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Research Article

Buckling Analysis of Embedded Nanosize FG Beams Based on a Refined Hyperbolic Shear Deformation Theory

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Abstract. In this study, the mechanical buckling response of refined hyperbolic shear deformable (FG) functionally graded nanobeams embedded in an elastic foundation is investigated based on the refined hyperbolic shear deformation theory. Material properties of the FG nanobeam change continuously in the thickness direction based on the power-law model. To capture small size effects, Eringen's nonlocal elasticity theory is adopted. Employing Hamilton's principle, the nonlocal governing equations of FG nanobeams embedded in the elastic foundation are obtained. To predict the buckling behavior of embedded FG nanobeams, the Navier-type analytical solution is applied to solve the governing equations. Numerical results demonstrate the influences of various parameters such as elastic foundation, power-law index, nonlocal parameter, and slenderness ratio on the critical buckling loads of size dependent FG nanobeams.

Keywords: FG nanobeam; Elastic foundation; Buckling; Nonlocal elasticity theory; Shear deformation beam theory.

1. Introduction

Functionally graded structures have been widely used in various engineering fields such as constructions, aerospace, nuclear, civil, nuclear, marine, biomedical, automotive, and other engineering domain. FGMs structures with the continuous variation of materials properties possess the advantage of reducing residual and thermal stresses. Recently, considerable interests have been devoted to experimental and theoretical works of the mechanical response of graded structures. Since controlling the experimental conditions is not evident for nanoscale structures, theoretical models become necessary. Functionally graded materials (FGMs) are advanced composite materials that have continuous variation of material properties from one surface to another, and thus eliminate the stress concentration found in laminated composites [1]. With the increase in the application of functionally graded beams in various engineering fields, a variety of beams theories with different approaches have been developed to predict its behavior. These beam theories can be divided into three categories as follows: classical beam theory (CBT), first-order shear deformation beam theory (FSBT), and higher-order beam theory (HSBT). A general review and assessment of these theories for composite plates and beams can be found in previous research [2 & 3]. To

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investigate the mechanical behavior of nanoscale structures, the classical continuum theory cannot be implemented for the analysis of such structures [4 & 5]. Therefore, this problem has been examined in the context of nonlocal continuum theories such as the nonlocal elasticity theory of Eringen [6 &7] and the strain gradient theory [8]. However, the studies about size-dependent effects on the mechanical buckling behaviour of FG elastic beams with micro/nano-beam related to both internal lengths and external dimensions are always of fundamental significance. In this context, the importance of the nonlocal theory of elasticity has stimulated the researchers to investigate the behavior of the nanoscale beams as structural elements of nano-electromechanical systems (NEMS). According to the increasing usage of micro/nano FG structure [9-11], in the recent years, a large number of research has been conducted to study the mechanical responses of these structures [12-14]. So far, only a few works have been reported for FG nanobeams based on the nonlocal elasticity theory. Pisano et al. [15 & 16] exploited the nonlocal finite element method for analyzing homogeneous and nonhomogeneous nonlocal elastic 2D problems. Janghorban and Zare [17] investigated nonlocal free vibration axially FG nanobeams using differential quadrature method. Eltaher et al. [18] studied free vibration of FG nanobeam based on the nonlocal Euler-Bernoulli beam theory. Recently, Li et al.[19] carried out vibration investigation of beams composed of FGMs incorporating nonlocal and strain gradient effects. Ebrahimi et al. [20] presented a nonlocal strain gradient model for wave dispersion investigation of thermally affected graded nanoscale plates. Bouafia et al. [21] studied the static bending and buckling of a FG nanobeam using the nonlocal sinusoidal beam theory. Belifa et al. [22] studied the nonlinear postbuckling behavior of nanoscale beams using a nonlocal zeroth-order shear deformation theory.

The aim of this paper is to propose a refined nonlocal beam theory for the mechanical buckling of FG nanobeams resting on the elastic foundation. The small scale effect is taken into account by using nonlocal constitutive relations of Eringen. The most interesting feature of this theory is that it accounts for a hyperbolic variation of the transverse shear strains across the thickness, and satisfies the zero traction boundary conditions on the top and bottom surfaces of the beam without using shear correction factors. The material properties of the FG nanobeam are assumed to vary in the thickness direction. Based on the nonlocal constitutive relations of Eringen, the equations of motion of FG nanobeams are derived using Hamilton’s principle. The obtained results are compared with those predicted by the previous research. The influences of nonlocal parameter, power-law index, elastic foundations, and aspect ratio on the buckling responses of FG nanobeam are discussed.

