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Research Paper

Some New Existence, Uniqueness and Convergence Results for Fractional Volterra-Fredholm Integro-Differential Equations

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Abstract. This paper demonstrates a study on some significant latest innovations in the approximated techniques to find the approximate solutions of Caputo fractional Volterra-Fredholm integro-differential equations. To this aim, the study uses the modified Adomian decomposition method (MADM) and the modified variational iteration method (MVIM). A wider applicability of these techniques are based on their reliability and reduction in the size of the computational work. This study provides an analytical approximate to determine the behavior of the solution. It proves the existence and uniqueness results and convergence of the solution. In addition, it brings an example to examine the validity and applicability of the proposed techniques.

Keywords: Modified Adomian Decomposition Method; Modified Variational Iteration Method; Caputo Fractional Volterra-Fredholm Integro-Differential Equation.

1. Introduction

In recent years, the development of analytical and numerical methods for fractional integro-differential equations has been of great focus in numerous papers.

In this paper, Caputo fractional Volterra-Fredholm integro-differential equation is used as follows:

$$ {}^{c}D^{\alpha}u(x) = a(x)u(x) + g(x) + \int_0^x Z_1(x,r)F_1(u(r)dr + \int_0^1 Z_2(x, r)F_2(u(r))dr \tag{1}$$

with the initial condition

$$u(0) = u_0, \tag{2}$$

where ${}^{c}D^{\alpha}$ is the Caputo's fractional derivative, $0<\alpha\le 1$, and $u: J \to \mathbb{R}$. Where $J=[0,1]$ is the continuous function which has to be determined, $g: J\to\mathbb{R}$ and $Z_i: J\times J\to\mathbb{R}, i=1,2$ are continuous functions. $F_i:\mathbb{R}\to\mathbb{R}, i=1,2$ are Lipschitz continuous functions. Abel [1] was the first who gave an application of fractional derivatives and applied the fractional calculus to solve the integral equation that arises in the formulation of the Tautochrone problem. Many mathematicians and physicists paid much attention and interest to the fractional integro-differential equations. This provides efficiency for the description of such practical dynamical arising in engineering and scientific disciplines as physics, biology, electrochemistry, chemistry, economy, electromagnetic, control theory, and viscoelasticity [4, 8, 9, 10, 17, 19, 20, 21, 22, 23, 24, 25, 26, 27]. Recently, many scholars have concentrated on the development of numerical and analytical techniques for fractional integro-differential equations. For instance, Al-Samadi and Gumah [5] used the homotopy analysis technique for fractional SEIR epidemic model, Zurigat et al. [18] applied the same technique for system of fractional integro-differential equations. Yang and Hou [15] applied the Laplace decomposition method to find approximate solution for the fractional integro-differential equations. Mittal and Nigam [13] used it to find approximate solutions for fractional integro-differential equations, applied the ADM. Ma and Huang [12] applied hybrid collocation method to examine integro-differential equations of fractional order. Furthermore, several authors like [5, 16, 18] studied the properties of fractional integro-differential equations.

The present paper mainly aims to study the behavior of the solution that can be formally determined by analytical approximated techniques: the MADM and MVIM. The objective is to prove the existence, uniqueness results, and convergence of the solution of the Caputo fractional Volterra-Fredholm integro-differential equation.

2. Modified Adomian Decomposition Method

Consider the Eq. (1) with the initial condition (2). By operating with J^{α} on both sides of the Eq. (1), it yields

$$u(x)=u_0+J^{\alpha}\left(a(x)u(x)+g(x)+\int_0^x Z_1(x,r)F_1(u(r))dr+\int_0^1 Z_2(x,r)F_2(u(r))dr\right) \tag{3}$$

Adomian's method defines the solution $u(x)$ by the series

$$u=\sum_{n=0}^{\infty}u_n, \tag{4}$$

and the nonlinear function F_i is decomposed as

$$F_1=\sum_{n=0}^{\infty}A_n, \qquad F_2=\sum_{n=0}^{\infty}B_n \tag{5}$$

here A_n, B_n are the Adomian polynomials given by

$$A_n=\frac{1}{n!}\left[\frac{d^n}{d\phi^n}\left(F_1\sum_{i=0}^{n}\phi^i u_i\right)\right]_{\phi=0} \tag{6-a}$$

$$B_n=\frac{1}{n!}\left[\frac{d^n}{d\phi^n}\left(F_2\sum_{i=0}^{n}\phi^i u_i\right)\right]_{\phi=0}. \tag{6-b}$$

The Adomian polynomials were introduced in [2, 3, 6] as

$$\begin{aligned}A_0&=F_1(u_0),\\ A_1&=u_1F_1'(u_0),\\ A_2&=u_2F_1'(u_0)+\frac{1}{2}u_1^2F_1''(u_0),\\ A_3&=u_3F_1'(u_0)+u_1u_2F_1''(u_0)+\frac{1}{3}u_1^3F_1'''(u_0),\end{aligned} \tag{7-a}$$

and

$$\begin{aligned}B_0&=F_2(u_0),\\ B_1&=u_1F_2'(u_0),\\ B_2&=u_2F_2'(u_0)+\frac{1}{2}u_1^2F_2''(u_0),\\ B_3&=u_3F_2'(u_0)+u_1u_2F_2''(u_0)+\frac{1}{3}u_1^3F_2'''(u_0),\end{aligned} \tag{7-b}$$

The components $u_0,u_1,u_2,\ldots$ are determined recursively by

$$u_0(x)=u_0+J^{\alpha}g(x), \tag{8-a}$$

$$u_{k+1}(x)=J^{\alpha}\left(a(x)u_k(x)\right)+J^{\alpha}\left(\int_0^x Z_1(x,r)A_k dr+\int_0^1 Z_2(x,r)B_k dr\right). \tag{8-b}$$

