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Research Paper

Magnetohydrodynamic Free Convection Flows with Thermal Memory over a Moving Vertical Plate in Porous Medium

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Abstract. The unsteady hydro-magnetic free convection flow with heat transfer of a linearly viscous, incompressible, electrically conducting fluid near a moving vertical plate with the constant heat is investigated. The flow domain is the porous half-space and a magnetic field of a variable direction is applied. The Caputo time-fractional derivative is employed in order to introduce a thermal flux constitutive equation with a weakly memory. The exact solutions for the fractional governing differential equations for fluid temperature, Nusselt number, velocity field, and skin friction are obtained by using the Laplace transform method. The numerical calculations are carried out and the results are presented in graphical illustrations. The influence of the memory parameter (the fractional order of the time-derivative) on the temperature and velocity fields is analyzed and a comparison between the fluid with the thermal memory and the ordinary fluid is made. It was observed that due to evolution in the time of the Caputo power-law kernel, the memory effects are stronger for the small values of the time *t*. Moreover, it is found that the fluid flow is accelerated / retarded by varying the inclination angle of the magnetic field direction.

Keywords: Free convection; Porous medium; Inclined magnetic field; Caputo time-fractional derivatives.

1. Introduction

The flows of fluids through a porous medium have many applications in different fields including, hydrogeology, fuel technology, soil science, and hydrometallurgy along with many engineering branches such as civil, biomedical, chemical, and petroleum engineering [1, 2]. Convective flows through porous media are widely encountered in fluid flow models with heat and mass transfer [3]. One of the convection flow types is the free convection flow in which the fluid motion occurs only through the density differences in the fluid due to large temperature/concentration gradients. Free convection flow models through porous media are of great practical and technological importance and have many applications in chemistry, biomedicine, and engineering [4-7]. Several mathematical models of natural convection flows through porous media have been studied in Refs. [8-11]. Electrically conducting fluids subjected to magnetic fields are common in the universe and the problem of their origin and evolution is important for understanding various physical models. The interaction between an electrically conducting fluid and a magnetic field is adequately characterized by the Magneto-hydrodynamic (MHD) equations, which are a combination of the Navier-Stokes equations of the fluid dynamics and the Maxwell's equations of the electromagnetism. MHD flows have abundant applications in astrophysics, geophysics, and metallurgy [12].

In the last years, a proper interest has given by many researchers to study the MHD convection flows through the porous media past a vertically lying infinite plate. MHD free convection of fluctuating flows through a porous medium past a vertically lying porous plate with a variable heat source and temperature was analyzed by Acharya et al. [13]. They found that heating of the plate causes back flow and the Lorentz force has retarding effects in the velocity profile. Khan and Solanki [14] investigated free convective flows over a vertical plate with heat and mass transfer in the presence of the thermal radiation. Using similarity transformation, they proposed solutions with the repeated integrals of the complementary function. The effect of Dufour number and ramp temperature on transient MHD natural convection flow past an infinite vertically lying plate in a porous medium was investigated by Marneni et al. [15]. A comparison between ramped temperature and isothermal temperature influences on the MHD free convection unsteady flow of a second grade fluid near an infinite vertical plate embedded in a porous medium was conducted by Samiulhaq et al. [16]. They illustrated that both the skin friction and the velocity values are quite less in the case of ramped temperature than in the case of isothermal temperature. Pattnaik and Biswal [17] utilized the Laplace transform method in order to obtain analytical solutions to the unsteady MHD free convective flow through porous media. They noted that both the permeability of the porous media and the magnetic field can enhance the fluid flow. Seth et al. [18] used the Laplace transform technique to solve the problem of the unsteady hydromagnetic natural convection flow with heat and mass transfer of a viscous, incompressible, electrically conducting, chemically reactive, and optically thin radiating fluid past an exponentially accelerated moving vertical plate embedded in a porous medium. Gaffar et al. [19] introduced a numerical solution for the problem of hydromagnetic free convective steady flow of non-Newtonian Eyring–Powell electrically conducting fluid over the non-isothermal vertical surface in a porous media in the presence of ion slip and Hall currents by using the Keller-Box implicit difference method. Many other analytical [20-27] and numerical [28-32] studies were conducted to investigate magnetohydrodynamic flows over a vertical plate in the porous media.

The generalization of the ordinary derivatives to the fractional derivatives with arbitrary non integer order has the advantage of having the ability to describe the memory of materials and processes and it has potential applications in physics, chemistry, and engineering [33, 34]. Several mathematicians contributed to this topic over the years and a proper attention has been given by many researchers to the application of the fractional calculus to study the new mathematical and physical fluid flow models [35-43].