2. Governing Equations

2.1. Power-law functionally graded material (P-FGM) beam

Consider a uniform FG nanobeam of thickness h, length L, and width b made by mixing two distinct materials (metal & ceramic). The coordinate x is along the longitudinal direction and z is along the thickness direction (Fig. 1). It is assumed that material properties of the FGM nanobeam, such as Young's modulus (E), Poisson's ratio (ν), and length scale parameter vary continuously through the nanobeam thickness according to a power-law form which can be described as:

$$E(z) = \left(E_2 - E_1\right)\left(\frac{z}{h}+\frac{1}{2}\right)^k + E_1 \tag{1}$$

where k is the material distribution parameter which takes the value to be greater or equal to zero.

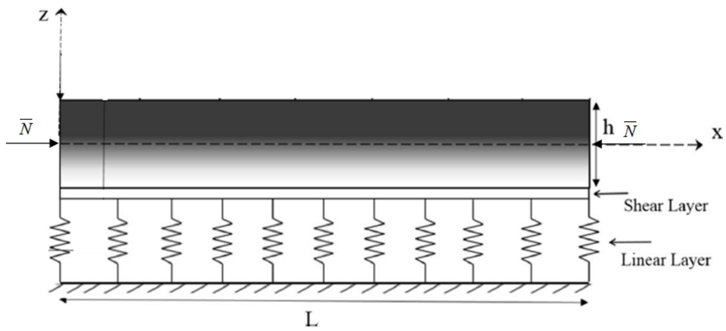

Fig. 1. Configuration of an embedded functionally graded nanobeam

2.2. Kinematic relations

Based on the refined hyperbolic shear deformation beam theory, the displacement field can be written as:

$$u(x,z) = u_0(x,y) - z\frac{\partial w_b}{\partial x} - f(z)\frac{\partial w_s}{\partial x}, \; w(x,z) = w_b + w_s \tag{2}$$

where $f(z) = h\sinh(z/h) - z\cosh(1/2)$, u and w are the longitudinal and transverse displacements, respectively, and ' is the rotation of the cross section. The nonzero strains are expressed as follows:

$$\varepsilon_x = \varepsilon_x^0 + z\, k_x^b + f(z)\, k_x^s \quad \text{and} \quad \gamma_{xz} = g(z)\, \gamma_{xz}^s \tag{3}$$

The governing equations for the buckling of refined shear deformation nanobeam can be expressed by [9]:

$$\frac{dN}{dx} = 0\,,\quad \frac{d^2 M_b}{dx^2} - N_0 \frac{d^2 (w_b + w_s)}{dx^2} - k_w w + k_s \frac{d^2 w}{dx^2} = 0\,,\quad \frac{d^2 M_s}{dx^2} + \frac{dQ}{dx} - N_0 \frac{d^2 (w_b + w_s)}{dx^2} - k_w w + k_s \frac{d^2 w}{dx^2} = 0 \tag{4}$$

The stress resultants are defined as:

$$(N, M_b, M_s) = \int_{-\frac{h}{2}}^{\frac{h}{2}} (1, z, f)\, \sigma_x dz \quad \text{and} \quad Q = \int_{-\frac{h}{2}}^{\frac{h}{2}} g \tau_{xz} dz \tag{5}$$

2.3. The nonlocal elasticity model for FG nanobeam

The response of materials at the nanoscale is different from those of their bulk counterparts. The nonlocal elasticity is first considered by Eringen [6 &7]. According to Eringen's nonlocal elasticity model, the stress state at a point inside a body is regarded as the function of strains of all points in the neighbor regions. Eringen [6 &7] proposed a differential form of the nonlocal constitutive relation as:

$$\sigma_x - \mu \frac{d^2 \sigma_x}{dx^2} = E(z) \varepsilon_x\,,\quad \tau_{xz} - \mu \frac{d^2 \tau_{xz}}{dx^2} = G(z) \gamma_{xz} \tag{6}$$

