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Research Paper

High Order Compact Finite Difference Schemes for Solving Bratu-Type Equations

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Abstract. In the present study, high order compact finite difference methods are used to solve one-dimensional Bratu-type equations numerically. The convergence analysis of the methods is discussed and it is shown that the theoretical order of the methods are consistent with their numerical rate of convergence. The maximum absolute errors in the solution at grid points are calculated and it is shown that the presented methods are efficient and applicable for lower and upper solutions.

Keywords: Bratu-type equations, Compact finite difference methods, Lower and upper solutions, Convergence.

1. Introduction

In this study, the following one-dimensional classical nonlinear Bratu-type equation is considered:

$$u''(x) + \lambda e^{u(x)} = 0, \quad 0 \le x \le 1, \quad \lambda > 0, \tag{1}$$

subjecting to the homogeneous Dirichlet boundary conditions as

$$u(0) = u(1) = 0. \tag{2}$$

for problem (1) with boundary conditions (2), the exact solution is given by [1-4]

$$u(x) = -2\ln\left[\frac{\cosh(\frac{\theta}{2}(\chi - 0.5))}{\cosh(\frac{\theta}{4})}\right], \tag{3}$$

where θ satisfies $\theta = \sqrt{2\lambda}\cosh(\frac{\theta}{4})$. Bratu problem has no solution, one solution, and two solutions when $\lambda > \lambda_c, \lambda = \lambda_c$, and $\lambda < \lambda_c$, respectively. The critical value $\lambda_c \approx 3.513830719$ can be computed from $1 = \frac{1}{4}\sqrt{2\lambda_c}\sinh(\frac{\theta}{4})$. The exact solution given by (3) is symmetric about $x = \frac{1}{2}$.

The Bratu problem has a scientific importance and several chemical and physical processes in science and engineering can be modelled using Bratu-type equation. The Bratu-type equation is also used in a large variety of applications such as the modelling of thermal reaction process in combustible non-deformable materials, including the solid fuel ignition model, the

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electrospinning process for the production of ultra-fine polymer fibers, the modelling of some chemical reaction-diffusion, questions in geometry and relativity about the Chandrasekhar model, radiative heat transfer, and nanotechnology [5-13]. The Bratu problem simulates a thermal reaction process in a rigid material where the process depends on the balance between the chemically generated heat and the heat transferred by conduction.

Several numerical methods have been developed to approximate the solution of Bratu-type equation, but most of these methods converge only to the lower solution. A Laplace transform decomposition numerical algorithm was proposed in Ref. [14]. Remero [15] introduced the efficient fourth-order iterative method for the upper and lower solution. Finite difference methods were studied in Refs. [16] and [17]. For instance in Ref. [16] and based on the Newton-Raphson-Kantorovich approximation method, an iterative finite difference scheme was proposed for the lower solution.For lower and upper solutions, a nonstandard finite difference method with a simple sinusoidal starting function was recommended. Ragb and et al. proposed a numerical scheme for the lower and upper solution [18] based on the differential quadrature methods. Also, Caglar et al. suggested the B-spline method [19]. The non-polynomial spline method was applied in Ref. [20] and recently, the block Nyström method to obtain the lower solution was introduced [21]. The other methods that were applied for Bratu problem are the Chebyshev polynomial expansions method [4], the multigrid-based methods [22], the Lie group shooting method [23], the perturbation-iteration method [24], the decomposition method [15], the differential transformation method [25] and the modified wavelet Galerkin method [26].

The main motive of this study is to obtain numerical solutions for the one-dimensional Bratu problem using high order compact finite difference methods that converge to both upper and lower solutions.

The present study is organized as follows: In section 2, the compact finite difference method is applied for solving Eqs. (1) and (2) and its convergence is discussed. Then we try to improve the accuracy of method. In section 3, the numerical results of applying the methods of this study are presented. Finally, the conclusion is drawn in Section 4.

2. Numerical Method

To approximate the solution, first, the range of $0 \le x \le 1$ is subdivided into N subintervals of width $h = \frac{1}{N}$, and $x_i = ih, i = 0, \dots, N$ is used to denote the points of subdivision. Moreover, the quantity u_i represents the numerical solution at x_i.

2.1 Compact finite difference method

The standard compact finite difference formula for the first derivatives is

$$u'_{i-1} + 4u'_i + u'_{i+1} = \frac{3}{h}(-u_{i-1} + u_{i+1}), \quad i = 2, \dots, N-2, \tag{4}$$

where the truncation error for equation (4) is $O(h^4)$. The coefficients of the standard compact finite difference formula are determined by Taylor expansion so that the formula can yield high accuracy. For more details on how to generate the compact finite difference formula, refer to [24, 27, 28].

The standard fourth-order compact finite difference formula (4) is used for interior points. Since the boundary conditions are known, it is required to adjust the compact finite difference formula to the boundary points. For $i = 1$, we use

$$4u'_1 + u'_2 = \frac{1}{h}\left(-\frac{11}{12}u_0 - 4u_1 + 6u_2 - \frac{4}{3}u_3 + \frac{1}{4}u_4\right), \tag{5}$$

and when $i = N - 1$, we use

$$u'_{N-2} + 4u'_{N-1} = \frac{1}{h}\left(-\frac{1}{4}u_{N-4} + \frac{4}{3}u_{N-3} - 6u_{N-2} + 4u_{N-1} + \frac{11}{12}u_N\right). \tag{6}$$

Equations (5) and (6) have the accuracy of $O(h^4)$. Therefore, the following system is obtained:

$$\begin{cases} 4u'_1 + u'_2 = \frac{1}{h}\left(-\frac{11}{12}u_0 - 4u_1 + 6u_2 - \frac{4}{3}u_3 + \frac{1}{4}u_4\right), \\ u'_{i-1} + 4u'_i + u'_{i+1} = \frac{3}{h}(-u_{i-1} + u_{i+1}), \qquad i = 2, \dots, N-2, \\ u'_{N-2} + 4u'_{N-1} = \frac{1}{h}\left(-\frac{1}{4}u_{N-4} + \frac{4}{3}u_{N-3} - 6u_{N-2} + 4u_{N-1} + \frac{11}{12}u_N\right). \end{cases} \tag{7}$$

