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Modelling of Crack Growth Using a New Fracture Criteria Based Peridynamics

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Abstract. Peridynamics (PD) is a nonlocal continuum theory based on integro-differential equations without spatial derivatives. The elongation fracture criterion is implicitly incorporated in the PD theory, and fracture is a natural outcome of the simulation. On the other hand, a new fracture criterion based on the crack opening displacement combined with peridynamic (PD-COD) is proposed. When the relative deformation of the PD bond between two particles reaches the critical crack tip opening displacement of the fracture mechanics, we assume that the bond force vanishes. The new damage rule of fracture criteria similar to the local damage rule in conventional PD is introduced to simulate the fracture. In this paper, first, a comparative study between XFEM and PD is presented. Then, four examples, i.e., a bilateral crack problem, double parallel crack, monoclinic crack, and the double inclined crack are given to demonstrate the effectiveness of the new criterion.

Keywords: Peridynamic; Castigliano's theorem; Critical extension; COD; PD-COD.

1. Introduction

For crack propagation problems in the engineering structure, it is difficult to obtain an analytical solution for the stress, strain, and displacement field and determine the direction of crack propagation. Many fracture, multiscale modeling of fracture, and numerical methods have been proposed such as Fracture modeling [1-4] and Multiscale modeling [5-10], including fourth order phase-field model, a novel two-stage discrete crack method, a higher-order stress-based gradient-enhanced damage model, an adaptive multiscale method, an efficient coarse graining in multiscale modeling, and so on.

The fracture finite element methods (FEM) include efficient remeshing techniques [10-16], the extended finite element method (XFEM) [17-18] or XIGA [19-20], the numerical manifold method (NMM) [21-22], the element-free Galerkin (EFG) methods [23], the reproducing kernel particle method (RKPM) [24], along with many other meshless methods [25-27], and enriched meshfree method (MM) (see the contributions by Rabczuk [28-36]). The cracking particle method (CPM) [37-39] is a method that can easily deal with complex crack patterns as fracture is a natural outcome of the simulation. For all other methods, the mixed mode fracture is usually studied theoretically based on different failure criteria [40-43] or using test methods. However, researchers prefer to put theories into practice and design the specimens since they will be able to provide the same states (e.g. the centrally cracked Brazilian disk specimen [44-47]), the compact tension shear specimen [48-50], but the experimental fracture studies on real components are very expensive and difficult.

With the advancements in computer science and computational science, the finite element method has become a powerful tool which was also applied to fracture problems. Analyzing the extension characteristics of cracks and predicting the service life of structures are of high importance in engineering. The traditional finite element method (FEM) is not well suited for crack propagation problems as the crack can only propagate along the existing element edges by node splitting. The results will highly depend on the mesh topology; therefore, remeshing techniques are commonly exploited. Besides some recent advancements in remeshing techniques, see for instance the work by Areias et al. [11-16]; their implementation to 3D and complex fracture patterns still remains a challenge.



The extended finite element method (XFEM) [51] allows crack propagation without remeshing. Though XFEM can principally capture crack propagation without remeshing, a certain mesh refinement is still required in order to obtain results with satisfactory accuracy. For problems in linear elastic fracture mechanics (LEFM), it was shown that its accuracy was related to the position of the crack tip [52-53]. It is also complicated to deal with complex fracture patterns such as crack coalescence or crack branching. Daux et al. developed strategies to deal with such type of problems [52]. Nevertheless, reliable fracture criteria for such applications are still missing. While the original XFEM was developed for problems in LEFM, it was meanwhile applied to non-linear problems including cohesive cracks; see for instance the work by Jin Feng et al. [54-55]. The phantom node method [56-57] is similar to XFEM, and it was shown in [58] that the crack kinematics of the phantom node method can be derived from XFEM. However, it has some advantages: (1) It models the crack by overlapping elements (not by enrichment), and therefore, existing element formulations can be readily exploited. In contrast, efficient FEM formulations for problems with constraints, for instance, need to be redeveloped and tested for the 'standard' XFEM, (2) No 'mixed terms' are needed in the phantom node method, and (3) The implementation is easier. Although, the phantom node method requires that the crack propagates through an entire element, some special crack tip elements have been developed which allow the crack tip inside an element [59-60]. Both the phantom node method as well as XFEM have been implemented in the commercial software package ABAQUS.

Meshless methods (MM) eliminate the mesh and can deal well with fracture problems as well (see the contributions by Rabczuk [28-36] or Zhuang et al. [27,61-62]). However, as XFEM, they still require fracture criteria. An interesting and powerful meshfree method for fracture is the cracking particles method (CPM) [37-39] which can –similar to PD– also complex fracture pattern, such as crack coalescence, crack nucleation, and crack coalescence without any special criteria. In the CPM, the crack is modelled by a set of crack segments, and therefore, the crack path is not continuous. It has been shown that certain meshless methods are nearly identical to the so-called state-based PD formulation [63-64].

Peridynamics (PD), which was first proposed in 2000 by Silling [67-68], is a new numerical method for characterizing materials based on nonlocal models, so that fracture is a natural outcome of the simulation [65-66]. For a linear elastic solid, the original bond-based PD formulation [67] modeled fracture by equating the critical energy release rate of the continuum with the one from PD theory which finally led to a critical stretch between the particles as fracture criterion which can be implemented with ease. However, the extension to nonlinear materials and complex mixed-mode fracture cannot easily be treated with the bond-based PD. Therefore, the state-based PD was developed which can handle more complex constitutive models. However, it was shown that such models are quite similar to meshfree methods [63-64]. Nonetheless, the theory has been further developed and applied to some numerical simulations of discontinuous damage problems in recent years [69]. The third generation of PD is called dual-horizon peridynamics (DH-PD) and allows for adaptive refinement [70-72]; it can be applied to bond-based PD as well as state-based PD.

Peridynamics [65, 73] has attracted great attention due to its flexibility in modeling complex fracture patterns. Silling and Bobaru [74] proposed a weighted local function of the particle weight method to determine the particle damage problem. Weckner et al. [75] used the Laplace and Fourier transforms in three-dimensional PD. Yu et al. [76] proposed an adaptive trapezoidal integration scheme. Kilic [77] described an efficient load distribution scheme. Silling and Askari [73] derived the critical energy release rate for bond-based PD in integral form. Foster et al. [78] proposed the critical energy density as a failure criterion in ate-dependent situations. Silling and Bobaru [79] used the function ϕ to modify the force density vector. Ayatollahi and Aliha [80] demonstrated the failure parameter of a critical stretch by experiments. Feng and Zhang simulated the cracking process of concrete [81]. The article by Zhou [82] examined rock-like materials. While Ren et al. [83] proposed a new criterion for damage determination of shear deformation.

In multi-scale problems, there are some big challenges, which have been studied by some researchers [5,7-9,39,43,84-86], see also the contribution by Costa and Bond [87] on a multi-scale PD finite element model. Jung and Seok [88] modelled fatigue cracks and computed the stress intensity factors of stratified inhomogeneous materials. Feng and Zhang [81] simulated the cracking process of concrete. Gu and Zhou [89] captured the fracture process of a plate with a round hole and provided new ideas and methods for material failure. Cheng and Liu [90] focused on dynamic fracture of functionally graded materials and analyzed the dynamic response of those materials. The article of Zhou [91] deals with rock-like materials. While Ren et al. [83] proposed a new criteria for damage determination of shear deformation and derived the Peridynamics theory using the principle of virtual work [92].

This paper mainly consists of two fracture criteria. First, a comparative study of PD and XFEM is done for problems involving crack propagation which include crack interactions for quasi-static problems. Two examples are considered: Crack propagation in a double-notched specimen under uniaxial tension and a specimen with a double center crack. Both specimens are made of Q345 steel, and experimental data is available. We use the XFEM capabilities in ABAQUS to solve this problem, while our own PD program is employed to solve such problems. Second, we present a new fracture criterion in PD. Though the formulation is devised for linear elastic solids, it can easily be extended to non-linear materials. This new idea is comparable to the crack opening displacements (COD) criterion employed in the fracture mechanics theory. When the relative deformation of two adjacent particles in PD reaches the critical COD value, the PD forces between the two particles vanish, and a crack is formed.

The contents of this article are summarized as follows: Section 2 introduces the PD theory. In section 3 the elongation fracture criterion is described, and two examples are simulated. In section 4 the COD criterion is derived in the context of PD. Also, the critical value of peridynamic crack tip opening displacement (PD-COD), which governs the crack propagation, is provided. At the same time, four examples are presented which verify the new formulation. The results of the first example are compared with the experimental data and results obtained by other methods including the 'classical' PD method and XFEM.

The subsequent three examples include the center double parallel crack problem, the center monoclinic crack problem, and the center double oblique crack problem. Finally, conclusions are provided in section 5.