where $\mu = (e_0 a)^2$ is the nonlocal parameter, e_0 is a constant appropriate to each material, and a is an internal characteristic length. The force-strain and the moment-strain of the nonlocal FG beam theory can be obtained as follows [9]:

$$\begin{gathered} N - \mu \frac{d^2 N}{dx^2} = A \frac{du_0}{dx} - B \frac{d^2 w_b}{dx^2} - B_s \frac{d^2 w_s}{dx^2}\,, M_b - \mu \frac{d^2 M_b}{dx^2} = A \frac{du_0}{dx} - D \frac{d^2 w_b}{dx^2} - D_s \frac{d^2 w_s}{dx^2}\,, \\ M_s - \mu \frac{d^2 M_s}{dx^2} = B_s \frac{du_0}{dx} - D_s \frac{d^2 w_b}{dx^2} - H_s \frac{d^2 w_s}{dx^2}\,,\quad Q - \mu \frac{d^2 Q}{dx^2} = A_s \frac{dw_s}{dx} \end{gathered} \tag{7}$$

where

$$\left(A, B, D, B_s, D_s, H_s\right) = \int_{-\frac{h}{2}}^{\frac{h}{2}} E(z) \left(1, z, z^2, f, z\, f, f^2\right) dz\,,\ A_s = \int_{-\frac{h}{2}}^{\frac{h}{2}} G(z) g^2 dz \tag{8}$$

The nonlocal governing equations can be derived by substituting Eq. (7) into Eq. (4) as:

$$A \frac{d^2 u_0}{dx^2} - B \frac{d^3 w_b}{dx^3} - B_s \frac{d^3 w_s}{dx^3} = 0 \tag{9-a}$$

$$B \frac{d^3 u_0}{dx^3} - D \frac{d^4 w_b}{dx^4} - D_s \frac{d^4 w_s}{dx^4} - N_0 \left(\frac{d^2 (w_b + w_s)}{dx^2} - \mu \frac{d^4 (w_b + w_s)}{dx^4} \right) + \left(1 - \mu \nabla^2\right) \left(-k_w + k_s \frac{d^2 w}{dx^2} \right) = 0 \tag{9-b}$$

$$B_s \frac{d^3 u_0}{dx^3} - D_s \frac{d^4 w_b}{dx^4} - H_s \frac{d^4 w_s}{dx^4} + A_s \frac{d^2 w_s}{dx^2} - N_0 \left(\frac{d^2 (w_b + w_s)}{dx^2} - \mu \frac{d^4 (w_b + w_s)}{dx^4} \right) + \left(1 - \mu \nabla^2\right) \left(-k_w + k_s \frac{d^2 w}{dx^2} \right) = 0 \tag{9-c}$$

3. Solution Procedure

To satisfy the governing equations of motion, the displacement variables are adopted as:

$$\{\boldsymbol{u}_0, \boldsymbol{w_b}, \boldsymbol{w_s}\} = \sum_{\boldsymbol{m}=1}^{\infty} \{\boldsymbol{U_n} \cos(\lambda\, \boldsymbol{x}), \boldsymbol{W_{bn}} \sin(\lambda\, \boldsymbol{x}), \boldsymbol{W_{sn}} \sin(\lambda\, \boldsymbol{x})\} \tag{10}$$

where ($U_n, W_{bn},\ \& W_{sn}$) are the unknown Fourier coefficients, and $\alpha = \boldsymbol{n}\pi/\boldsymbol{L}$. Substituting Eq. (10) into Eq. (9) and setting the determinant of coefficient matrix of obtained equations to be zero, the critical buckling load is found as:

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{12} & C_{22}-\overline{P} & C_{23}-\overline{P} \\ C_{13} & C_{23}-\overline{P} & C_{33}-\overline{P} \end{bmatrix} \begin{bmatrix} U_n \\ W_{bn} \\ W_{sn} \end{bmatrix} = 0 \tag{11}$$

where C_{ij} are given in Appendix.