The system (7) can be written in a matrix form as:

$$A_1 U' = B_1 U + H_1, \tag{8}$$

where

$$A_1 = \begin{pmatrix} 4 & 1 & 0 & \dots & 0 \\ 1 & 4 & 1 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & 1 & 4 & 1 \\ 0 & \dots & 0 & 1 & 4 \end{pmatrix}_{(N-1)\times(N-1)}, B_1 = \frac{3}{h}\begin{pmatrix} \frac{-4}{3} & 2 & \frac{-4}{9} & \frac{1}{12} & 0 & \dots & 0 \\ -1 & 0 & 1 & 0 & 0 & & \vdots \\ 0 & -1 & 0 & 1 & 0 & \dots & 0 \\ & \ddots & \ddots & \ddots & & & \\ \vdots & & 0 & -1 & 0 & 1 & 0 \\ & & 0 & 0 & -1 & 0 & 1 \\ 0 & \dots & 0 & \frac{-1}{12} & \frac{4}{9} & -2 & \frac{4}{3} \end{pmatrix}_{(N-1)\times(N-1)},$$

$$U' = \begin{pmatrix} u'_1 \\ u'_2 \\ \vdots \\ u'_{N-2} \\ u'_{N-1} \end{pmatrix}_{(N-1)\times 1}, \quad U = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-2} \\ u_{N-1} \end{pmatrix}_{(N-1)\times 1} \quad \text{and} \quad H_1 = \frac{11}{12h}\begin{pmatrix} -u_0 \\ 0 \\ \vdots \\ 0 \\ u_N \end{pmatrix}_{(N-1)\times 1}.$$

Based on the boundary condition (2), $u_0 = u_N = 0$, therefore, $H_1 = \vec{0}$.

Lemma 2.1. *[29] The coefficient matrix A_1 is invertible.*

According to the above-mentioned lemma, $U' = A_1^{-1}B_1U$. Similarly, for the second derivatives, the following finite difference schemes are obtained:

$$\begin{cases} 14u_1'' - 5u_2'' + 4u_3'' - u_4'' = \dfrac{12}{h^2}(u_0 - 2u_1 + u_2), \\ u_{i-1}'' + 10u_i'' + u_{i+1}'' = \dfrac{12}{h^2}(u_{i-1} - 2u_i + u_{i+1}), \qquad i = 2, \dots, N-2, \\ -u_{N-4}'' + 4u_{N-3}'' - 5u_{N-2}'' + 14u_{N-1}'' = \dfrac{12}{h^2}(u_{N-2} - 2u_{N-1} + u_N). \end{cases} \tag{9}$$

All the above-mentioned relations have fourth order accuracy. The matrix form for Eq. (9) is

$$A_2U'' = B_2U + H_2, \tag{10}$$

where

$$A_2 = \begin{pmatrix} 14 & -5 & 4 & -1 & 0 & \dots & 0 \\ 1 & 10 & 1 & 0 & 0 & & 0 \\ 0 & 1 & 10 & 1 & 0 & \dots & 0 \\ & \ddots & \ddots & \ddots & & & \\ \vdots & 0 & 1 & 10 & 1 & 0 \\ & 0 & 0 & 1 & 10 & 1 \\ 0 & \dots & 0 & -1 & 4 & -5 & 14 \end{pmatrix}_{(N-1)\times(N-1)}, B_2 = \frac{12}{h^2}\begin{pmatrix} -2 & 1 & 0 & \dots & 0 \\ 1 & -2 & 1 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & 1 & -2 & 1 \\ 0 & \dots & 0 & 1 & -2 \end{pmatrix}_{(N-1)\times(N-1)},$$

$$U = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-2} \\ u_{N-1} \end{pmatrix}_{(N-1)\times 1}, \quad U'' = \begin{pmatrix} u_1'' \\ u_2'' \\ \vdots \\ u_{N-2}'' \\ u_{N-1}'' \end{pmatrix}_{(N-1)\times 1} \quad \text{and} \quad H_2 = \frac{12}{h^2}\begin{pmatrix} u_0 \\ 0 \\ \vdots \\ 0 \\ u_N \end{pmatrix}_{(N-1)\times 1}.$$

Again for boundary condition (2), we have $H_2 = \vec{0}$.

2.2 Compact finite difference method for Bratu-type equations

By inserting the relation $u_i'' = -\lambda e^{u_i}$ in the system (9), the following nonlinear system is obtained:

$$\begin{cases} -\lambda(14e^{u_1} - 5e^{u_2} + 4e^{u_3} - e^{u_4}) = \dfrac{12}{h^2}(-2u_1 + u_2), \\ -\lambda(e^{u_{i-1}} + 10e^{u_i} + e^{u_{i+1}}) = \dfrac{12}{h^2}(u_{i-1} - 2u_i + u_{i+1}), \quad i = 2, \dots, N-2, \\ -\lambda(-e^{u_{N-4}} + 4e^{u_{N-3}} - 5e^{u_{N-2}} + 14e^{u_{N-1}}) = \dfrac{12}{h^2}(u_{N-2} - 2u_{N-1}). \end{cases} \tag{11}$$

This technique transforms the problem into a system of nonlinear equations and the resulting nonlinear system can be solved by using an appropriate nonlinear solver. This nonlinear system can be written as

$$-\lambda A_2 e^U = \frac{1}{h^2} M_0 U, \tag{12}$$

where $e^U = [e^{u_1}, \dots, e^{u_{N-1}}]^T$ and

$$M_0 = \begin{pmatrix} -24 & 12 & 0 & \dots & 0 \\ 12 & -24 & 12 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & 12 & -24 & 12 \\ 0 & \dots & 0 & 12 & -24 \end{pmatrix}_{(N-1)\times(N-1)}.$$

where $\|M_0^{-1}\|_\infty \le c$ and c is a fixed number. Next, the convergence analysis is going to be conducted. For this purpose, let $\bar{U} = [u(x_1), \dots, u(x_{N-1})]^T$ be the vector of exact solution . Moreover, consider $\|.\|$ as $\|.\|_\infty$, $s = \max_{i=1,\dots,N-1} \left\{ \left| \frac{\partial e^{u(x)}}{\partial x} \right| : x = x_i \right\}$ and $E = \bar{U} - U$.

