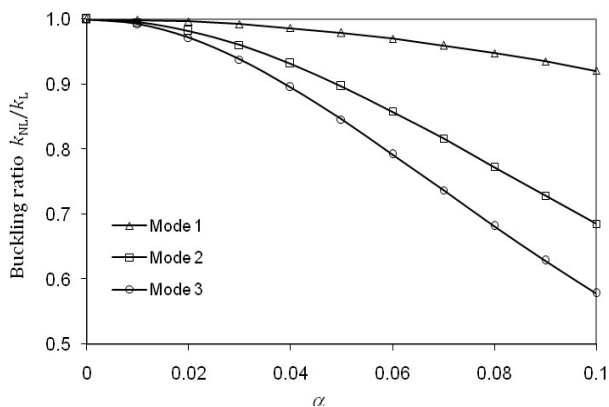


Fig. 3. The variations of buckling ratio with scaling effect parameter for nonlocal Timoshenko beam (A)

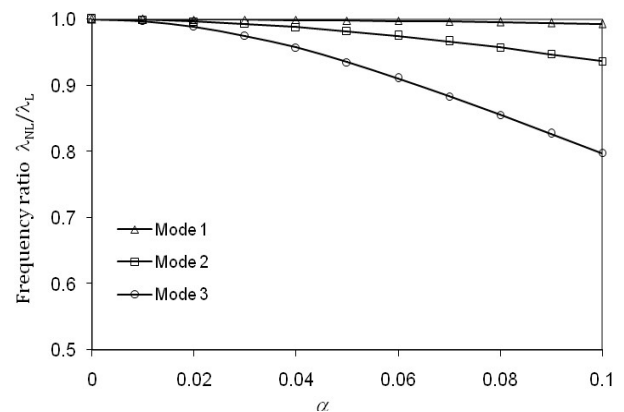


Fig. 4. The variations of frequency ratio with scaling effect parameter for nonlocal Timoshenko beam (A)

The first three mode shapes of buckling and vibration of the nonlocal beam (A) are presented in Figures 5 and 6, respectively. Each mode is obtained for three levels of scaling effect parameter ($\alpha=0, 0.05, 0.1$). Note that the mode shape associated with $\alpha=0$ corresponds to the mode shape for the local beam. It can be seen from these figures that the buckling mode shapes of this special case (beam (A)) are not affected by various values of the scaling effect parameters, while the vibration mode shapes of beam (A) are significantly affected by these values.