4. Numerical Results and Discussions

In this section, the analytical solutions obtained in previous sections are presented. The results obtained by Rahmani and Jandaghian [12] using the third order shear deformation beam theory are compared with those of the present study. The dimensionless quantity is used in the following form:

$$\overline{N} = \frac{N L^2}{E_1 I}, K_w = \frac{k_w L^4}{D_1}, K_s = \frac{k_s L^2}{D_1}, \quad D_1 = \frac{E_1 h^3}{12(1-\nu^2)} \tag{12}$$

The parameters used in this example are:
$E_1 = 1$ TPa , $E_2 = 0.25$ TPa , $\nu_1 = \nu_2 = 0.3$, L(length) = 10 (nm) , b(width) = 1000 (nm) , h(thickness) = varied .

The dimensionless critical buckling loads for FG nanobeam are tabulated in Table 1. The comparisons show a good agreement between the obtained results of this study and those obtained by Rahmani and Jandaghian [12]. According to this table, by increasing the nonlocal parameter (μ), the buckling loads decrease. However, the increase of power-law index (k) leads to an increase of dimensionless critical buckling loads.

Table 1. Dimensionless critical buckling load of the FG nanobeam

L/h	k	Nonlocal parameter, $\sqrt{\mu}\,(nm)$									
		0		0.5		1		1.5		2	
		TSBT(a)	Present	TSBT(a)	Present	TSBT(a)	Present	TSBT(a)	Present	TSBT(a)	Present
10	0	2.4057	2.4057	2.3478	2.3477	2.1896	2.1895	1.9685	1.9685	1.7248	1.7248
	1	5.3084	5.3084	5.1806	5.1806	4.8316	4.8316	4.3438	4.3438	3.8059	3.8059
	10	8.1337	8.1337	7.9378	7.9378	7.4030	7.4030	6.6557	6.6557	5.8315	5.8315
100	0	2.4668	2.4668	2.4662	2.4662	2.4643	2.4643	2.4613	2.4613	2.4571	2.4571
	1	5.4271	5.4271	5.4257	5.4257	5.4217	5.4217	5.4150	5.4150	5.4057	5.4057
	10	8.3158	8.3158	8.3137	8.3137	8.3076	8.3076	8.2973	8.2973	8.2831	8.2831

TSBT(a): Taken from Rahmani and Jandaghian. [12]

Table 2 compares dimensionless buckling loads of the present model with those presented by Rahmani and Jandaghian. [12] for a FG nanobeam. In order to compare the loads, the material selection is performed as follows: $E_1 = 210$ GPa , $\nu_1 = 0.3$, for steel and $E_2 = 390$ GPa and $\nu_2 = 0.3$ for alumina. It can be seen from Table 2 that the present nonlocal results display a good agreement with the results obtained from Rahmani and Jandaghian. [12].

Table 2. Comparison of dimensionless critical buckling load of FG nanobeam ($L/h = 20$)

k	Nonlocal parameter $\mu(nm)$							
	1		2		3		4	
	TSBT(a)	Present	TSBT(a)	Present	TSBT(a)	Present	TSBT(a)	Present
0	89,258	89,258	81,900	81,900	75,663	75,664	70,309	70,310
0,5	114,944	114,945	105,470	105,470	97,438	97,439	90,543	90,544
1	123,709	123,709	113,512	113,512	104,869	104,868	97,447	97,448
5	142,363	142,363	130,629	130,629	120,682	120,682	112,142	112,142
10	150,216	150,216	137,834	137,835	127,338	127,339	118,328	118,328

Table 3 presents the influences of elastic foundation parameters (K_w, K_s), nonlocal parameter (μ), gradient index (k), and slenderness ratio (L/h) on the non-dimensional buckling load of S–S FG nanobeams. It should be mentioned that the nonlocal parameter weakens the nanobeam structure. Therefore, it has a remarkable decreasing influence on the non-dimensional buckling loads. Moreover, it is found that the existence of elastic foundation makes the beam more rigid and increase the buckling loads.