Theorem 2.2. *Let $\bar{U} = [u(x_1), \dots, u(x_{N-1})]^T$ and $U = [u_1, \dots, u_{N-1}]^T$ be the vectors of exact solution of the boundary-value problem (1), (2), and the numerical solution obtained by solving the nonlinear system (12), respectively. Then, provided $\lambda h^2 s \|M_0^{-1}\| \|A_2\| \le 1$, we have*

$$\|E\| \le O(h^4). \tag{13}$$

Proof. According to Eq. (12),

$$M_0 U + h^2 \lambda A_2 e^U = \vec{0}, \tag{14}$$

and for the exact solution,

$$M_0 \bar{U} + h^2 \lambda A_2 e^{\bar{U}} = T, \tag{15}$$

where

$$T = \frac{h^6}{20} [-19u^{(6)}(x_1), u^{(6)}(x_2), \dots, u^{(6)}(x_{N-2}), -19u^{(6)}(x_{N-1})]^T, \tag{16}$$

is the vector of local truncation error. By using Eqs. (14) and (15),

$$M_0(\bar{U} - U) + h^2 \lambda A_2 (e^{\bar{U}} - e^U) = T,$$

$$(M_0 + \lambda h^2 A_2 J)E = T,$$

where

$$E = \bar{U} - U, e^{\bar{U}} - e^U = JE,$$

and $J = \mathrm{diag}\left\{ \left| \frac{\partial e^{u(x)}}{\partial x} \right| : x = x_i, i = 1, \dots, N-1 \right\}$ is a diagonal matrix of order $N-1$. Now if $\lambda h^2 s \|M_0^{-1}\| \|A_2\| \le 1$, then $(I + h^2 \lambda M_0^{-1} A_2 J)$ is invertible and

$$E = (I + h^2 \lambda M_0^{-1} A_2 J)^{-1} M_0^{-1} T,$$

$$\|E\| \le \|(I + h^2 \lambda M_0^{-1} A_2 J)^{-1}\| \|M_0^{-1}\| \|T\|.$$

It follows that

$$\|E\| \le \frac{\|M_0^{-1}\| \|T\|}{1 - \lambda h^2 \|M_0^{-1}\| \|A_2\| \|J\|}, \tag{17}$$

From Eq. (16), we have

$$\|\mathrm{T}\| \leq \frac{h^6 M_6}{20}, \tag{18}$$

where $M_6 = \max_{0\leq\xi\leq1} |u^{(6)}(\xi)|$.

According to Eqs. (17), (18), $\|\mathrm{M}_0^{-1}\| \leq c$, $\|\mathrm{J}\| \leq s = \max_{i=1,\dots,N-1} \left\{ \left|\frac{\partial e^{u(x)}}{\partial x}\right| : x = x_i \right\}$ and $\|\mathrm{A}_2\| \leq 24$, one can obtain

$$\|\mathrm{E}\| \leq \frac{h^6 M_6 c}{20(1 - 24\lambda \mathrm{h}^2 cs)} \equiv \frac{O(h^6)}{O(h^2)} \equiv O(h^4).$$

□

2.3 Fifth-order compact finite difference method for the Bratu-type equations

In this section, we try to improve the results of the pervious method by applying the compact finite difference method. The system of equations (9) are considered and the sides of the equations are extended by using the Taylor expansion at point x_i, $i = 1, \dots, N-1$. Therefore, the following system is obtained:

$$\begin{cases} 14u_1'' - 5u_2'' + 4u_3'' - u_4'' + \tau_1 = \dfrac{12}{h^2}(u_0 - 2u_1 + u_2), \\ u_{i-1}'' + 10u_i'' + u_{i+1}'' + \tau_i = \dfrac{12}{h^2}(u_{i-1} - 2u_i + u_{i+1}), \qquad i = 2, \dots, N-2, \\ -u_{N-4}'' + 4u_{N-3}'' - 5u_{N-2}'' + 14u_{N-1}'' + \tau_{N-1} = \dfrac{12}{h^2}(u_{N-2} - 2u_{N-1} + u_N), \end{cases} \tag{19}$$

where

$$\begin{cases} \tau_1 = -\dfrac{19}{20} h^4 u_1^{(6)} + O(h^5), \\ \tau_i = \dfrac{1}{20} h^4 u_i^{(6)} + O(h^6), \qquad i = 2, \dots, N-2 \\ \tau_{N-1} = -\dfrac{19}{20} h^4 u_{N-1}^{(6)} + O(h^5). \end{cases} \tag{20}$$

Now, $u_i^{(k)}, k = 3,4,5,6$ can be obtained with a derivative from $u_i'' = -\lambda e^{u_i}$ as follows:

$$\begin{cases} u_i''' = u_i' u_i'', \\ u_i^{(4)} = (u_i'')^2 + (u_i')^2 u_i'', \\ u_i^{(5)} = 4u_i'(u_i'')^2 + (u_i')^3 u_i'', \\ u_i^{(6)} = 4(u_i'')^3 + 11(u_i' u_i'')^2 + (u_i')^4 u_i'' \end{cases} \tag{21}$$

Therefore, the following system is obtained:

$$\begin{cases} 14u_1'' - 5u_2'' + 4u_3'' - u_4'' - \dfrac{19}{20} h^4 (11(u_1' u_1'')^2 + 4(u_1'')^3 + (u_1')^4 u_1'') = \dfrac{12}{h^2}(-2u_1 + u_2), \\ u_{i-1}'' + 10u_i'' + u_{i+1}'' + \dfrac{1}{20} h^4 (11(u_i' u_i'')^2 + 4(u_i'')^3 + (u_i')^4 u_i'') = \dfrac{12}{h^2}(u_{i-1} - 2u_i + u_{i+1}), \\ \quad i = 2, \dots, N-2, \\ -u_{N-4}'' + 4u_{N-3}'' - 5u_{N-2}'' + 14u_{N-1}'' - \dfrac{19}{20} h^4 (11(u_{N-1}' u_{N-1}'')^2 + 4(u_{N-1}'')^3 + (u_{N-1}')^4 u_{N-1}'') \\ \qquad = \dfrac{12}{h^2}(u_{N-2} - 2u_{N-1} + u_N). \end{cases} \tag{22}$$