Having defined the components $u_0, u_1, u_2, \ldots,$ the solution u in a series form defined by (4) follows immediately. It is important to note that the ADM suggests that the 0^{th} component u_0 be defined by the initial conditions and the function $g(x)$ as described above. The other components namely $u_1, u_2, \ldots,$ are derived recurrently. The MADM was introduced by Wazwaz [14]. This method is based on the assumption that the function $J^{\alpha} g(x) = R(x)$ can be divided into two parts, namely $R_1(x)$ and $R_2(x)$, under this assumption we set

$$R(x) = R_1(x) + R_2(x). \tag{9}$$

This decomposition is applied when the function onsists of several parts and can be decomposed into two different parts [2, 3, 11, 14]. In this case, $R(x)$ is usuall $R(x)$ by a summation of a polynomial and trigonometric or transcendental functions. A proper choice for the part R_1 is important. For the method to be more efficient, R_1 as one term of $R(x)$ for at least a number of terms if possible and consisted of the remaining terms of $R(x)$ are selected. Consequently, the following modified recursive relation is developed as:

$$\begin{aligned} u_0(x) &= u_0 + R_1(x), \\ u_1(x) &= R_2(x) + J^{\alpha}\left(a(x)u_0(x)\right) + J^{\alpha}\left(\int_0^x Z_1(x,r)A_0 dr + \int_0^1 Z_2(x,r)B_0 dr\right) \\ u_{k+1}(x) &= J^{\alpha}\left(a(x)u_k(x)\right) + J^{\alpha}\left(\int_0^x Z_1(x,r)A_k dr + \int_0^1 Z_2(x,r)B_k dr\right), k \ge 1. \end{aligned} \tag{10}$$

3. Modified Variational Iteration Method

The following fractional equation is considered:

$$^{c}D^{\alpha} u + Mu + Nu = g(x), \tag{11}$$

where $^{c}D^{\alpha}$ is the Caputo fractional order derivative, is a linear differential operator, N represents the nonlinear terms, and g is the source term. The basic character of He's method is the construction of a correction functional for Eq. (11), which reads

$$u_{n+1}(x) = u_n(x) + \int_0^x [\lambda(s)\, {}^{c}D^{\alpha} u_n(s) + M\tilde{u}_n(s) + N\tilde{u}_n(s) - g(s)] ds, \tag{12}$$

where λ is a Lagrange multiplier which can be identified optimally via variational theory [9], u_n is the n^{th} approximate solution, and $\tilde{u}_n$ denotes a restricted variation, i.e., $\delta\tilde{u}_n = 0$. To solve Eq. (9) by VIM, the Lagrange multiplier λ is determined and identified optimally via integration by parts. Then, the successive approximations $u_n(x), n \ge 0,$ of the solution $u(x)$ is readily obtained by using the obtained Lagrange multiplier and any selective function u_0. The approximation u_0 may be selected by any function that just satisfies at least the initial and boundary conditions. With determined λ several approximations $u_n(x),\ n \ge 0,$ follow immediately. The following variational iteration formulais obtained:

$$\begin{aligned} &u_0(x) \quad \text{is an arbitrary initial guess,} \\ &u_{n+1}(x) = u_n(x) + \int_0^x \lambda(s)[{}^{c}D^{\alpha} u_n(s) + Mu_n(s) + Nu_n(s) - g(s)] ds. \end{aligned} \tag{13}$$

By applying the operator J^{α} to both sides of Eq. (11) and using the given conditions, we obtain

$$u = R - J^{\alpha}[Mu] - J^{\alpha}[Nu], \tag{14}$$

where the function R represents the terms arising from integrating the source term g and from using the given conditions, all are assumed to be prescribed. The following variational iteration formula in obtained for Eq. (14):

$$\begin{aligned} &u_0(x) \quad \text{is an arbitrary initial guess,} \\ &u_{n+1}(x) = R(x) - J^{\alpha}\left[Mu_n(x)\right] - J^{\alpha}\left[Nu_n(x)\right]. \end{aligned} \tag{15}$$

In this paper, MVIM is used that was introduced by Ghorbani et al. [7] and can be settled based on the supposition that the function $R(x)$, the iterative relation (15) can be split into two parts, namely $R_1(x)$ and $R_2(x)$. Under this an assumption, we set

$$R(x) = R_1(x) + R_2(x). \tag{16}$$

Now, the following modified variational iteration formula is structured:

$$\begin{aligned} &u_0(x) = R_1(x), \\ &u_1(x) = R(x) - J^{\alpha}\left[Mu_0(x)\right] - J^{\alpha}\left[Nu_0(x)\right], \\ &. \\ &u_{n+1}(x) = R(x) - J^{\alpha}\left[Mu_n(x)\right] - J^{\alpha}\left[Nu_n(x)\right]. \end{aligned} \tag{17}$$

Consequently, the approximate solution is given by

$$\lim_{n\to\infty} u_n(x) = u(x). \tag{18}$$

4. Main Results

This section gives existence and uniqueness results of Eq. (1) with the initial condition (2) and proves it. Before starting and proving the main results, the following hypotheses is introduced:

(A1) There exist two constants $L_{F_1}, L_{F_2} > 0$ such that, for any $u_1, u_2 \in C(J,\mathbb{R})$

$$\left|F_1(u_1(x)) - F_1(u_2(x))\right| \le L_{F_1}\left|u_1 - u_2\right| \tag{19-a}$$

and

$$\left|F_2(u_1(x)) - F_2(u_2(x))\right| \le L_{F_2}\left|u_1 - u_2\right| \tag{19-b}$$

(A2) There exist two functions $Z_1^*, Z_2^* \in C(D,\mathbb{R}^+)$, the set of all positive functions are continuous on $D = \{(x,r) \in \mathbb{R}\times\mathbb{R} : 0 \le r \le x \le 1\}$ uch that

$$\begin{aligned} Z_1^* &= \sup_{x\in[0,1]} \int_0^x \left|Z_1(x\ \ r)\right| dr < \infty, \\ Z_2^* &= \sup_{x\in[0,1]} \int_0^1 \left|Z_2(x,r)\right| dr < \infty. \end{aligned} \tag{20}$$

(A3) The two functions $a, g : J \to \mathbb{R}$ are continuous.