This study aims to use the time-fractional derivatives with a power-law kernel to develop a model with a weakly thermal memory of the convective flow of a linearly viscous fluid over a vertical plate. The flow is influenced by the Darcy's resistance of the porous medium as well as an external magnetic field with a variable direction. The plate is infinitely extended and has a time-dependent translational motion in its vertical plane. The fractional differential energy equation is obtained by introducing a generalized fractional Fourier's constitutive equation using the time-fractional Caputo derivative. This fractional constitutive equation provides a weighted average of the temperature gradient; therefore, in this model the thermal flux is damped.

The governing equations for the hydro-magnetic free convection flow are introduced under the usual Boussinesq's approximation [44]. Analytical solutions for the temperature and velocity fields are obtained using the Laplace transform method and are expressed through Wright functions [45] and generalized G-Lorenzo-Hartley functions [46]. The Nusselt number and the skin friction are also determined. Using the properties of Wright functions, the solutions corresponding to the ordinary fluid (fluid without memory effects) are obtained as a particular case of the fractional case by making the fractional parameter tend to 1. The influence of the fractional parameter and the magnetic field direction on the fluid behavior is studied by numerical simulations and graphical illustrations. It is found that the influence of the thermal damping is stronger for the small values of the time t. This fact is in accordance with the evolution in time of the Caputo power-law kernel which decreases with time t and tends to zero for large values of the time t. Moreover, by varying the inclination angle of the magnetic field, the fluid flow is accelerated / retarded.

2. Statement of the problem

Consider the unsteady laminar free convection flow with heat transfer of an electrically conducting incompressible viscous fluid in a porous medium in the presence of an inclined magnetic field with a constant intensity. The fixed Cartesian coordinate system is chosen in such a way that $x-$axis is taken along the plate in the vertically upward direction and $y-$axis is perpendicular to the plate (Fig. 1).

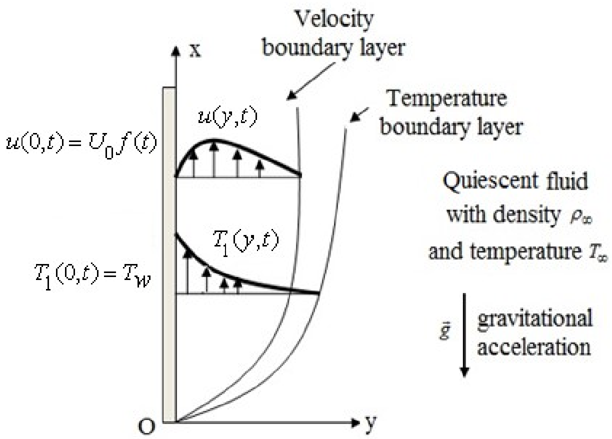

Fig. 1. Flow geometry of the problem.

The Fluid velocity is considered as $\vec{v}=(u(y,t),0,0)$ and the fluid motion is influenced by the applied magnetic field $\vec{B}=B_0\left(-\cos\theta\vec{i}+\sin\theta\vec{j}\right)$, $\theta\in(0,\pi/2]$, while the induced magnetic field and electric field are considered negligible. Generally, the magnetic body force is given by $\vec{F}=\vec{J}\times\vec{B}$, where $\vec{J}=\sigma(\vec{E}+\vec{v}\times\vec{B})$ is the current density vector, $\vec{E}$ is the electric field, and σ is electrical conductivity of the fluid. In the above-mentioned hypothesis, the *x*-component of magnetic force becomes $F_x=-\sigma B_0^2\sin^2\theta\, u\left(y,t\right)$. Initially, both the plate and the fluid are at rest and maintained at a uniform temperature T_∞. At the time $t=0^+$, the plate is moving in $x-$direction with the velocity $U_0 f(t)$ in its own plane, where $U_0>0$ is a constant with the dimension of velocity and $f(t)$ is a differentiable function of exponential order at infinite with $f(0)=0$.