By replacing $u_i'' = -\lambda e^{u_i}$ in $u_i^{(6)}$, it can be written as:

$$u_i^{(6)} = 11\lambda^2 (e^{u_i})^2 (u_i')^2 - 4\lambda^3 (e^{u_i})^3 - \lambda (u_i')^4 e^{u_i}, \tag{23}$$

Finally, the following nonlinear system is obtained:

$$-\lambda A_2 e^U = \frac{1}{h^2} M_0 U + h^4 C U^{(6)}, \tag{24}$$

where

$$C=\begin{pmatrix} -\frac{19}{20} & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{20} & 0 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & 0 & \frac{1}{20} & 0 \\ 0 & \dots & 0 & 0 & -\frac{19}{20} \end{pmatrix}_{(N-1)\times(N-1)},$$

and

$$U^{(6)}=\begin{pmatrix} u_1^{(6)} \\ \vdots \\ u_{N-1}^{(6)} \end{pmatrix}=11\lambda^2\begin{pmatrix} (e^{u_1}u_1')^2 \\ \vdots \\ (e^{u_{N-1}}u_{N-1}')^2 \end{pmatrix}-4\lambda^3\begin{pmatrix} (e^{u_1})^3 \\ \vdots \\ (e^{u_{N-1}})^3 \end{pmatrix}-\lambda\begin{pmatrix} e^{u_1}(u_1')^4 \\ \vdots \\ e^{u_{N-1}}(u_{N-1}')^4 \end{pmatrix}. \tag{25}$$

For the convergence analysis of the above-mentioned method, the first-order derivative error is obtained in the following lemma:

Lemma 2.3. *Let $\bar{U}=[u(x_1),\dots,u(x_{N-1})]^T$ and $U=[u_1,\dots,u_{N-1}]^T$ be the vectors of exact solution of the problem (1) with the boundary condition (2) and the numerical solution obtained by solving the nonlinear system (8), respectively. If we let $E'=\bar{U}'-U'$, then we have*

$$\|E'\|\leq O(h^3). \tag{26}$$

Proof. According to Eq. (8), one can conclude

$$A_1U'=B_1U, \tag{27}$$

and for the exact solution, it can be written as

$$A_1\bar{U}'=B_1\bar{U}+T_1, \tag{28}$$

where $T_1=O(h^4)$ is the vector of local truncation error. By using Eqs. (27) and (28), one can drive

$$A_1(\bar{U}'-U')=B_1(\bar{U}-U)+T_1,\quad A_1E'=B_1E+T_1.$$

Therefore,

$$E'=A_1^{-1}B_1E+A_1^{-1}T_1,$$

$$\|E'\|\leq\|A_1^{-1}\|\|B_1\|\|E\|+\|A_1^{-1}\|\|T_1\|\equiv O(h^{-1})O(h^4)+O(h^4)\equiv O(h^3).$$

□

Theorem 2.4. *Let $\bar{U}=[u(x_1),\dots,u(x_{N-1})]^T$ and $U=[u_1,\dots,u_{N-1}]^T$ be the vectors of exact solution of the boundary-value problem (1), (2), and the numerical solution obtained by solving the nonlinear system (24), respectively. Then, provided $\|\lambda h^2M_0^{-1}A_2J+h^6M_0^{-1}CM_1+h^6M_0^{-1}CM_2A_1^{-1}B_1\|<1$, we have*

$$\|E\|\leq O(h^5), \tag{29}$$

where

$$M_1=diag\{22\lambda^2e^{2U_i}(U_i')^2-12\lambda^3e^{3U_i}-\lambda e^{U_i}(U_i')^4\},$$

$$M_2=diag\{22\lambda^2e^{2U_i}U_i'-4\lambda e^{U_i}(U_i')^3\}\quad and\quad E=\bar{U}-U.$$

Proof. By using the relation Eq. (23), one can obtain:

$$u_i^{(6)}-U_i^{(6)}=11\lambda^2(e^{2u_i}(u_i')^2-e^{2U_i}(U_i')^2)-4\lambda^3(e^{3u_i}-e^{3U_i})-\lambda(e^{u_i}(u_i')^4-e^{U_i}(U_i')^4)$$

$$=11\lambda^2(2e^{2U_i}(U_i')^2E_i+2e^{2U_i}U_i'E_i')-4\lambda^3(3e^{3U_i}E_i)-\lambda(e^{U_i}(U_i')^4E_i+4e^{U_i}(U_i')^3E_i')$$

$$= (22\lambda^2 e^{2U_i}(U_i')^2 - 12\lambda^3 e^{3U_i} - \lambda e^{U_i}(U_i')^4)E_i + (22\lambda^2 e^{2U_i}U_i' - 4\lambda e^{U_i}(U_i')^3)E_i',$$

where $E_i = u_i - U_i$ and $E_i' = u_i' - U_i'$. By considering M_1 and M_2, we have

$$u_i^{(6)} - U_i^{(6)} = M_1 E + M_2 E'. \tag{30}$$

On the other hand, according to Eq. (24), it can be driven as

$$h^2\lambda A_2 e^U + M_0 U + h^6 C U^{(6)} = 0, \tag{31}$$

and for the exact solution, we have

$$h^2\lambda A_2 e^{\bar{U}} + M_0\bar{U} + h^6 C\bar{U}^{(6)} = Th^2, \tag{32}$$

where the vector $\bar{U} = u(x_i), i = 1, \dots, N-1$ is the exact solution and $T = O(h^5)$ is the local truncation error. By using Eqs. (31) and (32) the following relation is obtained

$$h^2\lambda A_2\left(e^{\bar{U}} - e^U\right) + M_0(\bar{U} - U) + h^6 C\left(\bar{U}^{(6)} - U^{(6)}\right) = Th^2,$$