Lemma 4.1 *If $u_0(x) \in C(J,\mathbb{R})$, then $u(x) \in C(J,\mathbb{R}^+)$ is a solution of the problem $(1)-(2)$ if u satisfies*

$$\begin{aligned} u(x) = u_0 &+ \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1} a(s) u(s) ds + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1} g(s) ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left(\int_0^s Z_1(s,\tau)F_1(u(\tau))d\tau + \int_0^1 Z_2(s,\tau)F_2(u(\tau))d\tau\right)ds, \end{aligned} \tag{21}$$

or $x \in J$. Our first result is based on the Banach contraction principle.

Theorem 4.2 *Assume that (A1), (A2) and (A3) hold. If*

$$\left(\frac{\|a\|_\infty + Z_1^* L_{F_1} + Z_2^* L_{F_2}}{\Gamma(\alpha+1)}\right) < 1. \tag{22}$$

then, there exists a unique solution $u(x) \in C(J)$ to $(1)-(2)$.

Proof. By Lemma 4.1. It is obvious that a function u is a solution to $(1)-(2)$ if u satisfies

$$\begin{aligned} u(x) = u_0 &+ \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1} a(s) u(s) ds + \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1} g(s) ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left(\int_0^s Z_1(s,\tau)F_1(u(\tau))d\tau + \int_0^1 Z_2(s,\tau)F_2(u(\tau))d\tau\right)ds. \end{aligned} \tag{23}$$

Let the operator $T : C(J,\mathbb{R}) \to C(J,\mathbb{R})$ be defined by

$$(Tu)(x)=u_0+\frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}a(s)u(s)ds+\frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}g(s)ds+\frac{1}{\Gamma(\alpha)} \times\int_0^x (x-s)^{\alpha-1}\left(\int_0^s Z_1(s,\tau)F_1(u(\tau))d\tau+\int_0^1 Z_2(s,\tau)F_2(u(\tau))d\tau\right)ds, \tag{24}$$

we can be seen that, if $u\in C(J,\mathbb{R})$ is a fixed point of T, then u is a solution of $(1)-(2)$ It is proved that T as a fixed point u in $C(J,\mathbb{R})$. For that, let $u_1,u_2\in C(J,\mathbb{R})$ and for any $x\in[0,1]$ such that

$$u_1(x)=u_0+\frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}a(s)u_1(s)ds+\frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}g(s)ds+\frac{1}{\Gamma(\alpha)} \times\int_0^x (x-s)^{\alpha-1}\left(\int_0^s Z_1(s,\tau)F_1(u_1(\tau))d\tau+\int_0^1 Z_2(s,\tau)F_2(u_1(\tau))d\tau\right)ds \tag{25}$$

Consequently, we get

$$\begin{aligned}\left|(Tu_1)(x)-(Tu_2)(x)\right| &\le \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}|a(s)||u_1(s)-u_2(s)|ds \\ &+\frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left(\begin{aligned}&\sup_{s,\tau\in J}\int_0^\tau |Z_1(s,\tau)|\sup_{\tau\in J}|F_1(u_1(\tau))-F_1(u_2(\tau))|d\tau\\ &+\sup_{s,\tau\in J}\int_0^1 |Z_2(s,\tau)|\sup_{\tau\in J}|F_2(u_1(\tau))-F_2(u_2(\tau))|d\tau\end{aligned}\right)ds \\ &\le\left(\frac{\|a\|_\infty+Z_1^*L_{F_1}+Z_2^*L_{F_2}}{\Gamma(\alpha+1)}\right)|u_1(s)-u_2(s)|.\end{aligned} \tag{26}$$

From the inequality (17), we have

$$\|Tu_1-Tu_2\|_\infty\le\left(\frac{\|a\|_\infty+Z_1^*L_{F_1}+Z_2^*L_{F_2}}{\Gamma(\alpha+1)}\right)\|u_1-u_2\|_\infty. \tag{27}$$

This means that T is a contraction map. By the Banach contraction principle, it is concluded that T has a unique fixed point u in $C(J,\mathbb{R})$.

The existence result is studied here by means of Schauder's fixed point Theorem.

Theorem 4.3 *Assume that F_1,F_2 are continuous functions and (A2), (A3) hold, if*

$$\frac{\|a\|_\infty}{\Gamma(\alpha+1)}<1. \tag{28}$$

Then, there exists at least a solution $u(x)\in C(J,\mathbb{R})$ to problem $(1)-(2)$.

Proof. Let the operator $T:C(J,\mathbb{R})\to C(J,\mathbb{R})$, be defined as in Theorem 4.2.

Firstly, it is proved that the operator T is completely continuous.

(1) It is shown that T is continuous.