2. Peridynamic Theory

PD discretizes objects with many particles. In this paper, we employ the state-based PD formulation which is subsequently summarized. The key in state-based PD is how to compute the force state $\underline{\mathbf{T}}$ depending on the deformation state $\underline{\mathbf{Y}}$ between the particles in the current configuration. The initial relative position vector $(\mathbf{x}_{(j)} - \mathbf{x}_{(k)})$ prior deformation becomes $(\mathbf{y}_{(j)} - \mathbf{y}_{(k)})$ after deformation. The relative position vector $(\mathbf{y}_{(j)} - \mathbf{y}_{(k)})$ and the stretch between material points $\mathbf{x}_{(k)}$ and $\mathbf{x}_{(j)}$ can be defined as:

$$(\mathbf{y}_{(j)} - \mathbf{y}_{(k)}) = \underline{\mathbf{Y}}(\mathbf{x}_{(k)}, t) \langle \mathbf{x}_{(j)} - \mathbf{x}_{(k)} \rangle \quad (1)$$

and

$$s_{(k)(j)} = \frac{(|\mathbf{y}_{(j)} - \mathbf{y}_{(k)}| - |\mathbf{x}_{(j)} - \mathbf{x}_{(k)}|)}{|\mathbf{x}_{(j)} - \mathbf{x}_{(k)}|} \quad (2)$$

The force state for material point $\mathbf{x}_{(k)}$ depends on other material points within its horizon. It can also be expressed as:

$$\underline{\mathbf{T}}(\mathbf{x}_{(k)}, t) = \underline{\mathbf{T}}(\underline{\mathbf{Y}}_{(k)}, t) \quad (3)$$

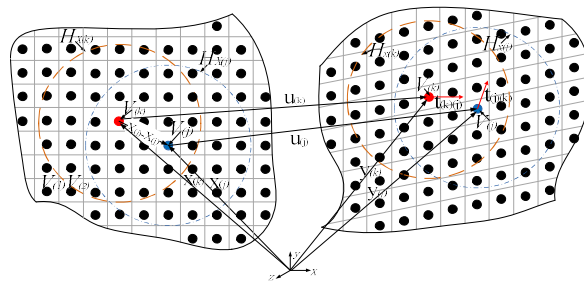


Fig. 1. Kinematics of PD material points

Fig. 1 shows the force density vector $\mathbf{t}_{(k)(j)}$ that the material point $\mathbf{x}_{(j)}$ exerts on the material point $\mathbf{x}_{(k)}$. It is given as:

$$\mathbf{t}_{(k)(j)}(\mathbf{u}_{(j)} - \mathbf{u}_{(k)}, \mathbf{x}_{(j)} - \mathbf{x}_{(k)}, t) = \underline{\mathbf{T}}(\mathbf{x}_{(k)}, t) \langle \mathbf{x}_{(j)} - \mathbf{x}_{(k)} \rangle \quad (4)$$

The interaction between the material points $\mathbf{x}_{(k)}$ and $\mathbf{x}_{(j)}$ can be derived from a scalar-valued micropotential, $w_{(k)(j)}$

$$w_{(k)(j)} = w_{(k)(j)}(\mathbf{y}_{(1^k)} - \mathbf{y}_{(k)}, \mathbf{y}_{(2^k)} - \mathbf{y}_{(k)}, \mathbf{L}) \quad (5)$$

where $\mathbf{y}_{(k)}$ is the position vector of point $\mathbf{x}_{(k)}$ in the deformed configuration, and $\mathbf{y}_{(1^k)}$ is the position vector of the first material point that interacts with point $\mathbf{x}_{(k)}$.

Then, the strain energy density, $W_{(k)}$, of material point $\mathbf{x}_{(k)}$ can be expressed as the summation of micro-potentials, $w_{(k)(j)}$, within its horizon of material points $\mathbf{x}_{(k)}$ and is given by:

$$W_{(k)} = \frac{1}{2} \sum_{j=1}^{\infty} \left(\frac{1}{2} w_{(k)(j)}(\mathbf{y}_{(1^k)} - \mathbf{y}_{(k)}, \mathbf{y}_{(2^k)} - \mathbf{y}_{(k)}, \mathbf{L}) + \frac{1}{2} w_{(j)(k)}(\mathbf{y}_{(1^j)} - \mathbf{y}_{(j)}, \mathbf{y}_{(2^j)} - \mathbf{y}_{(j)}, \mathbf{L}) \right) V_{(j)} \quad (6)$$

in which $w_{(k)(j)} = 0$ for $k = j$.

By applying the principle of virtual work, the PD equations of motion at material point $\mathbf{x}_{(k)}$ can be derived as:

$$\delta \int_{t_0}^{t_1} (T - U) dt = 0 \quad (7)$$

where T and U represent the total kinetic and potential energies in the body. By solving the Lagrange's equation, this principle is satisfied as:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\mathbf{u}}_{(k)}} \right) - \frac{\partial L}{\partial \mathbf{u}_{(k)}} = 0 \quad (8)$$

where the Lagrangian L is defined as:

$$L = T - U \quad (9)$$

By summing the kinetic and potential energies of all material points, the total kinetic and potential energies in the body are

obtained by:

$$T = \sum_{i=1}^{\infty} \frac{1}{2} \rho_{(i)} \dot{\mathbf{u}}_{(i)} \cdot \dot{\mathbf{u}}_{(i)} V_{(i)} \quad (10)$$

and

$$U = \sum_{i=1}^{\infty} W_{(i)} V_{(i)} - \sum_{i=1}^{\infty} (\mathbf{b}_{(i)} \cdot \mathbf{u}_{(i)}) V_{(i)} \quad (11)$$

Through the Eqs. **Error! Reference source not found.**, **Error! Reference source not found.**, and **Error! Reference source not found.**, Lagrange's equation of the material point $\mathbf{x}_{(k)}$ can be recast as:

$$\begin{aligned} \rho_{(k)} \mathbf{u}_{(k)} = \sum_{j=1}^{\infty} & \left[\mathbf{t}_{(k)(j)} (\mathbf{u}_{(j)} - \mathbf{u}_{(k)}, \mathbf{x}_{(j)} - \mathbf{x}_{(k)}, t) \right. \\ & \left. - \mathbf{t}_{(j)(k)} (\mathbf{u}_{(k)} - \mathbf{u}_{(j)}, \mathbf{x}_{(k)} - \mathbf{x}_{(j)}, t) \right] V_{(j)} + \mathbf{b}_{(k)} \end{aligned} \quad (12)$$

where

$$\mathbf{t}_{(k)(j)} (\mathbf{u}_{(j)} - \mathbf{u}_{(k)}, \mathbf{x}_{(j)} - \mathbf{x}_{(k)}, t) = \frac{1}{2} \frac{1}{V_{(k)}} \left(\sum_{i=1}^{\infty} \frac{\partial w_{(k)(i)}}{\partial (\mathbf{y}_{(i)} - \mathbf{y}_{(k)})} V_{(i)} \right) \quad (13)$$

and

$$\mathbf{t}_{(j)(k)} (\mathbf{u}_{(k)} - \mathbf{u}_{(j)}, \mathbf{x}_{(k)} - \mathbf{x}_{(j)}, t) = \frac{1}{2} \frac{1}{V_{(j)}} \left(\sum_{i=1}^{\infty} \frac{\partial w_{(i)(k)}}{\partial (\mathbf{y}_{(k)} - \mathbf{y}_{(i)})} V_{(i)} \right) \quad (14)$$

Finally, the PD equations of motion at material point $\mathbf{x}_{(k)}$ can be rewritten as:

$$\rho_{(k)} \ddot{\mathbf{u}}_{(k)} = \sum_{j=1}^{\infty} \left(\mathbf{T}(\mathbf{x}_{(k)}, t) \langle \mathbf{x}_{(j)} - \mathbf{x}_{(k)} \rangle - \mathbf{T}(\mathbf{x}_{(j)}, t) \langle \mathbf{x}_{(k)} - \mathbf{x}_{(j)} \rangle \right) V_{(j)} + \mathbf{b}_{(k)} \quad (15)$$

Because the volume of each material point $V_{(j)}$ is infinitesimally small, the infinite summation can be expressed as integration while considering only the material points within the horizon,

$$\sum_{j=1}^{\infty} (\cdot) V_{(j)} \rightarrow \int_H (\cdot) dV' \rightarrow \int_H (\cdot) dH \quad (16)$$

Using the expression of Eq. **Error! Reference source not found.** with the domain of integral H and the thickness of the plate, the PD dynamic equation is expressed as:

$$\rho(\mathbf{x}) \ddot{\mathbf{u}}_{(k)} = \int_H \left(\mathbf{t}(\mathbf{u}' - \mathbf{u}, \mathbf{x}' - \mathbf{x}, t) - \mathbf{t}'(\mathbf{u} - \mathbf{u}', \mathbf{x} - \mathbf{x}', t) \right) dH + \mathbf{b}(\mathbf{x}, t) \quad (17)$$

in which $\mathbf{b}(\mathbf{x}, t)$ denotes the body force.

3. Critical stretch fracture criteria

3.1 Basic model

Fracture is modelled by simply removing the interaction of two particles when a certain stretch is exceeded as illustrated in Fig. 2. All of the micropotentials between the material points $\mathbf{x}_{(k^+)}$ and $\mathbf{x}_{(j^-)}$ which are located above and below the new crack surface can be obtained as:

$$w_{(k^+)(j^-)} = \left(4ad^2 \delta^2 \left(\sum_{i=1}^{K^-} s_c^2 V_{(i)} \right) + 4\delta b s_c^2 \left| \mathbf{x}_{(j^-)} - \mathbf{x}_{(k^+)} \right| \right) \quad (18)$$

and

$$w_{(j^-)(k^+)} = \left(4ad^2 \delta^2 \left(\sum_{i=1}^{J^+} s_c^2 V_{(i)} \right) + 4\delta b s_c^2 \left| \mathbf{x}_{(k^+)} - \mathbf{x}_{(j^-)} \right| \right) \quad (19)$$

in which a , b and d are the parameters of PD.

Then, the total strain energy required to remove all of the interactions across the newly created crack surface can be recast as:

$$W = \frac{1}{2} \sum_{k=1}^{K^+} \frac{1}{2} \sum_{j=1}^{J^-} w_{(k^+)(j^-)} V_{(k^+)} V_{(j^-)} + \frac{1}{2} \sum_{k=1}^{K^+} \frac{1}{2} \sum_{j=1}^{J^-} w_{(j^-)(k^+)} V_{(j^-)} V_{(k^+)} \quad (20)$$

where K^+ and J^- indicate the number of material points $\mathbf{x}_{(k^+)}$ and $\mathbf{x}_{(j^-)}$ above and below the crack surface. The corresponding strain energy release rate G_c is given by:



$$G_c = \frac{s_c^2 \sum_{k=1}^{K^+} \sum_{j=1}^{J^-} \left(2\delta b |x_{(j)} - x_{(k^+)}| + ad^2 \delta^2 \left(\sum_{i=1}^{K^-} V_{(i)} + \sum_{i=1}^{J^+} V_{(i)} \right) \right) V_{(k^+)} V_{(j)}}{A} \quad (21)$$

where s_c indicates the critical stretch.