where $e^{\bar{U}} - e^U = JE$, and $\mathrm{J} = \mathrm{diag}\left\{\left|\frac{\partial e^{u(x)}}{\partial x}\right| : x = x_i, i = 1, \dots, N-1\right\}$ is a diagonal matrix of order $N-1$. Therefore,

$$h^2\lambda A_2 JE + M_0 E + h^6 C(M_1 E + M_2 E') = Th^2. \tag{33}$$

By replacing $E' = A_1^{-1}B_1 E + A_1^{-1}T_1$ in Eq. (33), one can conclude

$$(h^2\lambda A_2 J + h^6 CM_1 + h^6 CM_2 A_1^{-1}B_1 + M_0)E = Th^2 - h^6 CM_2 A_1^{-1}T_1, \quad or$$

$$M_0(I + h^2\lambda M_0^{-1}A_2 J + h^6 M_0^{-1}CM_1 + h^6 M_0^{-1}CM_2 A_1^{-1}B_1)E = Th^2 - h^6 CM_2 A_1^{-1}T_1.$$

Now, if $\|\lambda h^2 M_0^{-1}A_2 J + h^6 M_0^{-1}CM_1 + h^6 M_0^{-1}CM_2 A_1^{-1}B_1\| < 1$, then, $(I + h^2\lambda M_0^{-1}A_2 J + h^6 M_0^{-1}CM_1 + h^6 M_0^{-1}CM_2 A_1^{-1}B_1)$ is invertible and

$$\|E\| \le \frac{\|M_0^{-1}\|(h^2\|T\| + h^6\|C\|\|M_2\|\|A_1^{-1}\|\|T_1\|)}{1 - h^2\lambda\|M_0^{-1}A_2 J\| - h^6\|M_0^{-1}CM_1\| - h^6\|M_0^{-1}CM_2 A_1^{-1}B_1\|} \equiv \frac{O(h^7)}{O(h^2)} \equiv O(h^5).$$

□

2.4 Sixth-order compact finite difference method for the Bratu-type equation

In this section, the compact finite difference method of the sixth order is obtained for the Bratu problem. The system of linear Eqs. (9) is considered and sentences to the seventh derivative of the Taylor expansion are added at point $x_i, i = 1, \dots, N-1$ relative to h. For inner points $x_i, i = 2, \dots, N-2$, the seventh derivative is deleted. Therefore, the following relations are obtained

$$\begin{cases} 14u_1'' - 5u_2'' + 4u_3'' - u_4'' + \tau_1 = \dfrac{12}{h^2}(u_0 - 2u_1 + u_2), \\ u_{i-1}'' + 10u_i'' + u_{i+1}'' + \tau_i = \dfrac{12}{h^2}(u_{i-1} - 2u_i + u_{i+1}), \quad i = 2, \dots, N-2, \\ -u_{N-4}'' + 4u_{N-3}'' - 5u_{N-2}'' + 14u_{N-1}'' + \tau_{N-1} = \dfrac{12}{h^2}(u_{N-2} - 2u_{N-1} + u_N), \end{cases} \tag{34}$$

where

$$\begin{cases} \tau_1 = -\dfrac{19}{20}h^4 u_1^{(6)} - h^5 u_1^{(7)} + O(h^6), \\ \tau_i = \dfrac{1}{20}h^4 u_i^{(6)} + O(h^6), \qquad i = 2, \dots, N-2 \\ \tau_{N-1} = -\dfrac{19}{20}h^4 u_{N-1}^{(6)} + h^5 u_{N-1}^{(7)} + O(h^6). \end{cases} \tag{35}$$

Therefore, the following system is obtained:

$$\begin{cases} 14u_1'' - 5u_2'' + 4u_3'' - u_4'' - \dfrac{19}{20}h^4u_1^{(6)} - h^5u_1^{(7)} = \dfrac{12}{h^2}(-2u_1 + u_2), \\ u_{i-1}'' + 10u_i'' + u_{i+1}'' + \dfrac{1}{20}h^4u_i^{(6)} = \dfrac{12}{h^2}(u_{i-1} - 2u_i + u_{i+1}), \qquad i = 2, \ldots, N-2, \\ -u_{N-4}'' + 4u_{N-3}'' - 5u_{N-2}'' + 14u_{N-1}'' - \dfrac{19}{20}h^4u_{N-1}^{(6)} + h^5u_{N-1}^{(7)} = \dfrac{12}{h^2}(u_{N-2} - 2u_{N-1}). \end{cases} \tag{36}$$

Now, $u_i^{(7)}$ is obtained as

$$u_i^{(7)} = 34u_i'(u_i'')^3 + 26(u_i')^3(u_i'')^2 + (u_i')^5u_i'', \tag{37}$$

by replacing $u_i'' = -\lambda e^{u_i}$ in $u_i^{(7)}$,

$$u_i^{(7)} = 26\lambda^2(e^{u_i})^2(u_i')^3 - 34\lambda^3(e^{u_i})^3u_i' - \lambda e^{u_i}(u_i')^5. \tag{38}$$

Finally, by replacing $u_i'' = -\lambda e^{u_i}$ in the system (36), the following nonlinear system is obtained:

$$-\lambda A_2 e^U = \frac{1}{h^2}M_0U + h^4CU^{(6)} + h^5DU^{(7)}, \tag{39}$$

where

$$\mathrm{D} = \begin{pmatrix} -1 & 0 & 0 & \ldots & 0 \\ & 0 & 0 & 0 & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & 0 & 0 & 0 \\ 0 & \ldots & 0 & 0 & 1 \end{pmatrix}_{(N-1)\times(N-1)}$$

and

$$U^{(7)} = \begin{pmatrix} u_1^{(7)} \\ \vdots \\ u_{N-1}^{(7)} \end{pmatrix} = 26\lambda^2 \begin{pmatrix} (e^{u_1})^2(u_1')^3 \\ \vdots \\ (e^{u_{N-1}})^2(u_{N-1}')^3 \end{pmatrix} - 34\lambda^3 \begin{pmatrix} (e^{u_1})^3u_1' \\ \vdots \\ (e^{u_{N-1}})^3u_{N-1}' \end{pmatrix} - \lambda \begin{pmatrix} e^{u_1}(u_1')^5 \\ \vdots \\ e^{u_{N-1}}(u_{N-1}')^5 \end{pmatrix}. \tag{40}$$