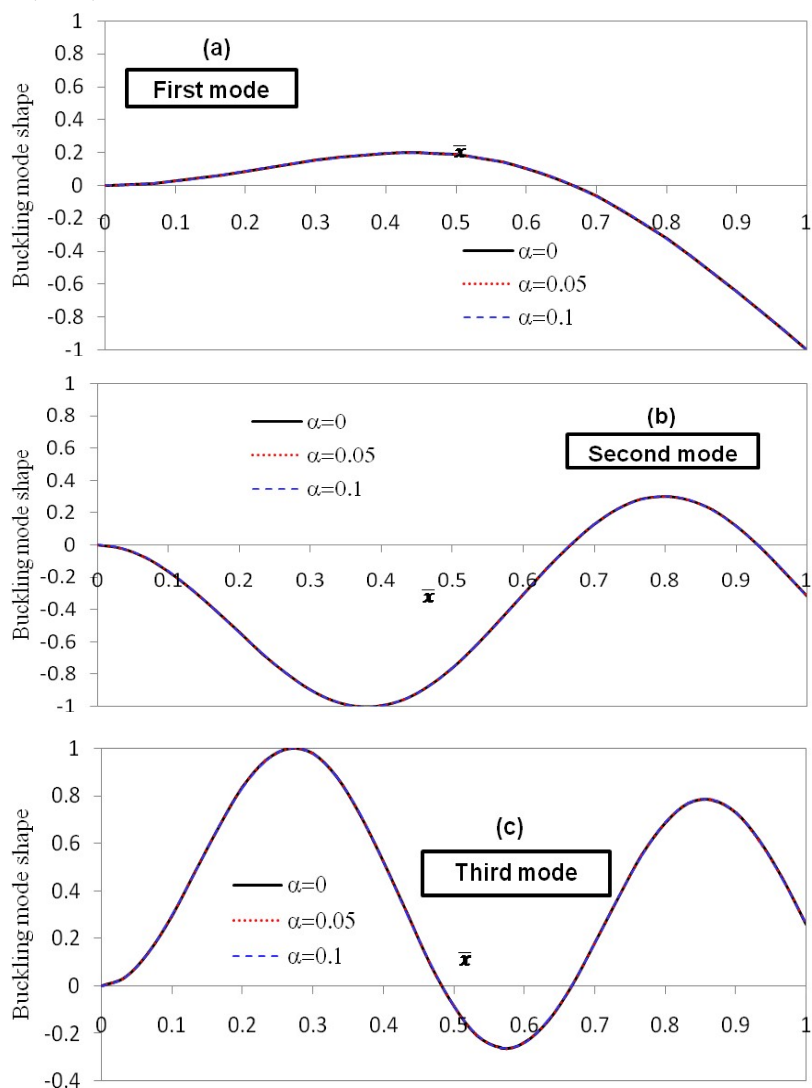
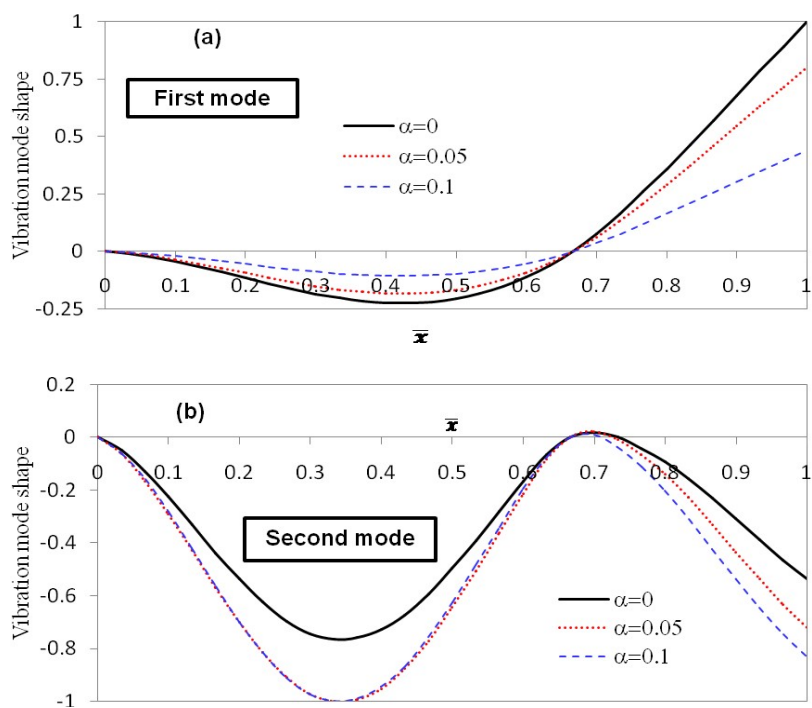


Fig. 5. First three buckling modes of nonlocal beam (A)



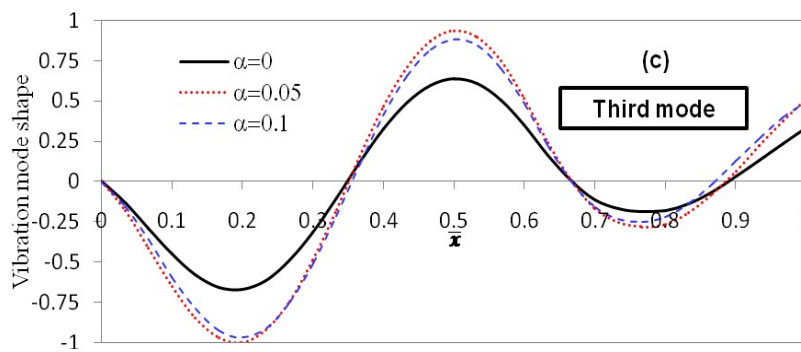


Fig. 6. First three vibration modes of nonlocal beam (A)

6. Conclusion

In this research, bending, buckling, and vibration behaviors of nonlocal Timoshenko beams were investigated using a variational approach. The governing equations of the nonlocal Timoshenko beams were obtained, and then the weak form of these equations was outlined. The Ritz technique was used to investigate the behavior of the nonlocal beams with arbitrary boundary conditions along them. The stiffness matrix, buckling stiffness matrix, and mass matrix of the nonlocal Timoshenko beams were derived. In the development process, it was found that there were two terms in the weak statement of the governing equations which had non-symmetric bilinear form. Therefore, it was impossible to construct the quadratic functional form. The obtained results were in excellent agreement with those reported in the literature.