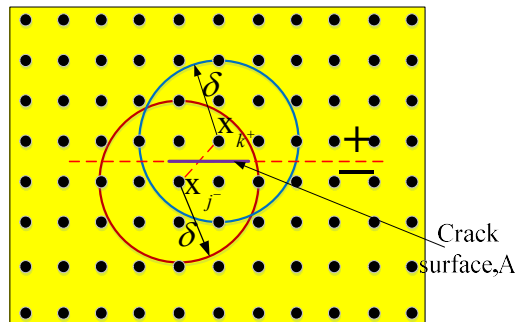


Fig. 2. Interaction between material points $x_{(k^+)}$ and $x_{(j^-)}$

For a 2D analysis, the expression for the critical energy release rate becomes:

$$G_c = \left(bh\delta^5 + \frac{8}{9}ad^2h^2\delta^7 \right) s_c^2 \quad (22)$$

in which h represents the thickness of the material. Hence, after substituting the peridynamic parameters a , b and d into Eq. **Error! Reference source not found.**, the critical stretch can be expressed as:

$$s_c = \sqrt{\frac{G_c}{\left(\frac{6}{\pi}\mu + \frac{16}{9\pi^2}(\kappa - 2\mu) \right) \delta}} \quad (23)$$

In order to include damage initiation in the material response, a history-dependent scalar-valued function μ is defined as:

$$\mu(x_{(j)} - x_{(k)}, t) = \begin{cases} 1 & \text{if } s_{(k)(j)}(x_{(j)} - x_{(k)}, t) < s_c \\ 0 & \text{else} \end{cases} \quad (24)$$

Note that when the stretch between these material points exceeds its critical stretch, failure occurs and $\mu = 1$, otherwise $\mu = 0$. Then, the force density vector can be modified through μ :

$$t_{(k)(j)} = 2\delta \left\{ ad \frac{\Lambda_{(k)(j)}}{|x_{(j)} - x_{(k)}|} \theta_{(k)} + b\mu(x_{(j)} - x_{(k)}, t) s_{(k)(j)} \right\} \cdot \frac{y_{(j)} - y_{(k)}}{|y_{(j)} - y_{(k)}|} \quad (25)$$

Local damage at a point is defined through the weight ratio function. The local damage at a point can be quantified as:

$$\phi(x_{(k)}, t) = 1 - \frac{\int_H \mu(x_{(j)} - x_{(k)}, t) dV_{(j)}}{\int_H dV_{(j)}} \quad (26)$$

The local damage ranges from 0 to 1. When the damage is zero, it means that all interactions are intact, and when the local damage is one, all the interactions initially associated with the point have been eliminated. The crack has then been completed, formed, and propagated. Then, half of the interaction within its horizon results in a local damage value of one half.

3.2 Numerical example

Example 1: Double-notched specimen made of Q345 steel under uniaxial tension

Q345 material is a low alloy high strength structural steel, with elasticity modulus $E = 203\text{GPa}$, Poisson's ratio $\nu = 0.3$, yield strength $f = 410\text{Mpa}$, ultimate strength $\sigma_s = 572\text{Mpa}$, elongation $\delta = 27.96\%$, and density $\rho = 7850\text{kg/m}^3$. The length and width of the specimen are 70mm and 40mm, respectively, as shown in Fig. 3. The specimen is loaded under uniaxial tension with a constant loading rate of $2.217 \times 10^{-5}\text{m/s}$. The crack length is 10 mm, and three different crack size distances in loading directions are tested according to Table 1. The initial and fracture specimen from the experiments are illustrated in Figs. 4 and 5.

Table 1. Bilateral horizontal crack size (unit: mm)

left crack size	right crack size	crack longitudinal offset distance (specimen label)		
10	10	0 (10-00)	10 (10-10)	20 (10-20)

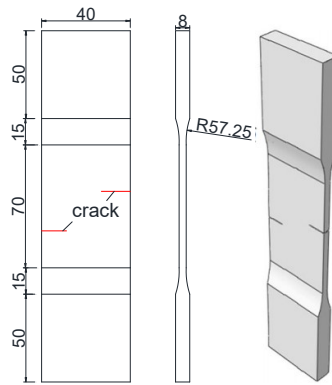


Fig. 3. Specimen style and size (unit: mm) [93]

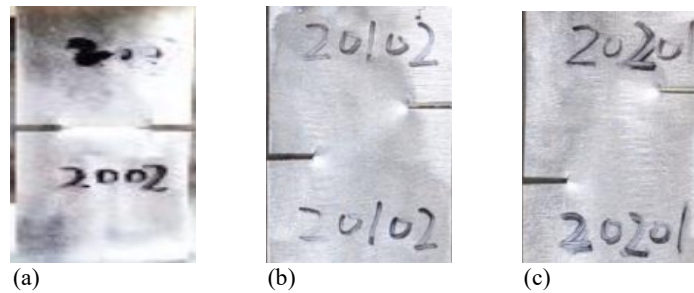


Fig. 4. Experimental fracture process [93]. (a) 10-00 test, (b) 10-10 test, (c) 10-20 test

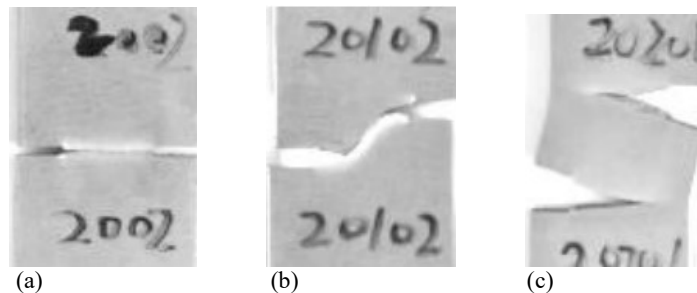


Fig. 5. Experimental fracture results [93]. (a) 10-00 test, (b) 10-10 test, (c) 10-20 test

The C3D8R XFEM element in ABAQUS is employed to simulate the crack propagation of the three specimens. The XFEM region of the specimen is divided into two parts. Each of them contains only one crack. Three kinds of specimen crack growth results are given in Fig. 6. The calculated results by XFEM are in good agreement with the experimental results in Fig. 5.

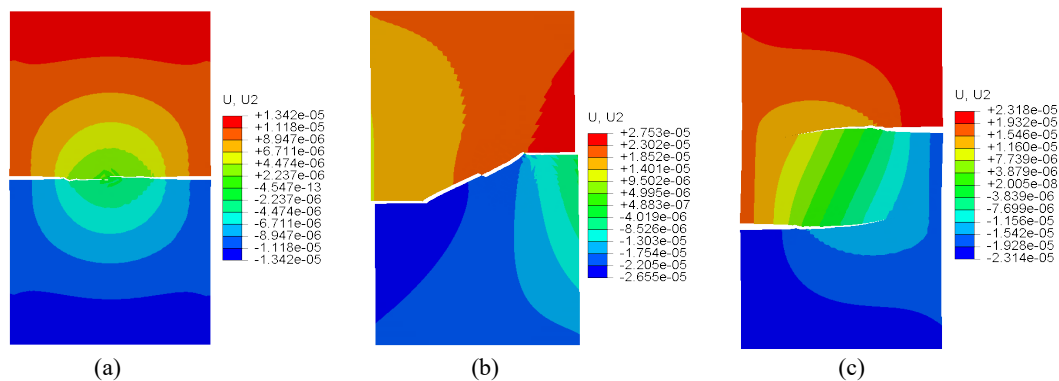


Fig. 6. Crack propagation results of XFEM. (a) 10-00 test, (b) 10-10 test, (c) 10-20 test

The Q345 steel specimen is a 2D simplified model in PD. A particle distance of $\Delta x=0.5\text{mm}$ is employed in PD, and the load is applied displacement-controlled over a width $d=3\Delta x$. To better visualize the crack, only the area near the crack, i.e., 40mm in longitudinal direction, is shown.

The fracture patterns obtained by the PD simulations for the three specimens are shown in Fig. 7. When the crack tip damage value is close to 1, the PD force of two particles disappears, and the crack propagates forward. By increasing the loading, the damage site expands and eventually runs through the entire specimen as shown in Fig. 8.