Theorem 2.5. *Let* $\bar{U} = [u(x_1), \ldots, u(x_{N-1})]^T$ *and* $U = [u_1, \ldots, u_{N-1}]^T$ *be the vectors of exact solution of the boundary-value problem (1), (2), and numerical solution obtained by solving the nonlinear system (39), respectively. Then provided* $\|\lambda h^2M_0^{-1}A_2J + h^6M_0^{-1}CM_1 + h^6M_0^{-1}CM_2A_1^{-1}B_1 + h^7M_0^{-1}DN_1 + h^7M_0^{-1}DN_2A_1^{-1}B_1\| < 1$, *we have*

$$\|E\| \le O(h^6), \tag{41}$$

where

$$\begin{cases} M_1 = diag\{22\lambda^2e^{2U_i}(U_i')^2 - 12\lambda^3e^{3U_i} - \lambda e^{U_i}(U_i')^4\}, \\ M_2 = diag\{22\lambda^2e^{2U_i}U_i' - 4\lambda e^{U_i}(U_i')^3\}, \\ N_1 = diag\{52\lambda^2e^{2U_i}(U_i')^3 - 102\lambda^3e^{3U_i}U_i' - \lambda e^{U_i}(U_i')^5\}, \\ N_2 = diag\{78\lambda^2e^{2U_i}(U_i')^2 - 34\lambda^3e^{3U_i} - 5\lambda e^{U_i}(U_i')^4\} \text{ and } E = \bar{U} - U. \end{cases}$$

Proof. The proof is similar to that of theorem 2.4.

□

3. Numerical results

In this section, the numerical results of the new proposed method are presented for various values of λ and N. By adopting a simple approach, similar to what used by Body [4], it was found that it is appropriate to consider $u_0(x) = a\sin(\pi x)$ as initial guess; because it satisfies the boundary conditions. The upper and lower numerical solutions of the systems (12), (24), and (39) were compared with the exact solution to the boundary-value problem (1) and (2) for the values of $\lambda = 0.5, 1, 2, 3, 3.51$ and $N = 10, 20, 40, 80$ and the maximum error was calculated. Finally, the numerical rate of convergence using the following formula was calculated:

$$ROC = \log_2\left(\frac{Error^{2h}}{Error^{h}}\right),$$

where $Error^h$ is the error obtained using the step size h. Tables 1 and 2 explain the maximum error and the rate of convergence of the upper and lower solutions for the Bratu problem using the method (12) versus various λ and N, respectively. Also Tables 3 and 4 show the maximum error and the rate of convergence of the upper and lower solutions for the Bratu problem using the method (24) versus various λ and N, respectively. In Tables 5 and 6 one can see the maximum error and the rate of convergence of the upper and lower solutions for the Bratu problem using the method (39) versus various λ and N, respectively. In Tables 7 and 8, the absolute errors between lower solutions of method (39) for $N = 10$ are compared with the Lie-group shooting, the Laplace, the B-spline, and the decomposition methods. Moreover, the computed and exact solutions of Bratu problem for $\lambda = 0.0001$ are compared in Fig. 1 (lower solution) and Fig. 2 (upper solution). The Bifurcated nature of the computed solution to Bratu problem for different values of $\lambda \in (0, 3.513830719]$ is plotted in Figure 3, for all values of λ, N is 20. Finally, the absolute errors between lower solutions of method (24) for $N = 10$ are compared with the Lie-group shooting and the B-spline methods in Table 9.

Table 1. Observed absolute error and *ROC* of upper solution for method (12)

λ	N	MaxError	ROC	λ	N	MaxError	ROC
	10	3.72×10^{-3}	-		10	1.56×10^{-3}	-
	20	1.65×10^{-4}	4.49		20	8.11×10^{-5}	4.26
0.5	40	1.03×10^{-5}	4.00	1	40	5.14×10^{-6}	3.98
	80	6.41×10^{-7}	4.00		80	3.22×10^{-7}	3.99
	10	2.14×10^{-3}	-		10	2.23×10^{-3}	-
	20	3.16×10^{-5}	6.08		20	2.25×10^{-5}	6.63
2	40	1.99×10^{-6}	3.98	3	40	1.02×10^{-6}	4.46
	80	1.26×10^{-7}	3.98		80	6.40×10^{-8}	3.99
	10	7.87×10^{-3}	-		20	1.63×10^{-4}	5.59
3.51	40	5.93×10^{-6}	4.78	3.51	80	3.22×10^{-7}	4.20

Table 2. Observed absolute error and *ROC* of lower solution for method (12)

λ	N	MaxError	ROC	λ	N	MaxError	ROC
	10	3.05×10^{-8}	-		10	2.31×10^{-7}	-
	20	1.36×10^{-9}	4.49		20	1.54×10^{-8}	3.90
0.5	40	1.12×10^{-10}	3.60	1	40	1.13×10^{-9}	3.77
	80	7.36×10^{-12}	3.92		80	7.15×10^{-11}	3.98
	10	1.67×10^{-6}	-		10	5.97×10^{-5}	-
	20	2.63×10^{-7}	2.67		20	3.11×10^{-6}	4.26
2	40	1.58×10^{-8}	4.05	3	40	1.47×10^{-7}	4.39
	80	9.58×10^{-10}	4.04		80	8.40×10^{-9}	4.13
	10	5.95×10^{-3}	-		20	1.38×10^{-4}	5.42
3.51	40	5.22×10^{-6}	4.72	3.51	80	2.87×10^{-7}	4.19

Table 3. Observed absolute error and *ROC* of upper solution for method (24)