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Appendix

The coefficients of the stiffness, buckling stiffness, and mass matrices of the nonlocal Timoshenko beams $K_{ij}^{11}, B_{ij}^{11}, M_{ij}^{11}, K_{ij}^{12}, K_{ij}^{21}, B_{ij}^{21}, M_{ij}^{21}, K_{ij}^{22}, M_{ij}^{22}, M_{ij}^{22}, Q_i^1$, and Q_i^2 can be obtained by the following relations.

$$\begin{aligned}
 K_{ij}^{11} &= \int_0^1 \left(\left(\frac{df}{dx} \right)^2 P_{i-1} P_{j-1} + f \frac{df}{dx} \left(\frac{dP_{i-1}}{dx} P_{j-1} + P_{i-1} \frac{dP_{j-1}}{dx} \right) + f^2 \frac{dP_{i-1}}{dx} \frac{dP_{j-1}}{dx} \right) dx \\
 B_{ij}^{11} &= \int_0^1 \Omega k \left(\left(\frac{df}{dx} \right)^2 P_{i-1} P_{j-1} + f \frac{df}{dx} \left(\frac{dP_{i-1}}{dx} P_{j-1} + P_{i-1} \frac{dP_{j-1}}{dx} \right) + f^2 \frac{dP_{i-1}}{dx} \frac{dP_{j-1}}{dx} \right) dx \\
 M_{ij}^{11} &= \int_0^1 \Omega f^2 P_{i-1} P_{j-1} dx \\
 K_{ij}^{12} &= \int_0^1 \left(\frac{df}{dx} g P_{i-1} P_{j-1} + f g \frac{dP_{i-1}}{dx} P_{j-1} \right) dx \\
 K_{ij}^{21} &= \int_0^1 \left(\frac{df}{dx} g P_{i-1} P_{j-1} + f g P_{i-1} \frac{dP_{j-1}}{dx} \right) dx \\
 B_{ij}^{21} &= \int_0^1 -\alpha^2 \Omega k \left(\frac{d^2 f}{dx^2} \frac{dg}{dx} P_{i-1} P_{j-1} + \frac{d^2 f}{dx^2} g \frac{dP_{i-1}}{dx} P_{j-1} + 2 \frac{df}{dx} \frac{dg}{dx} P_{i-1} \frac{dP_{j-1}}{dx} \right. \\
 &\quad \left. + 2 \frac{df}{dx} g \frac{dP_{i-1}}{dx} \frac{dP_{j-1}}{dx} + f \frac{dg}{dx} P_{i-1} \frac{d^2 P_{j-1}}{dx^2} + f g \frac{dP_{i-1}}{dx} \frac{d^2 P_{j-1}}{dx^2} \right) dx \\
 M_{ij}^{21} &= \int_0^1 \alpha^2 \Omega \left(f \frac{dg}{dx} P_{i-1} P_{j-1} + f g \frac{dP_{i-1}}{dx} P_{j-1} \right) dx \\
 K_{ij}^{22} &= \int_0^1 \left(\Omega \left[\left(\frac{dg}{dx} \right)^2 P_{i-1} P_{j-1} + g \frac{dg}{dx} \left(\frac{dP_{i-1}}{dx} P_{j-1} + P_{i-1} \frac{dP_{j-1}}{dx} \right) + g^2 \frac{dP_{i-1}}{dx} \frac{dP_{j-1}}{dx} \right] + g^2 P_{i-1} P_{j-1} \right) dx \\
 M_{ij}^{22} &= \int_0^1 \frac{\Omega}{\zeta^2} g^2 P_{i-1} P_{j-1} dx \\
 M_{ij}^{22} &= \int_0^1 \frac{\alpha^2 \Omega}{\zeta^2} \left(\left(\frac{dg}{dx} \right)^2 P_{i-1} P_{j-1} + g \frac{dg}{dx} \left(\frac{dP_{i-1}}{dx} P_{j-1} + P_{i-1} \frac{dP_{j-1}}{dx} \right) + g^2 \frac{dP_{i-1}}{dx} \frac{dP_{j-1}}{dx} \right) dx \\
 Q_i^1 &= \int_0^1 \Omega q f P_{i-1} dx; Q_i^2 = \int_0^1 \alpha^2 \Omega q \left(\frac{dg}{dx} P_{i-1} + g \frac{dP_{i-1}}{dx} \right) dx
 \end{aligned} \tag{A1}$$