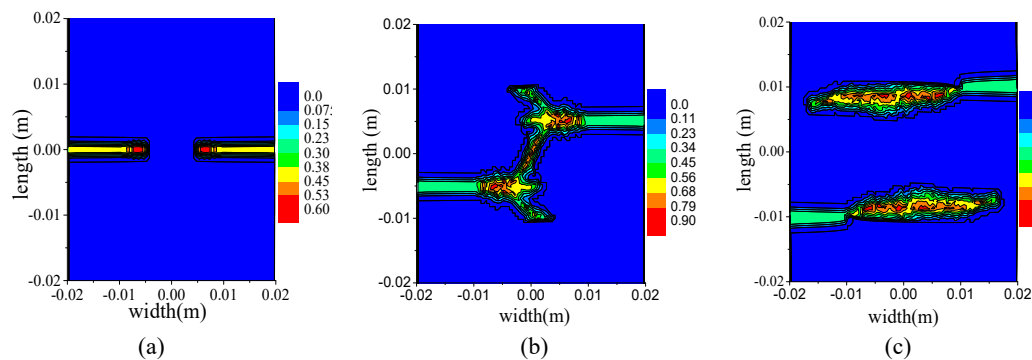


Fig. 7. Cracks propagation process of PD. (a) 10-00 test, (b) 10-10 test, (c) 10-20 test

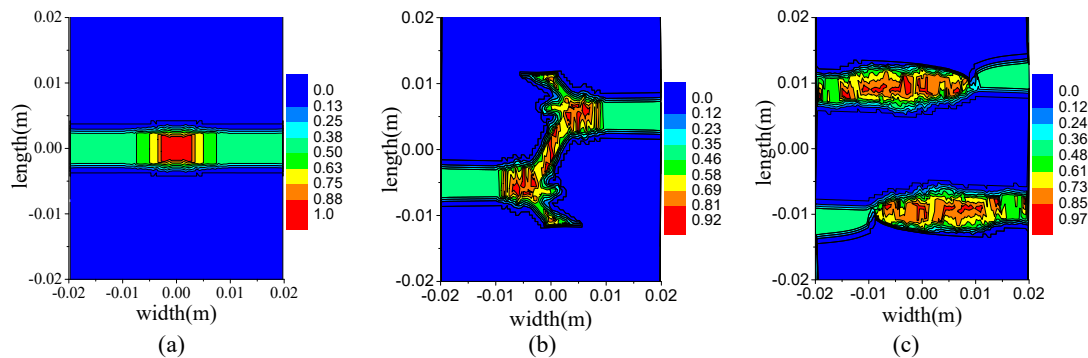


Fig. 8. Crack propagation results of PD. (a) 10-00 test, (b) 10-10 test, (c) 10-20 test

For a vertical crack tip distance of 0 mm, two cracks propagate expectedly along in the horizontal direction and eventually join into one single crack as depicted in Figs. 5 (a) and 8 (a). When the vertical crack tip distance is increased to 10 mm, the two cracks propagate initially in the horizontal direction and extend along the horizontal direction before they turn in a 45° angle versus the horizontal axis and finally join as shown in Figs. 5(b) and 8(b). Some artificial damage zones can be observed in the PD simulation which do not occur in XFEM, but the overall agreement is good. The specimen with a crack tip distance of 20 mm has a quite distinguished fracture pattern. Instead of crack coalescence, both cracks propagate through the entire specimen. No crack shielding occurs which is captured by both XFEM and PD as illustrated in Figs. 5(c) and 8(c).

Example 2: Double center crack problem

The specimen with two double-center cracks under uniaxial tension with a constant loading rate of 20m/s is used. The material parameters from the previous example are adopted. The thickness of the plate is 0.05m, the length of one crack is 0.01m. The longitudinal distance of the crack is varied from 5mm (case 1) to 10mm (case 2). A tensile load is applied at both ends of the plate in vertical direction. The particle distance in all PD simulations is 0.5mm. The influence of the longitudinal crack distance on damage rate and crack propagation path was analyzed.

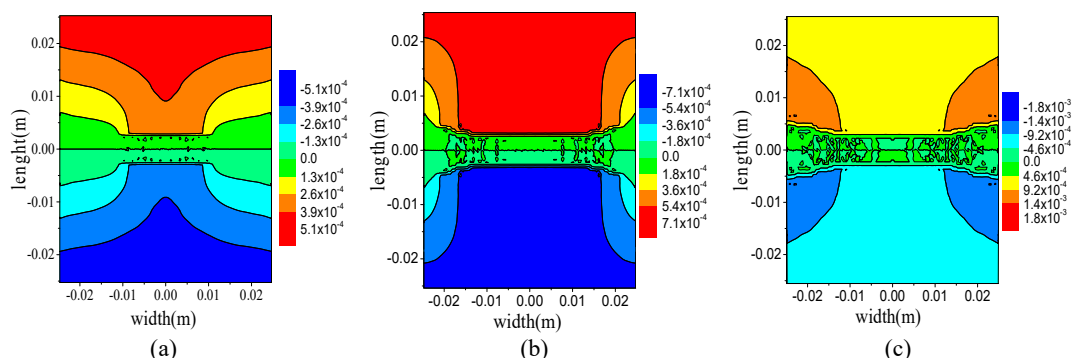


Fig. 9. Double crack distance 5 mm plate Y direction displacement. (a) $t=27.2 \mu\text{s}$, (b) $t=34 \mu\text{s}$, (c) $t=40.8 \mu\text{s}$

Figs. 9 and 10 depict the displacements y-direction at different load steps. Note that the maximum displacement is less for case 2 than case 1 after the crack has grown through the entire specimen. In both cases, the maximum displacement position changes from the ends of the plate to near the crack tip.

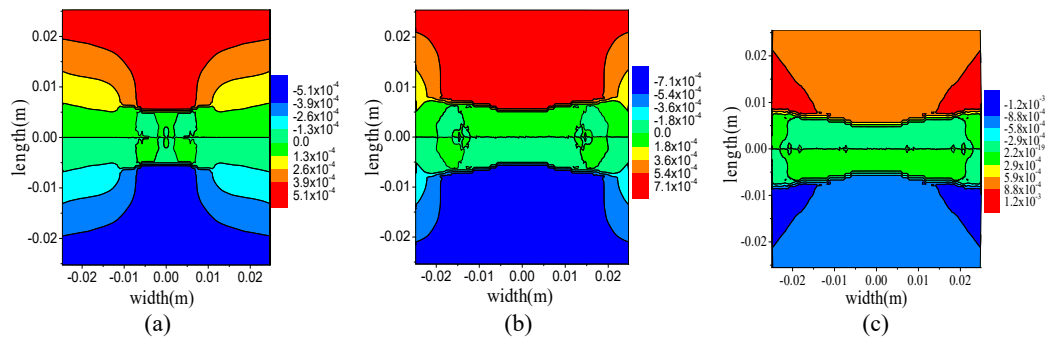


Fig. 10. Double crack distance 10 mm plate Y direction displacement. (a) $t=27.2 \mu s$, (b) $t=34 \mu s$, (c) $t=40.8 \mu s$

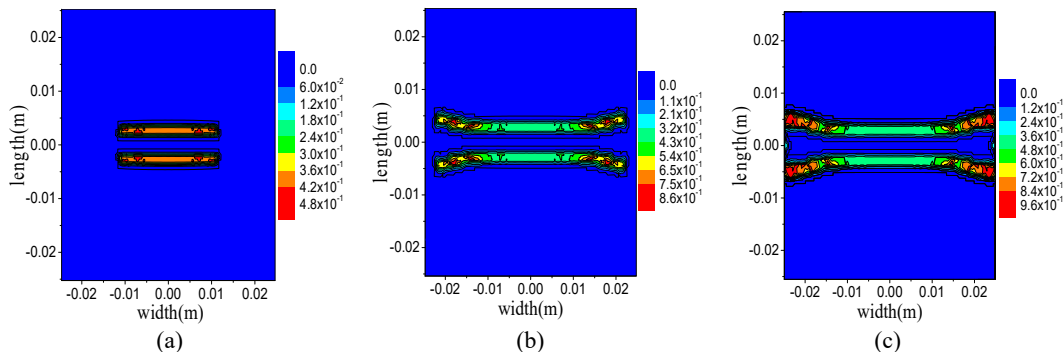


Fig. 11. Double crack distance 5 mm plate damage distribution. (a) $t=27.2 \mu s$, (b) $t=34 \mu s$, (c) $t=40.8 \mu s$

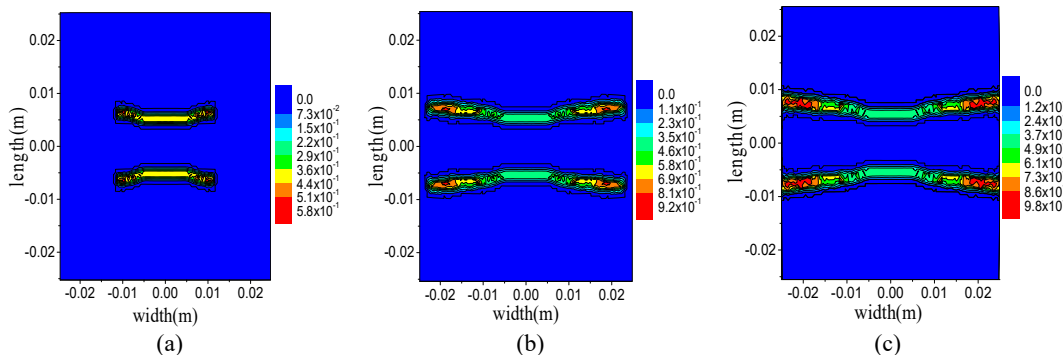


Fig. 12. Double crack distance 10 mm plate damage distribution. (a) $t=27.2 \mu s$, (b) $t=34 \mu s$, (c) $t=40.8 \mu s$

Figs. 11 and 12 show that the crack distance has a slight effect on the crack propagation path. The greater the crack distance is, the earlier the crack extension begins. At $27.2 \mu s$, the crack length increases concerning the increase of the crack length by 10mm, and the crack propagation direction changes as the crack tip expands, while the crack with the distance of 5mm spreads along the vertical load direction. At $34 \mu s$, cracks with 5mm distance propagation are changed to the opposite direction, and the crack with 10mm distance propagation direction is perpendicular to the load direction. At $40.8 \mu s$, the crack propagates through the entire length of the plate, and the distance between the crack tips in both plates is much larger than the initial crack distance.