Let u_n be a sequence such that $u_n\to u$ in $C(J,\mathbb{R})$ Then, for each $u_n,u\in C(J,\mathbb{R})$ and for any $x\in J$ we have

$$\begin{aligned}\left|(Tu_n)(x)-(Tu)(x)\right| &\le \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\sup_{s\in J}|a(s)|\sup_{s\in J}|u_n(s)-u(s)|ds \\ &+\frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\left(\begin{aligned}&\sup_{s,\tau\in J}\int_0^\tau |Z_1(s,\tau)|\sup_{\tau\in J}|F_1(u_n(\tau))-F_1(u(\tau))|d\tau\\ &+\sup_{s,\tau\in J}\int_0^1 |Z_2(s,\tau)|\sup_{\tau\in J}|F_2(u_n(\tau))-F_2(u(\tau))|d\tau\end{aligned}\right)ds \\ &\le\|a\|_\infty\|u_n(.)-u(.)\|_\infty\frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}ds\end{aligned} \tag{29}$$

$$+Z_1^* \| F_1(u_n(.)) - F_1(u(.)) \|_\infty \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} ds$$

$$+Z_1^* \| F_2(u_n(.)) - F_2(u(.)) \|_\infty \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} ds$$

Since $\int_0^x (x-s)^{\alpha-1} ds$ is bounded, $\lim_{n\to\infty} u_n(x) = u(x)$ and F_1, F_2 are continuous functions, it is concluded that $\|Tu_n - Tu\|_\infty \to 0$ as $n \to \infty$, therefore, T is continuous on $C(J,\mathbb{R})$.

(2) It is verified that T maps bounded sets into bounded sets in $C(J,\mathbb{R})$.

Indeed, it is just shown that for any $\lambda > 0$, there exists a positive constant r, such that for each $u \in \mathbb{B}_\lambda = \{u \in C(J,\mathbb{R}) : \|u\|_\infty \le \lambda\}$, one has $\|Tu\|_\infty \le r$.

Let $\mu_1 = \sup_{(u)\in J\times[0,\lambda]} F_1(u(x)) + 1$, and $\mu_2 = \sup_{(u)\in J\times[0,\lambda]} F_2(u(x)) + 1$ and for any $u \in \mathbb{B}_r$ and for each $x \in J$, we have

$$\begin{aligned} |(Tu)(x)| &= |u_0| + \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} |a(s)||u(s)| ds + \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} |g(s)| ds + \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} \left(\int_0^s |Z_1(s,\tau)||F_1(u(\tau))| d\tau + \int_0^1 |Z_2(s,\tau)||F_2(u(\tau))| d\tau \right) ds \\ &\le |u_0| + \|u\|_\infty \|a\|_\infty \frac{x^\alpha}{\Gamma(\alpha+1)} + \|g\|_\infty \frac{x^\alpha}{\Gamma(\alpha+1)} + \frac{Z_1^* \mu_1 x^\alpha}{\Gamma(\alpha+1)} + \frac{Z_2^* \mu_2 x^\alpha}{\Gamma(\alpha+1)} \\ &\le \left(|u_0| + \frac{\|a\|_\infty \lambda + \|g\|_\infty + Z_1^* \mu_1 + Z_2^* \mu_2}{\Gamma(\alpha+1)} \right) \end{aligned} \tag{30}$$

Therefore, $Tu \le r$ or every $u \in \mathbb{B}_r$, hich implies that $T\mathbb{B}_r \subset \mathbb{B}_r$.

(3) It is examined that T maps bounded sets into equicontinuous sets of $C(J,\mathbb{R})$.

Let $\mathbb{B}_\lambda$ be defined as in Eq. (2) and for each $u \in \mathbb{B}_\lambda$, $x_1, x_2 \in [0,1]$, with $x_1 < x_2$ be have

$$|(Tu)(x_2) - (Tu)(x_1)| \le \frac{1}{\Gamma(\alpha)} \left| \int_0^{x_2} (x_2-s)^{\alpha-1} a(s) u(s) ds - \int_0^{x_1} (x_1-s)^{\alpha-1} a(s) u(s) ds \right|$$

$$\begin{aligned} &+ \frac{1}{\Gamma(\alpha)} \left| \int_0^{x_2} (x_2-s)^{\alpha-1} g(s) ds - \int_0^{x_1} (x_1-s)^{\alpha-1} g(s) ds \right| + \frac{1}{\Gamma(\alpha)} \left| \begin{aligned} &\int_0^{x_2} (x_2-s)^{\alpha-1} \left(\int_0^s Z_1(s,\tau) F_1(u(\tau)) d\tau + \int_0^1 Z_2(s,\tau) F_2(u(\tau)) d\tau \right) ds \\ &- \int_0^{x_1} (x_1-s)^{\alpha-1} \left(\int_0^s Z_1(s,\tau) F_1(u(\tau)) d\tau + \int_0^1 Z_2(s,\tau) F_2(u(\tau)) d\tau \right) ds \end{aligned} \right| \\ &= \frac{1}{\Gamma(\alpha)} \left| \begin{aligned} &\int_0^{x_2} (x_2-s)^{\alpha-1} a(s) u(s) ds - \int_0^{x_1} (x_2-s)^{\alpha-1} a(s) u(s) ds \\ &+ \int_0^{x_1} (x_2-s)^{\alpha-1} a(s) u(s) ds - \int_0^{x_1} (x_1-s)^{\alpha-1} a(s) u(s) ds \end{aligned} \right| + \frac{1}{\Gamma(\alpha)} \left| \begin{aligned} &\int_0^{x_2} (x_2-s)^{\alpha-1} g(s) ds - \int_0^{x_1} (x_2-s)^{\alpha-1} g(s) ds \\ &+ \int_0^{x_1} (x_2-s)^{\alpha-1} g(s) ds - \int_0^{x_1} (x_1-s)^{\alpha-1} g(s) ds \end{aligned} \right| \\ &+ \frac{1}{\Gamma(\alpha)} \left| \begin{aligned} &\int_0^{x_2} (x_2-s)^{\alpha-1} \left(\int_0^s Z_1(s,\tau) F_1(u(\tau)) d\tau + \int_0^1 Z_2(s,\tau) F_2(u(\tau)) d\tau \right) ds \\ &- \int_0^{x_1} (x_2-s)^{\alpha-1} \left(\int_0^s Z_1(s,\tau) F_1(u(\tau)) d\tau + \int_0^1 Z_2(s,\tau) F_2(u(\tau)) d\tau \right) ds \\ &+ \int_0^{x_1} (x_2-s)^{\alpha-1} \left(\int_0^s Z_1(s,\tau) F_1(u(\tau)) d\tau + \int_0^1 Z_2(s,\tau) F_2(u(\tau)) d\tau \right) ds \\ &- \int_0^{x_1} (x_1-s)^{\alpha-1} \left(\int_0^s Z_1(s,\tau) F_1(u(\tau)) d\tau + \int_0^1 Z_2(s,\tau) F_2(u(\tau)) d\tau \right) ds \end{aligned} \right| \end{aligned} \tag{31}$$