Table 3. Influence of elastic foundation on the dimensionless critical buckling load of a S–S FG nanobeam

(K_w, K_s)	L/h	$\mu=1$			$\mu=2$			$\mu=3$		
		$k=0$	$k=1$	$k=5$	$k=0$	$k=1$	$k=5$	$k=0$	$k=1$	$k=5$
$(0,0)$	5	8.14771	11.32199	13.10123	7.47613	10.38876	12.02135	6.90683	9.59767	11.10593
	10	8.75834	12.14567	13.99365	8.03643	11.14455	12.84022	7.42446	10.29590	11.86244
	20	8.92576	12.37092	14.23634	8.19005	11.35124	13.06290	7.56638	10.48685	12.06817
$(25,5)$	5	16.42576	19.60004	21.37929	15.75419	18.66681	20.29941	15.18488	17.87572	19.38399
	10	17.03639	20.42372	22.27171	16.31448	19.42261	21.11827	15.70251	18.57396	20.14050
	20	17.20381	20.64897	22.51439	16.46810	19.62929	21.34095	15.84444	18.76490	20.34622
$(50,10)$	5	24.70382	27.87810	29.65734	24.03224	26.94487	28.57746	23.46294	26.15378	27.66205
	10	25.31445	28.70178	30.54976	24.59253	27.70066	29.39632	23.98057	26.85201	28.41855
	20	25.48187	28.92703	30.79245	24.74616	27.90734	29.61901	24.12249	27.04296	28.62428

The effects of gradient index (k) on variations of the critical buckling load of simply supported FG nanobeams at ($L/h=10$) and $\mu=1$ are presented in Fig. 2. It can be observed that for all values of elastic foundation constants, the critical buckling load increase with the increase of gradient index, where this augmentation is more sensible according to the lower values of the gradient index. Moreover, it is found that the shear layer or Pasternak parameter of the elastic foundation has a more remarkable impact on the critical buckling load than Winkler parameter. Therefore, with an increase of Pasternak constant, the critical buckling load increases significantly.

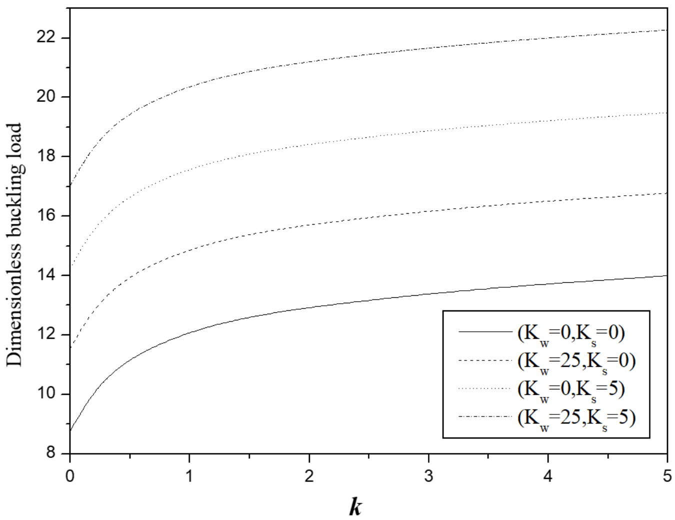

Fig. 2 Effect of Winkler and Pasternak parameters on the variation of dimensionless critical buckling load of the S–S FG nanobeam with respect to power-law index ($L/h=10$, $\mu=1$)

The variation of critical buckling load with nonlocal scaling parameters for different aspect ratios (L/h) of FG nanobeam is demonstrated in Fig. 3. It is obvious that the critical buckling load decreases with the increase in nonlocal scaling parameters while all aspect ratios are considered. The effect of mode number of FG nanobeam on the variation of buckling load ratio versus μ is illustrated in Fig. 4. For this purpose, the buckling load ratio is defined as follows:

$$\text{Buckling load ratio} = \frac{\text{Buckling load from nonlocal theory}}{\text{Buckling load from local theory}}$$

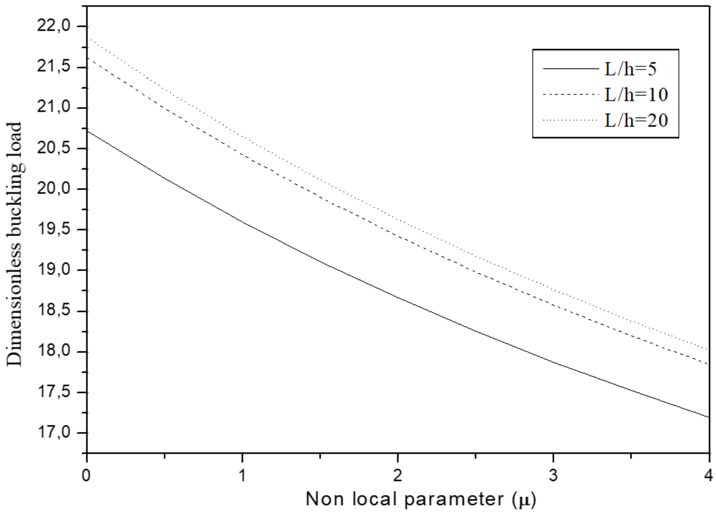

Fig. 3. Effect of the nonlocal parameter on the dimensionless critical buckling load $(K_w = 25, K_s = 5, k = 1)$