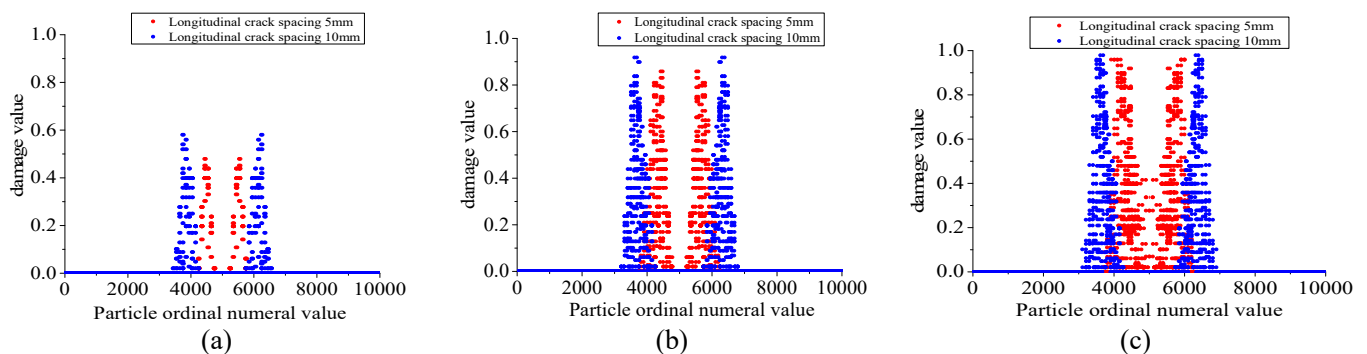


Fig. 13. The different time damage value of two kinds of plate. (a) $t=27.2 \mu s$, (b) $t=34 \mu s$, (c) $t=40.8 \mu s$

Fig. 13 shows the damage value of the material in the plates. The abscissa represents the ordinal value of the particle and the ordinate represents the damage value. The larger the damage value is, the faster the crack will propagate. So, the crack propagation rate of big crack distance is much higher than that of the small crack distance, and this phenomenon would



become weaker with the crack propagation. The damage rate of the two cracks with the crack distance of 10mm at $27.2 \mu\text{s}$ is significantly higher than that at the crack distance of 5mm, the gap between the two plates decreases gradually at $34 \mu\text{s}$, and the maximum of the two plates is nearly the same at $40.8 \mu\text{s}$.

4. PD-COD crack fracture criterion

4.1 Basic model

Consider a crack and a crack extension as illustrated in Fig. 4 (SZW: the length of extension zone, which is the amount of crack forward extension). The opening displacement of the crack tip is twice the height of the extension zone ($\text{COD} = 2 \times \text{SZD}$), and the stretch height is equal to the length of extension of the crack front ($\text{SZD} = \text{SZW}$). The actual crack tip opening displacement measurement is shown in Fig. 15 which exploits the crack tip symmetrical primitive crack at right angle to intersect up and down the crack surface points a and a' . The distance of the two points is the value of the opening displacement COD at the crack tip.

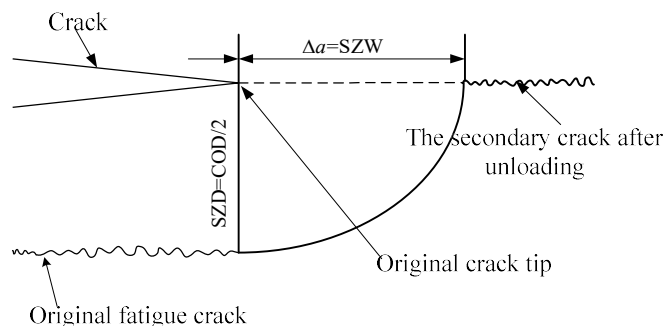


Fig. 14. Crack tip passivation model and extension zone

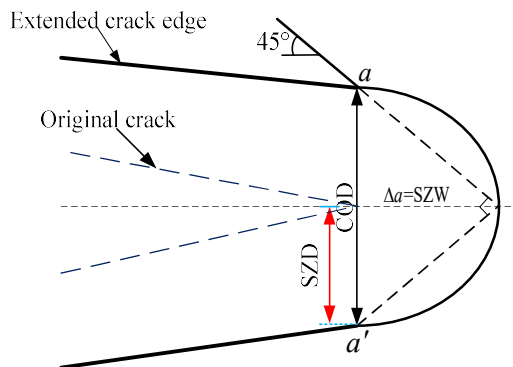


Fig. 15. Measures of the value of COD at the crack tip

As the load increases, the opening displacement of the crack tip δ increases, when a critical value δ_{cr} is reached. Therefore, the crack propagation criterion established by COD is written as:

$$\delta \geq \delta_{cr} \quad (27)$$

where δ_{cr} is the critical COD value, determining the beginning of the crack extension.

There are two main methods to calculate the COD value: one is the D-M model derived from the BCS formula, the other is the Wells formula. In this paper, the BCS formula is used to solve the crack propagation problem.

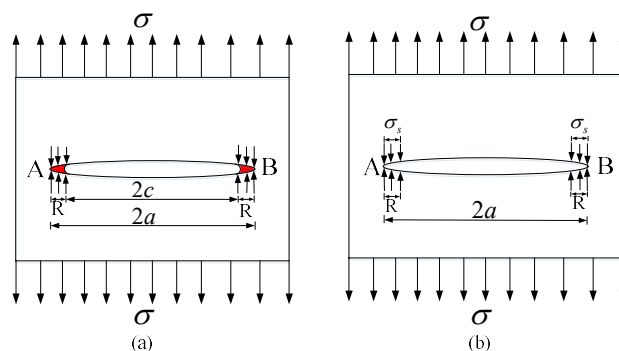


Fig. 16. Plane stress crack and D-M model

Previous studies on tensile tests of a large sheets have revealed a flat plastic zone as depicted in Fig. 16 (a). The plastic zone is simplified into a triangular curve. Assume that the material of the plastic zone is ideal plasticity, and the material surrounded by the plastic zones and cracks is the elastic zone. The plastic zone of the crack tip is excavated, forming a center through crack whose length is $2a = 2c + 2R$ in the elastic infinite plate, as shown in Fig.16 (b) above. So, the above problem is a simplified D-M elasticity model where the displacement of the crack tip δ can be obtained by the Paris displacement formula.

As shown in Fig. 17 by the Castigliano's theorem, the relative displacement between the two particles is obtained by derivative of the elastic strain energy as:

$$\delta_i = \frac{\partial U}{\partial P_i} \quad (28)$$

where U is the strain energy of the elastic body, P_i is the force, and δ_i is the displacement.

When $F \rightarrow 0$, the actual displacement between any two points is recast as:

$$\delta_i = \lim_{F \rightarrow 0} \frac{\partial U}{\partial F} \quad (29)$$

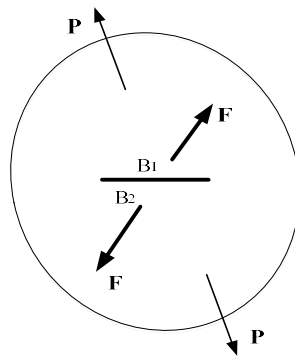


Fig. 17. The COD relative displacement of Castigliano's theorem

The relationship between the crack propagation force G_c and U can be obtained by:

$$G_c = \left(\frac{\partial U}{\partial A} \right)_p \quad (30)$$

and integrating Eq. **Error! Reference source not found.**

$$U = U_0 + \int_0^A G_c dA \quad (31)$$

$$G_c = G_I + G_{II} = \frac{K_c^2}{E} = \frac{K_I^2}{E} + \frac{K_{II}^2}{E} \quad (32)$$

with

$$K_I = K_{IP} + K_{IF} \text{ and } K_{II} = K_{IIP} + K_{IIF} \quad (33)$$

where U_0 is the strain energy of the elastic body with crack length $2a = 0$, and U is the strain energy of the elastomer with crack length $2a \neq 0$.

Substituting Eqs. **Error! Reference source not found.**, **Error! Reference source not found.**, and **Error! Reference source not found.** into Eq. **Error! Reference source not found.** leads to

$$\begin{aligned} \delta &= \lim_{F \rightarrow 0} \left[\frac{\partial U_0}{\partial F} + \frac{\partial}{\partial F} \int_0^A G_c dA \right] \\ &= \lim_{F \rightarrow 0} \left[\frac{\partial U_0}{\partial F} + \frac{\partial}{\partial F} \int_0^A \left(\frac{K_I^2}{E} + \frac{K_{II}^2}{E} \right) dA \right] \\ &= \lim_{F \rightarrow 0} \left[\frac{\partial U_0}{\partial F} + \frac{1}{E} \cdot \frac{\partial}{\partial F} \int_0^A (K_I^2 + K_{II}^2) dA \right] \\ &= \lim_{F \rightarrow 0} \left[\frac{\partial U_0}{\partial F} + \frac{1}{E} \cdot \frac{\partial}{\partial F} \left(\int_0^A (K_{IP} + K_{IF})^2 dA + \int_0^A (K_{IIP} + K_{IIF})^2 dA \right) \right] \\ &= \lim_{F \rightarrow 0} \left\{ \frac{\partial U_0}{\partial F} + \frac{2}{E} \cdot \left(\int_0^A (K_{IP} + K_{IF}) \frac{\partial K_{IF}}{\partial F} dA + \int_0^A (K_{IIP} + K_{IIF}) \frac{\partial K_{IIF}}{\partial F} dA \right) \right\} \end{aligned} \quad (34)$$

Since K_{IF} is proportional to F , when $F \rightarrow 0$, Eq. **Error! Reference source not found.** can be rewritten as:



$$\delta = \left(\frac{\partial U_0}{\partial F} \right)_{F=0} + \frac{2}{E} \int_0^A K_{IP} \frac{\partial K_{IF}}{\partial F} dA + \frac{2}{E} \int_0^A K_{IIP} \frac{\partial K_{IIF}}{\partial F} dA \quad (35)$$

where K_{IP}, K_{IIP} and K_{IF}, K_{IIF} are the stress intensity factors under force P and the virtual equilibrium force F , respectively.