Under the usual Boussinesq's approximation, the governing equation for the hydro-magnetic free convection fluid is as follows [15, 17]:

$$\frac{\partial u}{\partial t}=\nu\frac{\partial^2 u}{\partial y^2}+g\beta_T(T_1-T_\infty)-\frac{\nu}{k_1}u-\frac{\sigma B_0^2}{\rho}\sin^2\theta u, \tag{1}$$

where $T_1(y,t)$ is the temperature of the fluid, ν is the kinematics viscosity, g is the acceleration due to gravity, β_T is the thermal expansion coefficient, k_1 is the permeability of the porous medium, and ρ is the constant fluid density. The local thermal balance equation is:

$$\rho c_p\frac{\partial T_1}{\partial t}=-\frac{\partial q}{\partial y}, \tag{2}$$

where c_p is the specific heat at the constant pressure and $q(y,t)$ is the thermal heat flux. In this study, a thermal process with a power-law weakly memory is explored, therefore, regarding the thermal heat flux, the fractional generalized Fourier's law is considered as suggested by Henry et al. [47] and Hristov [48]:

$$q=-k_\beta\,{}^{C}D_t^{1-\beta}\left(\frac{\partial T_1}{\partial y}\right),\ 0<\beta\le 1, \tag{3}$$

where, the generalized thermal k_β conductivity has the dimension unit $\left[\frac{Kg\,m}{K\,s^{\beta+2}}\right]$, and ${}^{C}D_t^{\beta}$ denotes the Caputo fractional derivative operator defined as [48]:

$${}^{C}D_t^{\beta}f(t)=\begin{cases}\left(h_\beta * \dot{f}\right)(t),\ h_\beta(t)=\dfrac{t^{-\beta}}{\Gamma(1-\beta)},\dot{f}(t)=\dfrac{df(t)}{dt},0<\beta<1,\\ {}^{C}D_t^{0}f(t)=f(t),\ {}^{C}D_t^{1}f(t)=\dot{f}(t).\end{cases} \tag{4}$$

It is observed that for $\beta=1$, Eq. (3) becomes the well-known Fourier's law. Let's recall the Riemann-Liouville integral operator $I_t^{\gamma}f(t)$ defined as [49]

$$I_t^{\gamma}f(t)=\begin{cases}\left(h_{1-\gamma}*f\right)(t),\ \gamma>0,\\ I_t^{0}f(t)=f(t).\end{cases} \tag{5}$$

Using Eqs. (4) and (5), it is easy to obtain the following properties of the time-fractional operators:

$$\begin{aligned}&\left(h_\gamma * h_{1-\gamma}\right)(t)=1,\\ &{}^{C}D_t^{1-\gamma}f(t)=I_t^{\gamma}\dot{f}(t),\ {}^{C}D_t^{\gamma}f(t)=I_t^{1-\gamma}\dot{f}(t),\ 0\le\gamma\le 1,\\ &\left(I_t^{\gamma}\circ{}^{C}D_t^{\gamma}\right)f(t)=\left(h_{1-\gamma}*(h_\gamma*\dot{f})\right)(t)=(1*\dot{f})(t)=f(t)-f(0).\end{aligned} \tag{6}$$

Obviously, if $f(0)=0$, the Caputo derivative operator is invertible to the left, therefore,

$$\left(I_t^{\gamma}\circ{}^{C}D_t^{\gamma}\right)f(t)=f(t),f(0)=0. \tag{7}$$

Along with Eqs. (1)-(3), the following initial and boundary conditions are considered:

$$u(y,0)=0, T_1(y,0)=T_\infty,\ y\ge 0, \tag{8}$$

$$u(0,t)=U_0 f(t),\, T_1(0,t)=T_w,\, t>0, \tag{9}$$

$$u(y,t)\to 0,\, T_1(y,t)\to T_\infty \text{ as } y\to\infty. \tag{10}$$

Making the transformation yields

$$T_2(y,t)=\frac{T_1(y,t)-T_\infty}{T_w-T_\infty}, \tag{11}$$

and Eqs. (2) and (3) are written as

$$\rho c_p(T_w-T_\infty)\frac{\partial T_2(y,t)}{\partial t}=-\frac{\partial q}{\partial y}, \tag{12}$$

$$q=-k_\beta(T_w-T_\infty)\,{}^C D_t^{1-\beta}\left(\frac{\partial T_2(y,t)}{\partial y}\right), \tag{13}$$

with the corresponding initial and boundary conditions,

$$T_2(y,0)=0,\; T_2(0,t)=1,\; T_2(y,t)\to 0 \textit{ as } y\to\infty. \tag{14}$$

Eliminating the thermal flux q from Eqs. (12) and (13), the fractional energy equation is obtained as

$$\frac{\partial T_2(y,t)}{\partial t}=\alpha_\beta\,{}^C D_t^{1-\beta}\left(\frac{\partial^2 T_2(y,t)}{\partial y^2}\right), \tag{15}$$