As can be seen, the buckling load ratio decreases with increasing mode numbers. Moreover, the small-scale effects on the buckling load ratio become more distinguished at higher modes. Obviously, the difference between the buckling load ratios of the FG nanobeam is larger at higher nonlocal parameters. Furthermore, the buckling load ratio for all mode numbers decreases by increasing the μ value.

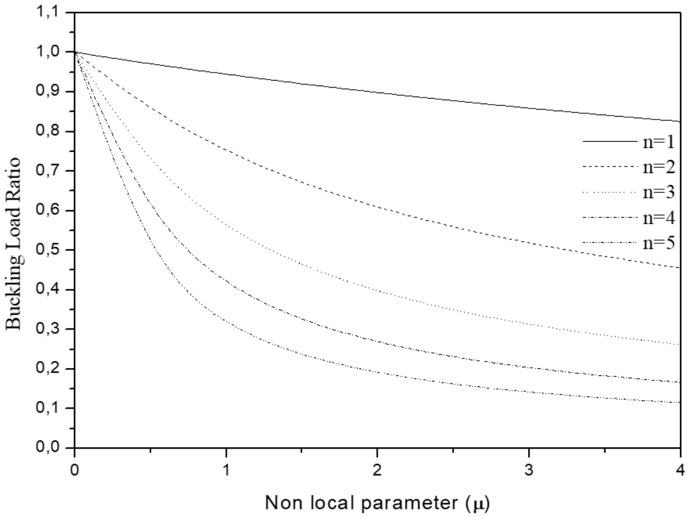

Fig. 4. Effect of mode number on the dimensionless critical buckling load ratio versus nonlocal parameter $(K_w = 25, K_s = 5, k = 1, L/h = 10)$

5. Conclusion

This study presents a nonlocal refined hyperbolic shear deformation beam theory for the buckling analysis of FG nanobeams embedded on a two-parameter elastic foundation. The present model is capable of capturing both small scale and shear deformation effects of FG nanobeams, and does not require shear correction factors. Governing equations based on the nonlocal elasticity theory are solved using Navier solution method. Mechanical properties of FG nanobeams are supposed to vary continuously through the thickness direction according to the power-law model. A detailed parametric study is conducted to investigate the influences of elastic foundation, nonlocal parameter, material composition, and slenderness ratio on the size-dependent buckling characteristics of FG nanobeams. It is found that for all values of elastic foundation parameters, increasing nonlocality and power-law exponent lead to reduction in rigidity of both nanobeam structure and buckling loads. However, with the rise in the magnitude of Winkler or Pasternak constants, the rigidity of the FG nanobeam as well as the buckling load results increase.

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APPENDIX

The stiffness coefficients $\boldsymbol{a}_{ij}$ appeared in governing equation (11) are as follows.

$$\boldsymbol{a}_{11} = -\boldsymbol{A}_{11}\lambda^2 , \quad \boldsymbol{a}_{12} = \boldsymbol{B}_{11}\lambda^3$$
$$\boldsymbol{a}_{13} = \boldsymbol{B}^s_{11}\lambda^3 , \quad \boldsymbol{a}_{22} = -\boldsymbol{D}_{11}\lambda^4$$
$$\boldsymbol{a}_{23} = -\boldsymbol{D}^s_{11}\lambda^4 , \quad \boldsymbol{a}_{33} = -\left(\boldsymbol{H}^s_{11}\lambda^4 + \boldsymbol{A}^s_{55}\lambda^2\right)$$
$$\overline{\boldsymbol{P}} = \alpha\, \boldsymbol{N}_0\, \lambda^2 , \quad \alpha = 1 + \mu\, \lambda^2$$