Consequently,

$$\left|(Tu)(x_2)-(Tu)(x_1)\right| \le \frac{1}{\Gamma(\alpha)}\left(\begin{array}{l}\int_{x_1}^{x_2}(x_2-s)^{\alpha-1}\left|a(s)\right|\left|u(s)\right|ds\\ +\int_0^{x_1}(x_1-s)^{\alpha-1}-(x_2-s)^{\alpha-1}\left|a(s)\right|\left|u(s)\right|ds\end{array}\right)$$

$$+\frac{1}{\Gamma(\alpha)}\left(\begin{array}{l}\int_{x_1}^{x_2}(x_2-s)^{\alpha-1}\left|g(s)\right|ds\\ +\int_0^{x_1}(x_1-s)^{\alpha-1}-(x_2-s)^{\alpha-1}\left|g(s)\right|ds\end{array}\right) \tag{32}$$

$$+\frac{1}{\Gamma(\alpha)}\left(\begin{array}{l}\int_{x_1}^{x_2}(x_2-s)^{\alpha-1}\left(\int_0^s\left|Z_1(s,\tau)\right|\left|F_1(u(\tau))\right|d\tau+\int_0^1\left|Z_2(s,\tau)\right|\left|F_2(u(\tau))\right|d\tau\right)ds\\ +\int_0^{x_1}(x_1-s)^{\alpha-1}-(x_2-s)^{\alpha-1}\left(\int_0^s\left|Z_1(s,\tau)\right|\left|F_1(u(\tau))\right|d\tau+\int_0^1\left|Z_2(s,\tau)\right|\left|F_2(u(\tau))\right|d\tau\right)ds\end{array}\right)=I_1+I_2+I_3,$$

where

$$\begin{aligned} I_1&=\frac{1}{\Gamma(\alpha)}\left(\int_{x_1}^{x_2}(x_2-s)^{\alpha-1}\left|a(s)\right|\left|u(s)\right|ds+\int_0^{x_1}(x_1-s)^{\alpha-1}-(x_2-s)^{\alpha-1}\left|a(s)\right|\left|u(s)\right|ds\right)\\ &\le\frac{(x_2-x_1)^\alpha}{\Gamma(\alpha+1)}\|a\|_\infty\lambda+\frac{x_1^\alpha}{\Gamma(\alpha+1)}\|a\|_\infty\lambda+\frac{(x_2-x_1)^\alpha}{\Gamma(\alpha+1)}\|a\|_\infty\lambda-\frac{x_2^\alpha}{\Gamma(\alpha+1)}\|a\|_\infty\lambda\\ &=\frac{\|a\|_\infty\lambda}{\Gamma(\alpha+1)}\left(2(x_2-x_1)^\alpha+\left(x_1^\alpha-x_2^\alpha\right)\right)\le\frac{\|a\|_\infty\lambda}{\Gamma(\alpha+1)}2(x_2-x_1)^\alpha,\end{aligned} \tag{33}$$

$$\begin{aligned} I_2&=\frac{1}{\Gamma(\alpha)}\left(\int_{x_1}^{x_2}(x_2-s)^{\alpha-1}\left|g(s)\right|ds+\int_0^{x_1}(x_1-s)^{\alpha-1}-(x_2-s)^{\alpha-1}\left|g(s)\right|ds\right)\\ &\le\frac{(x_2-x_1)^\alpha}{\Gamma(\alpha+1)}\|g\|_\infty+\frac{x_1^\alpha}{\Gamma(\alpha+1)}\|g\|_\infty+\frac{(x_2-x_1)^\alpha}{\Gamma(\alpha+1)}\|g\|_\infty-\frac{x_2^\alpha}{\Gamma(\alpha+1)}\|g\|_\infty\\ &=\frac{\|g\|_\infty}{\Gamma(\alpha+1)}\left(2(x_2-x_1)^\alpha+\left(x_1^\alpha-x_2^\alpha\right)\right)\le\frac{\|g\|_\infty}{\Gamma(\alpha+1)}2(x_2-x_1)^\alpha,\end{aligned} \tag{34}$$

and

$$\begin{aligned} I_3&=\frac{1}{\Gamma(\alpha)}\left(\begin{array}{l}\int_{x_1}^{x_2}(x_2-s)^{\alpha-1}\left(\int_0^s\left|Z_1(s,\tau)\right|\left|F_1(u(\tau))\right|d\tau+\int_0^1\left|Z_2(s,\tau)\right|\left|F_2(u(\tau))\right|d\tau\right)ds\\ +\int_0^{x_1}(x_1-s)^{\alpha-1}-(x_2-s)^{\alpha-1}\left(\int_0^s\left|Z_1(s,\tau)\right|\left|F_1(u(\tau))\right|d\tau+\int_0^1\left|Z_2(s,\tau)\right|\left|F_2(u(\tau))\right|d\tau\right)ds\end{array}\right)\\ &\le\frac{\left(Z_1^*\mu_1+Z_2^*\mu_2\right)}{\Gamma(\alpha+1)}\left(2(x_2-x_1)^\alpha+(x_1^\alpha-x_2^\alpha)\right)\le\frac{\left(Z_1^*\mu_1+Z_2^*\mu_2\right)}{\Gamma(\alpha+1)}2(x_2-x_1)^\alpha,\end{aligned} \tag{35}$$

It can be concluded the right-hand side of Eqs. (33), (34), and (35) is independent from $u\in\mathbb{B}_\lambda$ and tends to zero as $x_2-x_1\to 0$. This leads to $\left|(Tu)(x_2)-(Tu)(x_1)\right|\to 0$ as $x_2\to x_1$. i.e. the set $\{T\mathbb{B}_\lambda\}$ is equicontinuous.