λ	N	MaxError	ROC	λ	N	MaxError	ROC
	10	1.26×10^{-3}	-		10	2.90×10^{-3}	-
	20	2.22×10^{-5}	5.83		20	7.59×10^{-6}	8.57
0.5	40	1.27×10^{-7}	7.44	1	40	6.90×10^{-8}	6.78
	80	1.47×10^{-9}	6.43		80	6.24×10^{-10}	6.79
	10	2.70×10^{-3}	-		10	1.47×10^{-3}	-
	20	4.74×10^{-6}	9.15		20	8.34×10^{-6}	7.45
2	40	9.65×10^{-9}	8.94	3	40	4.11×10^{-8}	7.66
	80	1.29×10^{-10}	6.22		80	2.50×10^{-10}	7.66
	10	7.61×10^{-4}	-		20	2.40×10^{-5}	4.98
3.51	40	1.84×10^{-7}	7.03	3.51	80	1.33×10^{-9}	7.11

Table 4. Observed absolute error and *ROC* of lower solution for method (24)

λ	N	MaxError	ROC	λ	N	MaxError	ROC
	10	7.17×10^{-9}	-		10	1.39×10^{-7}	-
	20	1.29×10^{-10}	5.80		20	1.14×10^{-9}	6.92
0.5	40	6.63×10^{-13}	7.60	1	40	9.18×10^{-12}	6.96
	80	4.52×10^{-15}	7.20		80	7.30×10^{-14}	6.97
	10	3.67×10^{-6}	-		10	4.11×10^{-5}	-
	20	2.31×10^{-8}	7.31		20	8.50×10^{-8}	8.91
2	40	1.59×10^{-10}	7.18	3	40	8.47×10^{-11}	9.97
	80	1.08×10^{-12}	7.20		80	2.34×10^{-12}	5.17
	10	2.80×10^{-5}	-		20	1.57×10^{-5}	0.83
3.51	40	1.32×10^{-7}	6.89	3.51	80	9.78×10^{-10}	7.07

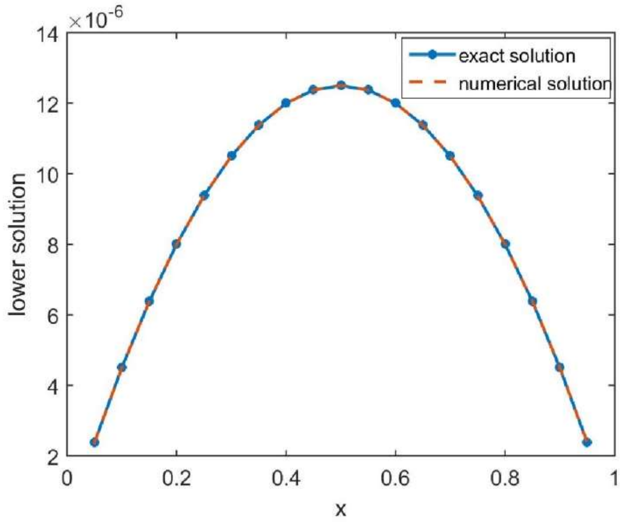

Fig. 1. Comparison of $u(x)$ and exact solution of lower solutions for $\lambda = 0.0001$ of method (39)

Table 5. Observed absolute error and *ROC* of upper solution for method (39)

λ	N	MaxError	ROC	λ	N	MaxError	ROC
	10	3.98×10^{-3}	-		10	3.67×10^{-3}	-
	20	7.69×10^{-6}	9.01		20	2.89×10^{-6}	10.30
0.5	40	9.79×10^{-8}	6.29	1	40	3.27×10^{-8}	6.46
	80	1.64×10^{-9}	5.90		80	5.44×10^{-10}	5.91
	10	1.93×10^{-3}	-		10	2.38×10^{-4}	-
	20	4.00×10^{-6}	8.91		20	2.64×10^{-6}	6.48
2	40	1.35×10^{-8}	8.16	3	40	9.42×10^{-9}	8.13
	80	1.29×10^{-10}	6.75		80	4.41×10^{-11}	7.73
	10	2.30×10^{-3}	-		20	4.71×10^{-7}	12.25
3.51	40	5.56×10^{-9}	6.40	3.51	80	9.83×10^{-12}	9.14

Table 6. Observed absolute error and *ROC* of lower solution for method (39)

λ	N	MaxError	ROC	λ	N	MaxError	ROC
	10	1.91×10^{-9}	-		10	6.87×10^{-9}	-
	20	2.57×10^{-12}	9.53		20	2.32×10^{-11}	8.20
0.5	40	6.50×10^{-15}	8.62	1	40	3.65×10^{-13}	5.99
	80	1.98×10^{-16}	5.03		80	4.69×10^{-15}	6.28
	10	5.88×10^{-7}	-		10	2.57×10^{-5}	-
	20	3.80×10^{-9}	7.27		20	8.81×10^{-8}	8.18
2	40	2.26×10^{-11}	7.39	3	40	3.44×10^{-10}	8.00
	80	1.65×10^{-13}	7.09		80	2.24×10^{-12}	7.26
	10	1.94×10^{-3}	-		20	1.74×10^{-6}	10.12
3.51	40	9.17×10^{-10}	10.89	3.51	80	2.88×10^{-11}	5.00

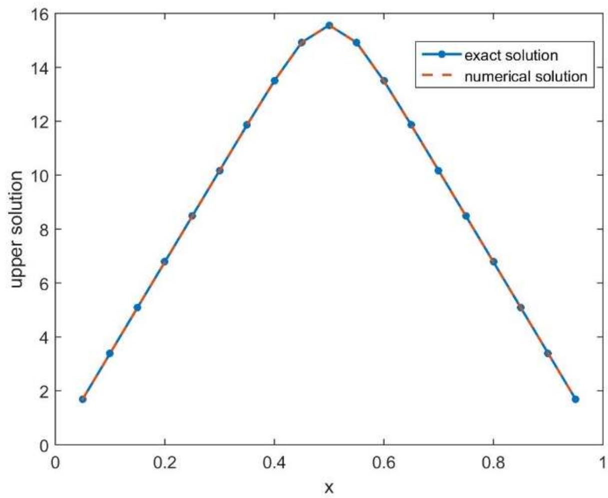

Fig. 2. Comparison of $u(x)$ and exact solution of upper solutions for $\lambda = 0.0001$ of method (39)