From the relationship between the critical energy release rate G_c and the strain energy U , the crack opening displacement δ_i is obtained as:

$$\delta = \frac{2}{E} \int_0^A K_{IP} \frac{\partial K_{IF}}{\partial F} dA + \frac{2}{E} \int_0^A K_{IIP} \frac{\partial K_{IIF}}{\partial F} dA \quad (36)$$

where $K_{IP} = \sigma \sqrt{\pi \xi}$, $K_{IF} = F / \sqrt{\pi \xi}$ and $K_{IIP} = \tau \sqrt{\pi \xi}$, $K_{IIF} = F / \sqrt{\pi \xi}$ are the stress intensity factors of the force and the virtual equilibrium force at the crack tip, and ξ is the instantaneous crack length.

Then, when the crack length $A = 2a$, the opening displacement of the crack tip can be expressed as:

$$\begin{aligned} \delta &= \frac{2}{E} \int_0^A \sigma \sqrt{\pi \xi} \frac{1}{\sqrt{\pi \xi}} dA + \frac{2}{E} \int_0^A \tau \sqrt{\pi \xi} \frac{1}{\sqrt{\pi \xi}} dA \\ &= 2 \frac{\sigma}{E} A + 2 \frac{\tau}{E} A \\ &= \frac{4}{E} a (\tau + \sigma) \end{aligned} \quad (37)$$

By combining PD with the COD method as shown in Fig. 18, the opening displacement between adjacent material points can be expressed as:

$$\Delta_{cr} = |\mathbf{y}_{(j)} - \mathbf{y}_{(k)}| - |\mathbf{x}_{(j)} - \mathbf{x}_{(k)}| = \delta_{cr} \quad (38)$$

where Δ_{cr} is the critical opening displacement of PD adjacent material points, and δ_{cr} is the critical crack opening displacement of the fracture mechanics.

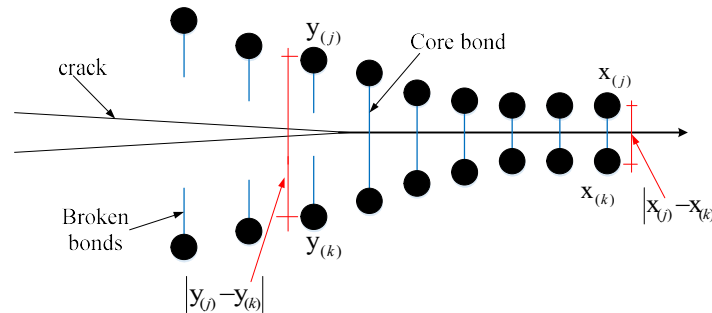


Fig. 18. Opened displacement of PD material points

In order to include damage initiation in the material response, a history-dependent scalar-valued functional ϕ_Δ can be introduced

$$\phi_\Delta(\mathbf{x}_{(j)} - \mathbf{x}_{(k)}, t) = \begin{cases} 1 & \text{if } \Delta < \Delta_{cr} \\ 0 & \text{else} \end{cases} \quad (39)$$

and the force density vector $\mathbf{t}_{(k)(j)}$ can be modified as:

$$\mathbf{t}_{(k)(j)} = 2\delta \left\{ ad \frac{\Lambda_{(k)(j)}}{|\mathbf{x}_{(j)} - \mathbf{x}_{(k)}|} \theta_{(k)} + b \cdot \Delta(\mathbf{x}_{(j)} - \mathbf{x}_{(k)}) s_{(k)(j)} \right\} \frac{\mathbf{y}_{(j)} - \mathbf{y}_{(k)}}{|\mathbf{y}_{(j)} - \mathbf{y}_{(k)}|} \quad (40)$$

where $\theta_{(k)}$ is the dilatation term

$$\theta_{(k)} = d\delta \sum_i^N \Lambda_{(k)(i)} \phi_\Delta(\mathbf{x}_{(j)} - \mathbf{x}_{(k)}, t) s_{(k)(i)} V_i \quad (41)$$

When the object under the external load with the time changes, we continue to calculate the open displacement of the crack tip. When the displacement between the two particles satisfies $\Delta \geq \Delta_{cr}$, the PD force $\mathbf{t}_{(k)(j)}$ is set to zero, and the crack propagates (based on the history value $\phi_\Delta = 0$). When the crack tip opened displacement $\Delta < \Delta_{cr}$, the force $\mathbf{t}_{(k)(j)}$ is unequal to zero and $\phi_\Delta = 1$.

When the crack does not propagate, the local damage function $\psi(\mathbf{x}, t)$ is introduced into the PD-COD model in order to

express the relationship between the crack opening displacement and damage of the particles

$$\psi(\mathbf{x}, t) = 1 - \frac{\int_H \phi_{\Delta}(\mathbf{x}_{(j)} - \mathbf{x}_{(k)}, t) dV}{\int_H dV} \quad (42)$$

The local damage of the PD-COD model ranges from 0 to 1 as shown in Fig. 19. When the crack opening displacement satisfies $\Delta \geq \Delta_{cr}$, the local damage is one, and all the interactions initially with the point have been eliminated. A local damage value of $\psi(\mathbf{x}, t) = 0$ means that all interactions are intact. However, the creation of a crack terminates half of the interactions with its horizon, resulting in a local damage value of one-half as shown in Fig. 19 (b).

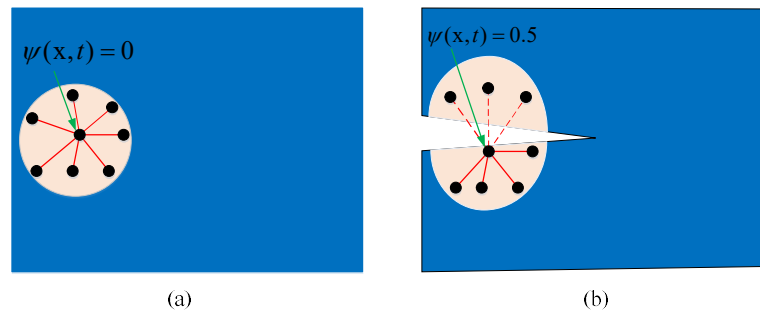


Fig. 19. Local damage value

4.2 Numerical example

In this paper, four numerical examples, as illustrated in Figures 3 and 20 to 22, are studied in order to demonstrate the performance of the new fracture criterion.

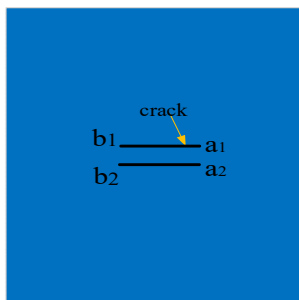


Fig. 20. Center double parallel crack

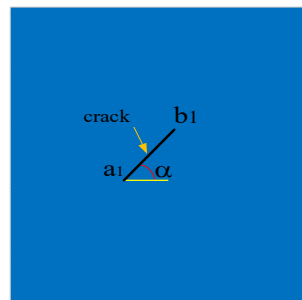


Fig. 21. Center monoclinic crack

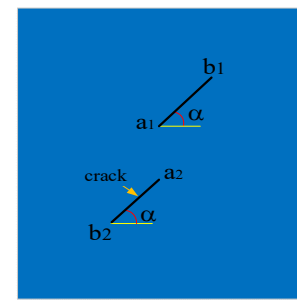


Fig. 22. Center double oblique

Example 1: Double-notched specimen made of Q345 steel under uniaxial tension

Q345 material is a low alloy high strength structural steel, with elasticity modulus $E = 203\text{GPa}$, poisson's ratio $\nu = 0.3$, elongation $\delta = 27.96\%$, and density $\rho = 7850\text{kg/m}^3$. The length and width of the specimen are 70mm and 40mm, respectively, as shown in Fig. 8. The specimen is loaded under uniaxial tension with a constant loading rate of $2.217 \times 10^{-5}\text{m/s}$ ensuring quasi-static conditions. The crack length is 10 mm, and three different crack size distances in loading directions are tested according to Table 1. The fractured specimens of these experiments are illustrated in Fig. 23 and compared with the fractured specimen from FEM simulations in Fig. 24. Furthermore, Fig. 14 shows the results obtained by the C3D8R XFEM element in ABAQUS. The calculated results by XFEM agree fairly well with the experimental results in Fig. 23 and FEM in Fig. 24 though both numerical simulations are not able to capture the curvature of the crack in the second specimen.

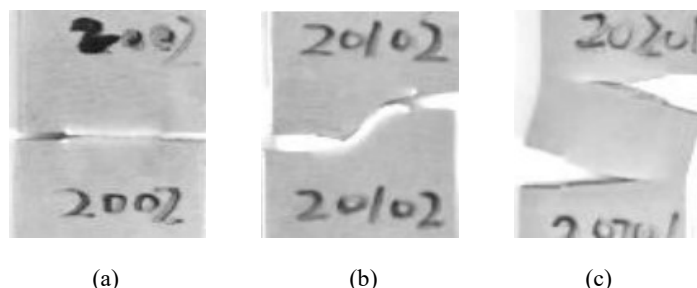


Fig. 23. Experimental fracture results [93]. (a) 10-00 test (b) 10-10 test (c) 10-20 test

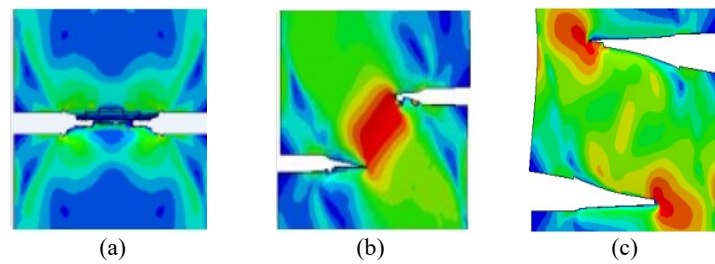


Fig. 24. FEM simulation fracture results [93]. (a) 10-00 test (b) 10-10 test (c) 10-20 test

Fig. 23 is made by experiment, and Fig. 24 is made by FEM. We used the maximum strain energy to model the crack propagation in FEM. When the strain energy of crack tip reached the critical strain energy, we scattered the units at crack tip. So, the crack will propagate forward for some distance.