From I_1 to I_3 together with the Arzela–Ascoli Theorem, it can be concluded that $T:C(J,\mathbb{R})\to C(J,\mathbb{R})$ is completely continuous.

Finally, it is required to investigate that there exists a closed convex bounded subset $\mathbb{B}_{\tilde\lambda}=\{u\in C(J,\mathbb{R}):u_\infty\le\tilde\lambda\}$ such that $T\mathbb{B}_{\tilde\lambda}\subseteq\mathbb{B}_{\tilde\lambda}$. For each positive integer $\tilde\lambda$, then $\mathbb{B}_{\tilde\lambda}$ is clearly closed, convex, and bounded of $C(J,\mathbb{R})$. It is claimed that there exists a positive integer ε such that $T\mathbb{B}_\varepsilon\subseteq\mathbb{B}_\varepsilon$ If this property is false, then for every positive integer $\tilde\lambda$, there exists $u_{\tilde\lambda}\in\mathbb{B}_{\tilde\lambda}$ such that $(Tu_{\tilde\lambda})\notin T\mathbb{B}_{\tilde\lambda}$, i.e. $Tu_{\tilde\lambda}(r)_\infty>\tilde\lambda$ for some $x_{\tilde\lambda}\in J$ where $x_{\tilde\lambda}$ denotes x depending on $\tilde\lambda$. Although by using the previous hypotheses we have

$$\begin{aligned}&|u_0|+\|u\|_\infty\|a\|_\infty\frac{t^\alpha}{\Gamma(\alpha+1)}+\|g\|_\infty\frac{t^\alpha}{\Gamma(\alpha+1)}+\frac{Z_1^*\mu_1x^\alpha}{\Gamma(\alpha+1)}+\frac{Z_2^*\mu_2x^\alpha}{\Gamma(\alpha+1)}\\&\le\left(|u_0|+\frac{\|a\|_\infty\lambda+\|g\|_\infty+Z_1^*\mu_1+Z_2^*\mu_2}{\Gamma(\alpha+1)}\right)\end{aligned}\tag{36}$$

Then,

$$\begin{aligned}\tilde{\lambda}&<\|Tu_{\tilde{\lambda}}\|_\infty\\&=\sup_{x\in J}\left|(Tu_{\tilde{\lambda}})(x)\right|\\&\le\sup_{x\in J}\left\{|u_0|+\|u\|_\infty\|a\|_\infty\frac{x^\alpha}{\Gamma(\alpha+1)}+\|g\|_\infty\frac{x^\alpha}{\Gamma(\alpha+1)}+\frac{Z_1^*\mu_1x^\alpha}{\Gamma(\alpha+1)}+\frac{Z_2^*\mu_2x^\alpha}{\Gamma(\alpha+1)}\right\}\\&\le\sup_{x\in J}\left(|u_0|+\frac{\|a\|_\infty\tilde{\lambda}+\|g\|_\infty+Z_1^*\mu_1+Z_2^*\mu_2}{\Gamma(\alpha+1)}\right)\end{aligned}\tag{37}$$

By dividing both sides by $\tilde{\lambda}$ and taking the limit as $\tilde{\lambda}\to+\infty$, we obtain

$$1<\frac{\|a\|_\infty}{\Gamma(\alpha+1)},\tag{38}$$

which contradicts our assumption (18). Therefore, for some positive integer $\tilde{\lambda}$ we must have $T\mathbb{B}_{\tilde{\lambda}}\subseteq\mathbb{B}_{\tilde{\lambda}}$. An application of Schauder's fixed point Theorem shows that there exists at least a fixed point u of T in $C(J,\mathbb{R})$ Then, u is the solution to $(1)-(2)$ in J, and the proof is completed.

Theorem 4.4 *Suppose that (A1)-(A3), and (22) hold, if the series solution* $u(x)=\sum_{i=0}^{\infty}u_i(x)$ *and* $\|u_1\|_\infty<\infty$ *obtained by the* m*-order deformation is convergent, then, it converges to the exact solution of the fractional Volterra-Fredholm integro-differential equation* $(1)-(2)$.

Proof. Denote as $(C[0,1],\|\|.\|\|)$, the Banach space of all continuous functions on J, with $|u_1(x)|\le\infty$ for all x in J.

First, the sequence of partial sums s_n is defined, letting s_n and s_m be arbitrary partial sums with $n\ge m$. It is proved that $s_n=\sum_{i=0}^{n}u_i(x)$ is a Cauchy sequence in this Banach space:

$$\begin{aligned}\|s_n-s_m\|_\infty&=\max_{\forall x\in J}|s_n-s_m|\\&=\max_{\forall x\in J}\left|\sum_{i=0}^{n}u_i(x)-\sum_{i=0}^{m}u_i(x)\right|\\&=\max_{\forall x\in J}\left|\sum_{i=m+1}^{n}u_i(x)\right|\\&=\max_{\forall x\in J}|\sum_{i=m+1}^{n}(\frac{1}{\Gamma(\alpha)}\int_0^x(x-r)^{\alpha-1}[a(r)u_i(r)+\int_0^tZ_1(t,s)A_i(s)ds+\int_0^1Z_2(t,s)B_i(s)ds]dr)|\\&=\max_{\forall x\in J}|\frac{1}{\Gamma(\alpha)}\int_0^x(x-r)^{\alpha-1}[a(r)\sum_{i=m}^{n-1}u_i(r)+\int_0^tZ_1(t\ s)\sum_{i=m}^{n-1}A_i(s)ds+\int_0^1Z_2(t,s)\sum_{i=m}^{n-1}B_i(s)ds)]dr|.\end{aligned}\tag{39}$$