Table 7. Comparison of the absolute error of different methods for $\lambda = 1$

x	Method (39)	LGSM [23]	B-spline [19]	Laplace [14]	Decomposition [15]
0.1	6.87×10^{-9}	7.51×10^{-7}	2.98×10^{-6}	1.98×10^{-6}	2.68×10^{-3}
0.2	6.83×10^{-9}	1.02×10^{-6}	5.46×10^{-6}	3.94×10^{-6}	2.02×10^{-3}
0.3	6.75×10^{-9}	9.05×10^{-7}	7.33×10^{-6}	5.85×10^{-6}	1.52×10^{-4}
0.4	6.64×10^{-9}	5.24×10^{-7}	8.50×10^{-6}	7.07×10^{-6}	2.20×10^{-3}
0.5	6.71×10^{-9}	5.07×10^{-9}	8.89×10^{-6}	9.47×10^{-6}	3.01×10^{-3}
0.6	6.64×10^{-9}	5.14×10^{-7}	8.50×10^{-6}	1.11×10^{-5}	2.20×10^{-3}
0.7	6.75×10^{-9}	8.95×10^{-7}	7.33×10^{-6}	1.26×10^{-5}	1.52×10^{-4}
0.8	6.83×10^{-9}	1.01×10^{-6}	5.46×10^{-6}	1.35×10^{-5}	2.02×10^{-3}
0.9	6.87×10^{-9}	7.42×10^{-7}	2.98×10^{-6}	1.20×10^{-5}	2.68×10^{-3}

Table 8. Comparison of the absolute error of different methods for $\lambda = 2$

x	Method (39)	LGSM [23]	B-spline [19]	Laplace [14]	Decomposition [15]
0.1	4.53×10^{-7}	4.03×10^{-6}	1.72×10^{-5}	2.13×10^{-3}	1.52×10^{-2}
0.2	4.98×10^{-7}	5.70×10^{-6}	3.26×10^{-5}	4.21×10^{-3}	1.47×10^{-2}
0.3	5.49×10^{-7}	5.22×10^{-6}	4.49×10^{-5}	6.19×10^{-3}	5.89×10^{-3}
0.4	5.73×10^{-7}	3.07×10^{-6}	5.28×10^{-5}	8.00×10^{-3}	3.25×10^{-3}
0.5	5.88×10^{-7}	1.45×10^{-8}	5.56×10^{-5}	9.60×10^{-3}	6.98×10^{-3}
0.6	5.73×10^{-7}	3.05×10^{-6}	5.28×10^{-5}	1.09×10^{-3}	3.25×10^{-3}
0.7	5.49×10^{-7}	5.19×10^{-6}	4.49×10^{-5}	1.19×10^{-2}	5.89×10^{-3}
0.8	4.98×10^{-7}	5.68×10^{-6}	3.26×10^{-5}	1.24×10^{-2}	1.47×10^{-2}
0.9	4.53×10^{-7}	4.01×10^{-6}	1.72×10^{-5}	1.09×10^{-2}	1.52×10^{-2}

Table 9. Comparison of the absolute error of present method with two other method for $\lambda = 3.51$

x	Present method (24)	LGSM [23]	B-spline [19]
0.1	9.41×10^{-6}	4.45×10^{-5}	3.84×10^{-2}
0.2	1.77×10^{-5}	7.12×10^{-5}	7.48×10^{-2}
0.3	2.51×10^{-5}	7.30×10^{-5}	1.06×10^{-1}
0.4	3.07×10^{-5}	4.47×10^{-5}	1.27×10^{-1}
0.5	3.28×10^{-5}	6.76×10^{-7}	1.35×10^{-1}
0.6	3.07×10^{-5}	4.56×10^{-5}	1.27×10^{-1}
0.7	2.51×10^{-5}	7.20×10^{-5}	1.06×10^{-1}
0.8	1.77×10^{-5}	7.05×10^{-5}	7.48×10^{-2}
0.9	9.41×10^{-6}	4.41×10^{-5}	3.84×10^{-2}

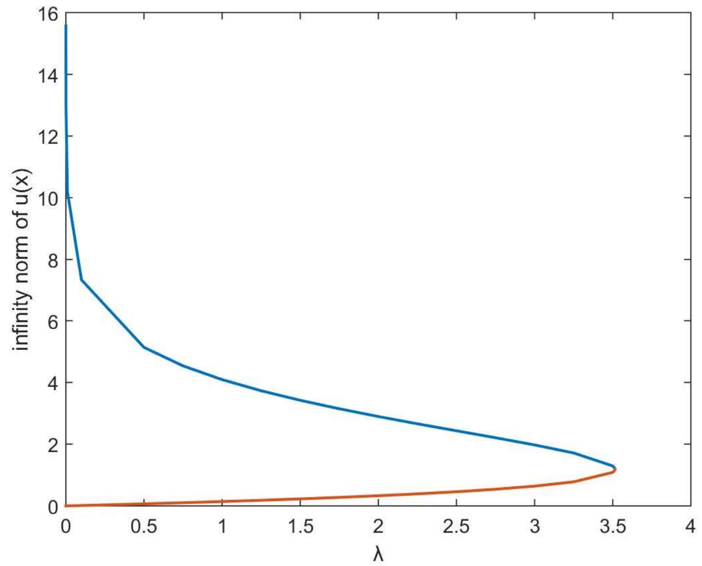

Fig. 3. The Bifurcated of the computed solution to Bratu problem for different values of $\lambda \in (0,3.513830719)$

4. Conclusion

In the present study, a high order compact finite difference method for Bratu problem was proposed and the convergence analysis was discussed. As pointed, many existing numerical methods for Bratu problem failed to compute the upper solution and the lower solution for $\lambda = \lambda_c$, but it was observed that the proposed methods of this study are in an excellent agreement with the exact values. The numerical results presented in Tables 1, 2, 3, 4, 5, 6, 7, 8, and 9 showed that the method is very accurate and the numerical experiment is extremely consistent with the theoretical analysis results of the present study.

Conflict of Interest

The authors declare no conflict of interest.

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