where $\alpha_\beta=\frac{k_\beta}{\rho c_p}\left[\frac{m^2}{s^\beta}\right]$ is the generalized thermal diffusion coefficient. Now, applying the fractional integral operator $I_t^{1-\beta}$ to Eq. (15) and using the properties of Eqs. (6) and (7) together with the initial condition (14), the following fractional differential equation of the thermal diffusion process is obtained:

$${}^C D_t^\beta T_2=\alpha_\beta\frac{\partial^2 T_2}{\partial y^2} \tag{16}$$

It is important to emphasize that Eq. (16) has the same form as the equation obtained through so called "the formalization fractionalization" used in many studies, cases in which it would seem that the temperature is damped by the fractional derivative. In the present study, Eq. (13) shows that the thermal flux is damped because the fractional derivative is applied to the temperature gradient $\partial T_2/\partial y$. Introducing the following dimensionless variables and parameters

$$y^*=\frac{yU_0}{\nu},\, t^*=\frac{tU_0^2}{\nu},\, u^*=\frac{u}{U_0},\, T^*(y^*,t^*)=T_2^*(\nu y^*/U_0,\nu t^*/U_0^2), \tag{17}$$

into Eqs. (1), (3) to (5), and (8) and dropping the “*” notation, the dimensionless governing equations are obtained as

$$\frac{\partial u(y,t)}{\partial t}=\frac{\partial^2 u(y,t)}{\partial y^2}+GrT(y,t)-\left(M\sin^2\theta+\frac{1}{K}\right)u(y,t), \tag{18}$$

$$\mathrm{P}_\beta\,{}^C D_t^\beta T(y,t)=\frac{\partial^2 T(y,t)}{\partial y^2}. \tag{19}$$

The dimensionless initial and boundary conditions are

$$u(y,0)=0,\; T(y,0)=0;\; y>0, \tag{20}$$

$$u(0,t)=f(t),\; T(0,t)=1;\; t>0, \tag{21}$$

$$u(y,t)\to 0,\; T(y,t)\to 0 \text{ as } y\to\infty, \tag{22}$$

where $\mathrm{P}_\beta=\frac{\nu^{2-\beta}}{U_0^{2(1-\beta)}\alpha_\beta}$ is the generalized Prandtl number, $Gr=\frac{g\beta_T\nu(T_w-T_\infty)}{U_0^3}$ is the thermal Grashof number,

$M=\dfrac{\sigma B_0^2\nu}{\rho U_0^2}$ is the magnetic parameter, and $K=\dfrac{k_1U_0^2}{\nu^2}$ is the permeability parameter.

3. Solution of the problem

3.1. Temperature field

Applying the Laplace transform to Eqs. (19), (21), (22) and using the initial condition (20), the following transformed problem is obtained:

$$\mathrm{P}_\beta s^\beta \bar{T}(y,s)=\frac{\partial^2\bar{T}(y,s)}{\partial y^2}, \tag{23}$$

$$\bar{T}(y,s)=\frac{1}{s},\ \bar{T}(y,s)\to 0,\ \text{as}\ \ y\to\infty. \tag{24}$$

The solution of the fractional differential Eq. (23) subjected to the conditions of Eq. (24) is given by

$$\bar{T}(y,s)=\frac{1}{s}e^{-y\sqrt{\mathrm{P}_\beta s^\beta}}. \tag{25}$$

Taking inverse Laplace transform yields

$$T(y,t)=\phi\left(1,-\frac{\beta}{2},-y\sqrt{\mathrm{P}_\beta}t^{-\frac{\beta}{2}}\right);\ 0<\beta<1 \tag{26}$$

where

$$t^{a-1}\phi\left(a,-\sigma,-bt^{-\sigma}\right)=L^{-1}\left\{\frac{1}{s^a}e^{-bs^\sigma}\right\},\ a\geq 0,\ b>0,\ 0<\sigma<1, \tag{27}$$

is the Wright's function [45]. It is obtained from Eq. (27) that for $a=0$ and $\sigma=1$

$$\phi\left(0,-1,-bt^{-1}\right)=L^{-1}\left\{e^{-bs}\right\}=\delta(t-b), \tag{28}$$

and for $a=1,\ \sigma=\dfrac{1}{2}$

$$\phi\left(1,-\frac{1}{2},-bt^{-\frac{1}{2}}\right)=L^{-1}\left\{\frac{1}{s}e^{-b\sqrt{s}}\right\}=erfc\left(\frac{b}{2\sqrt{t}}\right). \tag{29}$$