From Eqs. (4) and (5), we have

$$\begin{aligned}&\sum_{i=m}^{n-1}A_i=F_1(s_{n-1})-F_1(s_{m-1}),\\&\sum_{i=m}^{n-1}B_i=F_2(s_{n-1})-F_2(s_{m-1})\end{aligned}\tag{40}$$

Therefore,

$$\|s_n-s_m\|_\infty=\max_{\forall x\in J}(|\frac{1}{\Gamma(\alpha)}\int_0^x(x-r)^{\alpha-1}[a(r)(u(s_{n-1})-u(s_{m-1}))\tag{41}$$

$$+\int_0^t Z_1(t,s)\big(F_1(s_{n-1})-F_1(s_{m-1})\big)ds+\int_0^1 Z_2(t,s)\big(F_2(s_{n-1})-F_2(s_{m-1})\big)ds]dr\,|),$$

$$\leq \max_{\forall x\in J}\Big(\frac{1}{\Gamma(\alpha)}\int_0^x |\ x-t\,|^{\alpha-1}\,[|a(r)||u(s_{n-1})-u(s_{m-1})|$$

$$+\int_0^t |Z_1(t,s)||\big(F_1(s_{n-1})-F_1(s_{m-1})\big)|ds+\int_0^1 |Z_2(t,s)||\big(F_2(s_{n-1})-F_2(s_{m-1})\big)|ds]dr\Big)$$

$$\leq \frac{1}{\Gamma(\alpha+1)}\Big[\|a\|_\infty\|s_{n-1}-s_{m-1}\|_\infty+Z_1^*L_{F_1}\|s_{n-1}-s_{m-1}\|_\infty+Z_2^*L_{F_2}\|s_{n-1}-s_{m-1}\|_\infty\Big],$$

$$=\left(\frac{\|a\|_\infty+Z_1^*L_{F_1}+Z_2^*L_{F_2}}{\Gamma(\alpha+1)}\right)\|s_{n-1}-s_{m-1}\|_\infty,$$

$$=\delta\|s_{n-1}-s_{m-1}\|_\infty$$

where

$$\delta=\left(\frac{\|a\|_\infty+Z_1^*L_{F_1}+Z_2^*L_{F_2}}{\Gamma(\alpha+1)}\right)\tag{42}$$

Let $n=m+1$ then

$$\|s_n-s_m\|_\infty\leq\delta\|s_m-s_{m-1}\|_\infty\leq\delta^2\|s_{m-1}-s_{m-2}\|_\infty\leq\ldots\leq\delta^m\|s_1-s_0\|_\infty,\tag{43}$$

so,

$$\begin{aligned}\|s_n-s_m\|_\infty&\leq\|s_{m+1}-s_m\|_\infty+\|s_{m+2}-s_{m+1}\|_\infty+\ldots+\|s_n-s_{n-1}\|_\infty\\&\leq\left[\delta^m+\delta^{m+1}+\ldots+\delta^{n-1}\right]\|s_1-s_0\|_\infty\\&\leq\delta^m\left[1+\delta+\delta^2+\ldots+\delta^{n-m-1}\right]\|s_1-s_0\|_\infty\\&\leq\delta^m\left(\frac{1-\delta^{n-m}}{1-\delta}\right)\|u_1\|_\infty.\end{aligned}\tag{44}$$

Since $0<\delta<1$ we have $(1-\delta^{n-m})<1$ then

$$\|s_n-s_m\|_\infty\leq\frac{\delta^m}{1-\delta}\|u_1\|_\infty.\tag{45}$$

But $|u_1(x)|<\infty$, so, as $m\to\infty$ then, $\|s_n-s_m\|_\infty\to 0$. It is concluded that s_n is a Cauchy sequence in $C[0,1]$, therefore, $u=\lim_{n\to\infty}u_n$. Then, the series is convergence and the proof is complete.

5. Illustrative Example

In this section, the analytical technique based on modified variational iteration method is presented to solve Caputo fractional Volterra-Fredholm integro-differential equations.

Example 6.1 Consider the following Caputo fractional Volterra-Fredholm integro-differential equation:

$${}^cD^{0.5}\left[u(x)\right]=\frac{x^{0.5}}{\Gamma(1.5)}-\frac{x^2}{2}-\frac{x^2e^x}{3}u(x)+\int_0^x e^x su(s)ds+\int_0^1 x^2u(s)ds\tag{46}$$

with the initial condition

$$u(0)=0,\tag{47}$$

where the exact solution is $u(x)=x$.