As shown in Fig. 23 and 24, if the longitudinal initial crack distance is zero, the two propagating cracks join in a horizontal line. For a longitudinal crack distance of 10 mm, the crack initially propagates horizontally. When the horizontal and longitudinal distances of the two crack tips are equal, the cracks extend along a 45° direction into a slanted crack until they join. No crack coalescence is observed for the specimen with a 20 mm longitudinal crack distance.

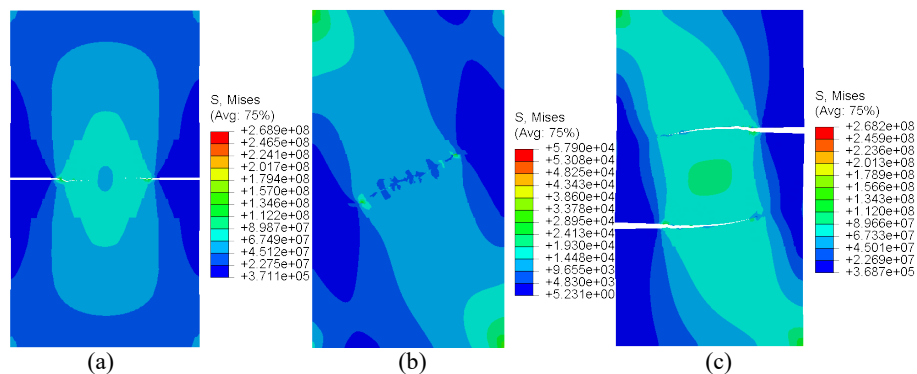


Fig. 25. XFEM simulation fracture results. (a) 10-00 test (b) 10-10 test (c) 10-20 test

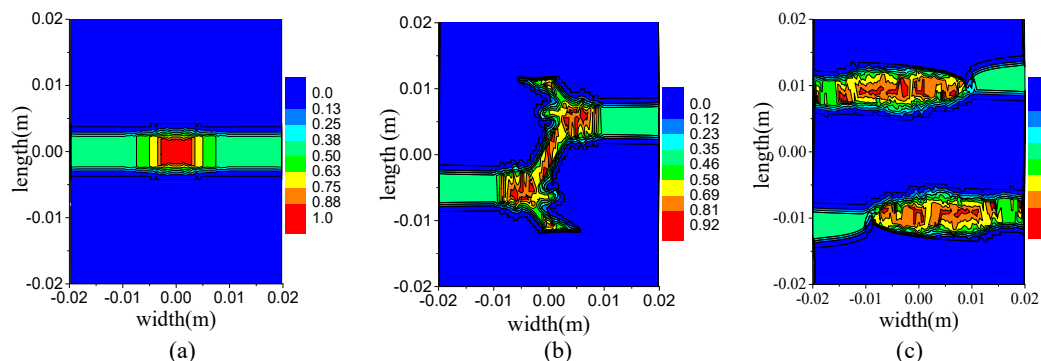


Fig. 26. Crack propagation results of s_{ij} . (a) 10-00 test (b) 10-10 test (c) 10-20 test

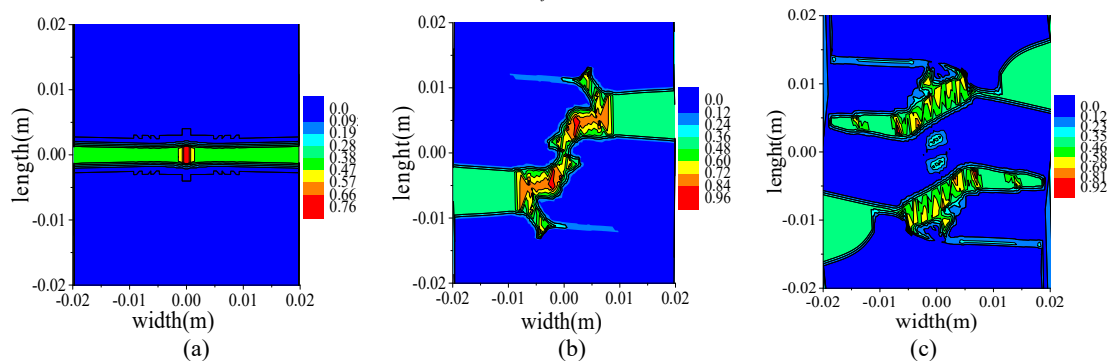


Fig. 27. Crack propagation results of PD-COD. (a) 10-00 test (b) 10-10 test (c) 10-20 test

The crack propagation results are shown in Fig. 26 using the PD elongation s_{ij} crack fracture criterion. The results of the three different specimens are consistent with the results of the experiment, FEM, and XFEM. It indicates that the PD elongation criterion can simulate the multi-crack propagation and fusion.

As shown in Fig. 27, the results of the crack propagation of the three specimens simulated by PD-COD fracture criterion are consistent with the results of FEM, XFEM, and PD elongation. The variety of simulation results is very similar. Therefore, we

can infer that the PD-COD fracture criterion can simulate the crack propagation and fusion.

Although both of the fracture criteria can simulate the fracture process, there are nuances between the two methods of simulating the multi-crack propagation and interaction process. It can be seen from Fig. 24 (c) and 25 (c) that the two cracks have a longitudinal distance of 20mm. Although the cracks propagation path is along the horizontal straight line, the two cracks have mutual influence during the expansion process.

The stress distribution in Fig. 24 (c) and 25 (c) is consistent. The stress in the middle of the two cracks is obviously higher than other parts. There is local damage in the middle test, but the damage value is small and no cracks are formed.

It can be seen that the two cracks propagation does not affect the stress state at the middle of the model in Fig. 26 (c). However, in Fig. 7 (c), the crack propagation is able to show not only the crack propagation path, but also the damage distribution in the middle part of the model. The most important result is that the damage distribution is consistent with the results of FEM and XFEM.

Example 2: Double center crack problem

We consider a specimen with two double-center cracks under uniaxial tension with a constant loading rate of 20m/s. The model is shown in Fig. 21. The material parameters are adopted from the previous example. The thickness of the plate is 0.05m, and the initial lengths are 0.01m for both cracks. The longitudinal distance of crack is 4mm (case 1), 5mm (case 2), and 10mm (case 3). A tensile load is applied at both ends of the plate in vertical direction.

The particle distance in all PD simulations is 0.5mm. The influence of the longitudinal crack distance on damage rate and crack propagation path is analyzed subsequently.

Fig. 8 and 9 present the fracture patterns for the PD extension s_{ij} crack propagation criterion and the PD-COD crack propagation criterion. By comparing Fig. 8 with Fig. 29 it is revealed that both methods yield similar results. An increasing distance of the crack leads to a higher curvature of the crack path.

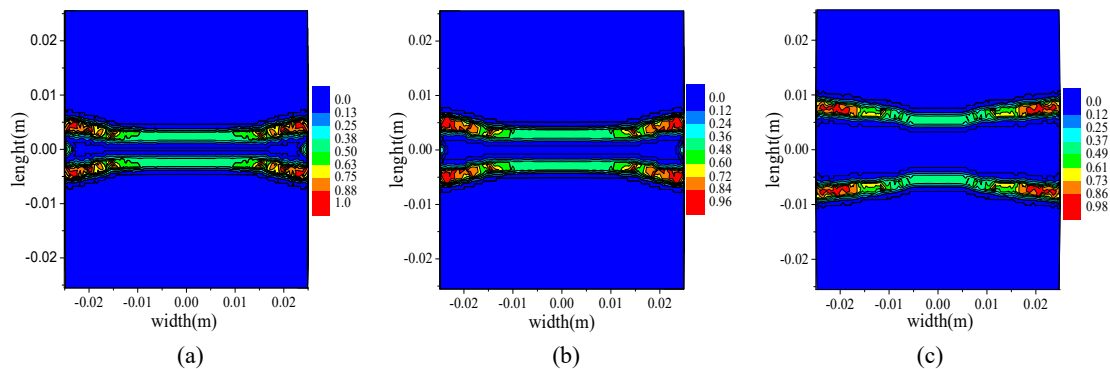


Fig. 28. Crack propagation results of s_{ij} . (a) d4 test (b) d5 test (c) d10 test

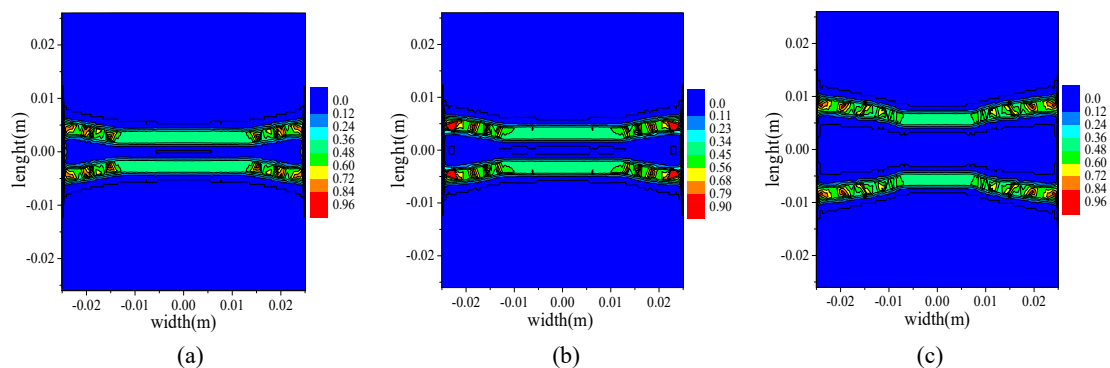


Fig. 29. Crack propagation results of PD-COD. (a) d4 test (b) d5 test (c) d10 test

Example 3: Slanted center crack problem

The third example is a specimen with a slanted center crack. Three different specimens with different crack angle versus the horizontal axis are tested. All specimens are loaded under uniaxial tension with a constant loading rate of 20m/s. The model is illustrated in Fig. 21. The material parameters are adopted from the first example. The thickness of the plate is 0.05m, and the length of one crack is 0.01m. The inclination angle α of the slanted crack is 30° (case 1), 45° (case 2), and 60° (case 3). A tensile load is applied at both ends of the plate in a vertical direction. The particle distance in all PD simulations is 0.5mm. The influence of the longitudinal crack distance on the damage rate and crack propagation path was analyzed by the PD elongation s_{ij} criterion and the new COD criteria.