Applying the operator $J^{0.5}$ to both sides of Eq. (46) yields

$$u(x)=0+J^{0.5}\left[\frac{x^{0.5}}{\Gamma(1.5)}-\frac{x^2}{2}\right]+J^{0.5}\left[-\frac{x^2e^x}{3}u(x)\right]+J^{0.5}\left[\int_0^x e^x su(s)ds+\int_0^1 x^2u(s)ds\right]\tag{48}$$

Then,

$$u(x)=J^{0.5}\left[\frac{x^{0.5}}{\Gamma(1.5)}-\frac{x^2}{2}\right]+J^{0.5}\left[-\frac{x^2e^x}{3}u(x)\right]+J^{0.5}\left[\int_0^x e^x su(s)ds+\int_0^1 x^2u(s)ds\right] \tag{49}$$

It can be seen in Eq.(22) that $g(x)=\frac{x^{0.5}}{\Gamma(1.5)}-\frac{x^2}{2}$ Suppose that $R(x)=J^{0.5}g(x)$, the following relations are obtained from Eq. (49):

$$\begin{aligned}R(x)&=J^{0.5}g(x)=J^{0.5}\left[\frac{x^{0.5}}{\Gamma(1.5)}-\frac{x^2}{2}\right],\\&=\frac{1}{\Gamma(1.5)\Gamma(0.5)}\int_0^x(x-s)^{-0.5}s^{0.5}ds-\frac{1}{2\Gamma(0.5)}\int_0^x(x-s)^{-0.5}s^2ds,\\&=\frac{1}{\Gamma(1.5)\Gamma(0.5)}\int_0^x(1-\frac{s}{x})^{-0.5}x^{-0.5}s^{0.5}ds-\frac{1}{2\Gamma(0.5)}\int_0^x(1-\frac{s}{x})^{-0.5}x^{-0.5}s^2ds,\\&=\frac{1}{\Gamma(1.5)\Gamma(0.5)}\int_0^1(1-\tau)^{-0.5}\tau^{0.5}xd\tau-\frac{1}{2\Gamma(0.5)}\int_0^1(1-\tau)^{-0.5}x^{2.5}\tau^2d\tau,\\&=\frac{x}{\Gamma(1.5)\Gamma(0.5)}\beta(0.5,1.5)-\frac{x^{2.5}}{2\Gamma(0.5)}\beta(0.5,3),\\&=x-\frac{x^{2.5}}{\Gamma(3.5)}.\end{aligned} \tag{50}$$

In this stage, the modified Adomian decomposition method is applied as

$$R(x)=R_1(x)+R_2(x)=J^{0.5}\left[\frac{x^{0.5}}{\Gamma(1.5)}-\frac{x^2}{2}\right]=x-\frac{x^{2.5}}{\Gamma(3.5)}. \tag{51}$$

And the modified recursive relation is used as

$$\begin{aligned}u_0(x)&=R_1(x)=x.\\u_1(x)&=R_2(x)+J^{0.5}\left(f(x)u_0(x)\right)+J^{0.5}\left(\int_0^x Z_1(x,s)A_0ds+\int_0^1 Z_2(x,s)B_0ds\right)\\&=-\frac{x^{2.5}}{\Gamma(3.5)}+J^{0.5}\left(-\frac{x^2e^x}{3}x\right)+J^{0.5}\left(\int_0^x e^xs^2ds+\int_0^1 x^2sds\right)\\&=-\frac{x^{2.5}}{\Gamma(3.5)}+J^{0.5}\left(-\frac{x^3e^x}{3}\right)+J^{0.5}\left(\frac{e^xx^3}{3}+\frac{x^2}{2}\right)\\&=-\frac{x^{2.5}}{\Gamma(3.5)}+J^{0.5}\left(-\frac{x^3e^x}{3}\right)+J^{0.5}\left(\frac{e^xx^3}{3}\right)+J^{0.5}\left(\frac{x^2}{2}\right)=0.\\u_2(x)&=0.\\u_n(x)&=0.\end{aligned} \tag{52}$$

Therefore, the obtained solution is

$$u(x)=\sum_{i=0}^{\infty}u_i(x)=x \tag{53}$$

Here, the modified variational iteration technique is applied. Considering the modified iterative relations as

$$\begin{aligned}u_0(x)&=R_1(x)=x.\\u_1(x)&=R(x)+J^{0.5}\left(f(x)u_0(x)\right)+J^{0.5}\left(\int_0^x Z_1(x,s)u_0ds+\int_0^1 Z_2(x,s)u_0ds\right)\\&=x-\frac{x^{2.5}}{\Gamma(3.5)}+J^{0.5}\left(-\frac{x^2e^x}{3}u_0(x)\right)+J^{0.5}\left(\int_0^x e^xsu_0(s)ds+\int_0^1 x^2u_0(s)ds\right)\end{aligned} \tag{54}$$

$$= x - \frac{x^{2.5}}{\Gamma(3.5)} + J^{0.5}\left(-\frac{x^2 e^x}{3}x\right) + J^{0.5}\left(\int_0^x e^x s^2 ds + \int_0^1 x^2 s ds\right)$$

$$= x - \frac{x^{2.5}}{\Gamma(3.5)} + J^{0.5}\left(-\frac{x^3 e^x}{3}\right) + J^{0.5}\left(\frac{e^x x^3}{3} + \frac{x^2}{2}\right)$$

$$= x - \frac{x^{2.5}}{\Gamma(3.5)} + J^{0.5}\left(-\frac{x^3 e^x}{3}\right) + J^{0.5}\left(\frac{e^x x^3}{3}\right) + J^{0.5}\left(\frac{x^2}{2}\right)$$

$$= x.$$

$$u_2(x) = x.$$

$$\vdots$$

$$u_n(x) = x.$$

Therefore, the obtained solution is

$$u(x) = x. \tag{55}$$

6. Conclusions

This paper concludes with significant findings. The study successfully applies the modified Adomian decomposition and modified variational iteration methods to find the approximate solutions of Caputo fractional Volterra-Fredholm integro-differential equation. The presentation of this study proves that these methods have a wider applicability that is based on the reliability of the methods and reduction in the size of the computational work. The methods are effective in the sense that they find analytical along with numerical solutions for wide classes of linear and nonlinear fractional Volterra-Fredholm integro-differential equations. Importantly, this study proves the existence and uniqueness of the solution. The example presented in this paper illustrates further and establishes the precision and efficiency of the proposed techniques.

Conflict of Interest

The authors declare no conflict of interest.

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