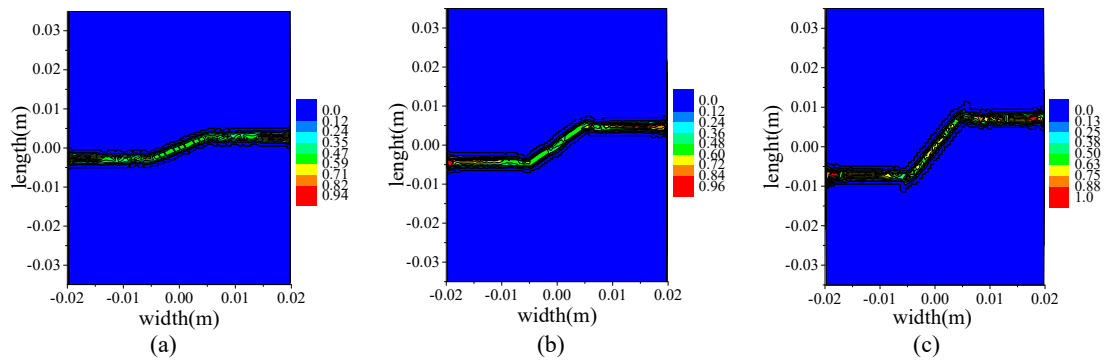


Fig. 30. Crack propagation results of s_{ij} . (a) 30° test (b) 45° test (c) 60° test

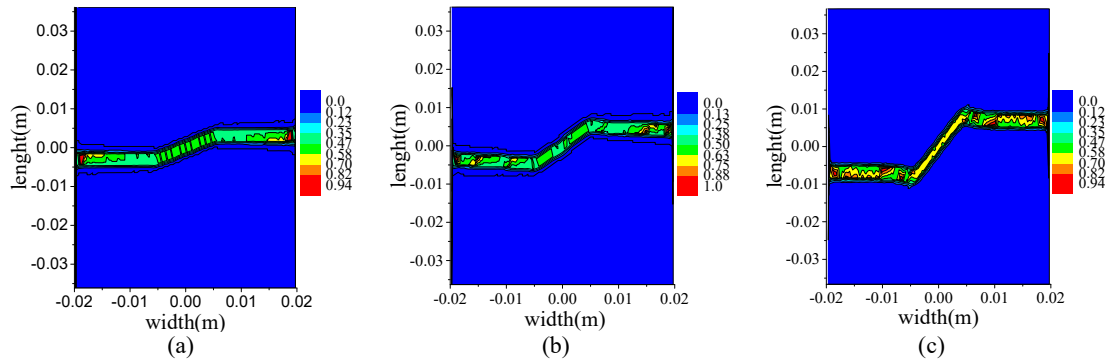


Fig. 31. Crack propagation results of PD-COD. (a) 30° test (b) 45° test (c) 60° test

Fig. 30 and 31 show the final fracture pattern from the PD simulations based on the extension s_{ij} crack propagation criterion and the COD criterion. As expected, the final crack path is perpendicular to the loading direction in this mode I dominated fracture problem. The results of both cracking criteria agree well.

Example 4: Specimen with two slanted initial cracks

The fourth example is a specimen with two slanted initial cracks. The specimen is again loaded under uniaxial tension with a constant loading rate of 20m/s, and the material parameters are the same as in all other examples. The model is presented in Fig. 22. The thickness of the plate is 0.05 m, and the length of one crack is 0.01m. The inclination angle α of both slanted cracks are 30° (case 1), 45° (case 2), and 60° (case 3). The particle distance in all PD simulations is 0.5mm. The influence of the longitudinal crack distance on damage rate and crack propagation path was analyzed by PD elongation s_{ij} criterion and PD-COD criterion and compared with two kinds of crack propagation paths.

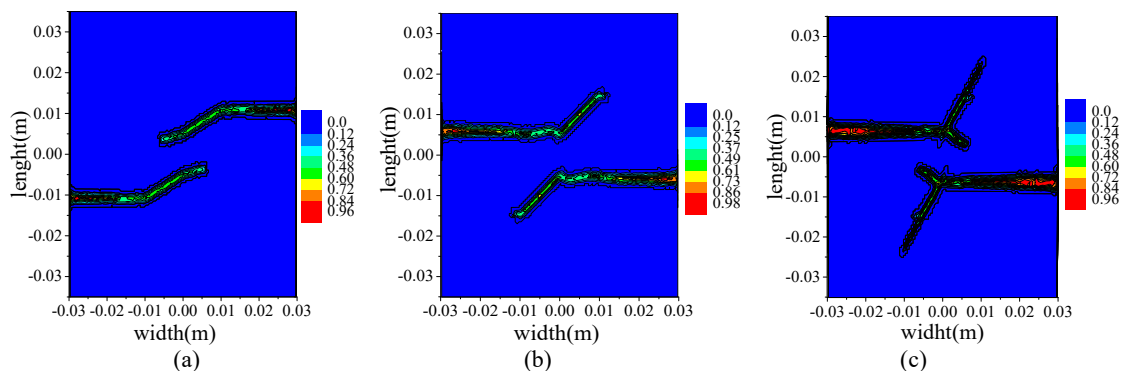


Fig. 32. Crack propagation results of s_{ij} . (a) 30° test (b) 45° test (c) 60° test

Fig. 32 and 33 illustrate the crack propagation path for the three cases considered here. The results from the PD simulations based on the extension s_{ij} crack propagation criterion and our novel criterion agree well. In all cases, crack shielding occurs, and the crack propagates towards the boundary of the plates. The crack shielding is more pronounced with increasing inclination angle for our new cracking criterion. Such a tendency is not apparent for the standard cracking criterion in PD. For case 3, the standard cracking criterion provides a short slightly inclined crack path which does not seem reasonable due to the mode I dominated fracture mode. Such an artifact does not occur for our new fracture criterion.

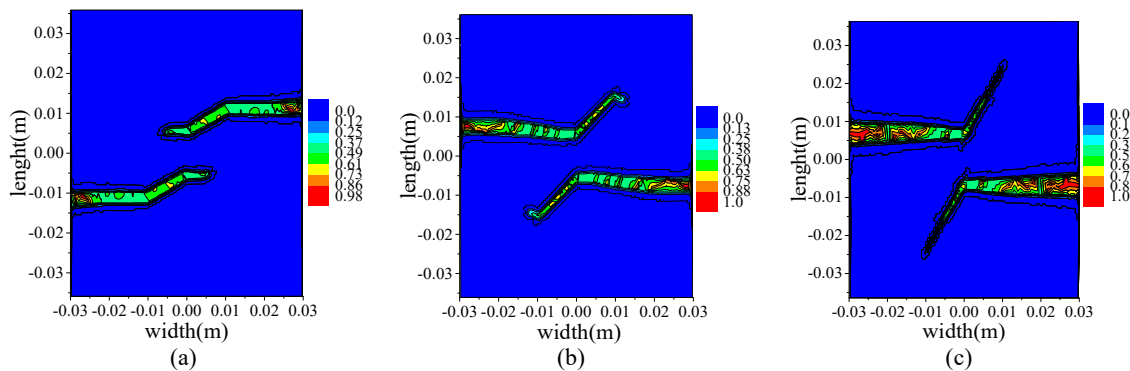


Fig. 33. Crack propagation results of PD-COD. (a) 30° test (b) 45° test (c) 60° test

5. Conclusions

There are two parts in this paper. In the first part, the influence of bilateral crack and central longitudinal crack on the crack propagation path is simulated by PD theory. At the same time, the experiment and XFEM are used to validate the simulation of the bilateral crack. The accuracy of the crack propagation law of PD theory and the advantage of the XFEM method are obtained. The PD results are consistent with the XFEM results demonstrating that PD is a good competitor to XFEM. For the center double crack distance, the bigger the distance is, the earlier the crack initiation time and the faster the initial expansion rate would be. As the crack propagates, this phenomenon is progressively weakened. The results show that the PD theory can accurately simulate the relationship between crack propagation and crack propagation with time.

In the second part, a new PD crack propagation criterion based on the COD method is proposed. The key idea is when the opened displacement of PD adjacent material points equal to the critical crack opened displacement at crack tip (the critical COD), the bonds between particles are broken, and a crack is formed. A local damage formulation $\psi(x, t)$ is also introduced in analogy to the original PD damage formulation to simulate the local damage. Four examples are studied to verify the correctness of the novel cracking criterion. The first example is compared with experimental data and the results of other numerical methods and shows excellent agreement. The following three examples are the double parallel crack, the monoclinic crack, and the double oblique crack. The four examples indicate that the crack propagation path will be affected by both the crack longitudinal distance and the crack oblique angle. The results of comparative analysis with the standard criterion show that PD-COD can accurately simulate the crack propagation more intuitive, and can be easily understood and more suitable to be used in engineering.

Conflict of Interest

The author(s) declared no potential conflicts of interest with respect to the research, authorship and publication of this article.

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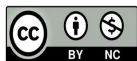


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