Now, for $\beta=1$, the temperature field to the ordinary case is obtained as

$$T(y,t)=erfc\left(\frac{y\sqrt{\mathrm{P}_1}}{2\sqrt{t}}\right), \tag{30}$$

Where $P_1=\dfrac{\mu c_p}{k_1}=Pr$ is the classical Prandtl number. The Nusselt number is given by:

$$Nu=-\left.\frac{\partial T(y,t)}{\partial y}\right|_{y=0}=-L^{-1}\left\{\left.\frac{\partial\bar{T}(y,s)}{\partial y}\right|_{y=0}\right\}=L^{-1}\left\{\frac{\sqrt{\mathrm{P}_\beta}}{s^{1-\frac{\beta}{2}}}\right\}=\frac{\sqrt{\mathrm{P}_\beta}t^{-\frac{\beta}{2}}}{\Gamma\left(1-\frac{\beta}{2}\right)};\ 0<\beta\leq 1. \tag{31}$$

3.2 Velocity field

Applying the Laplace transform to Eqs. (18), $(21)_1$, $(22)_1$ and using the initial condition $(20)_1$, the following differential equation for the Laplace transform of velocity is obtained:

$$s\bar{u}(y,s)=\frac{\partial^2\bar{u}(y,s)}{\partial y^2}+Gr\bar{T}(y,s)-\left(M\sin^2\theta+\frac{1}{K}\right)\bar{u}(y,s), \tag{32}$$

with the boundary conditions

$$u(0,s)=F(s)\ ,\bar{u}(y,s)\to 0\ \ as\ \ y\to\infty, \tag{33}$$

$F(s)$ being the Laplace transform of the function $f(t)$. Using Eq. (25) for the Laplace transform of the temperature, Eq. (32) becomes:

$$\left[s+\left(M\sin^2\theta+\frac{1}{K}\right)\right]\bar{u}(y,s)=\frac{\partial^2\bar{u}(y,s)}{\partial y^2}+Gr\frac{e^{-y\sqrt{\mathrm{P}_\alpha s^\beta}}}{s} \tag{34}$$

The solution of Eq. (34) along with the boundary conditions (33) is

$$\bar{u}(y,s)=F(s)e^{-y\sqrt{s+\left(M\sin^2\theta+\frac{1}{K}\right)}}-\frac{Gr}{\left[s-\mathrm{P}_\beta s^\beta+\left(M\sin^2\theta+\frac{1}{K}\right)\right]}\left[\frac{e^{-y\sqrt{s+\left(M\sin^2\theta+\frac{1}{K}\right)}}}{s}-\frac{e^{-y\sqrt{\mathrm{P}_\beta s^\beta}}}{s}\right]. \tag{35}$$

In order to obtain the inverse Laplace transform, Eq. (35) is written in the equivalent form as:

$$\bar{u}(y,s)=\left(s+M\sin^2\theta+\frac{1}{K}\right)F(s)\frac{e^{-y\sqrt{s+\left(M\sin^2\theta+\frac{1}{K}\right)}}}{s+M\sin^2\theta+\frac{1}{K}}+A(s)\frac{e^{-y\sqrt{\mathrm{P}_\beta s^\beta}}}{s}-B(s)\frac{e^{-y\sqrt{s+\left(M\sin^2\theta+\frac{1}{K}\right)}}}{s+M\sin^2\theta+\frac{1}{K}}, \tag{36}$$

with

$$A(s)=\frac{Gr}{\left[s-\mathrm{P}_\beta s^\beta+\left(M\sin^2\theta+\frac{1}{K}\right)\right]}=Gr\sum_{n=0}^{\infty}\frac{(-1)^n\left(M\sin^2\theta+\frac{1}{K}\right)^n s^{-(n+1)\beta}}{\left(s^{1-\beta}-\mathrm{P}_\beta\right)^{n+1}}, \tag{37}$$

$$B(s)=\left[1+\left(M\sin^2\theta+\frac{1}{K}\right)\frac{1}{s}\right]A(s). \tag{38}$$

The inverse Laplace transform of the functions $A(s)$ and $B(s)$ are

$$a(t)=L^{-1}\{A(s)\}=Gr\sum_{n=0}^{\infty}(-1)^n\left(M\sin^2\theta+\frac{1}{K}\right)^n G_{1-\beta,-(n+1)\beta,n+1}\left(t,\mathrm{P}_\beta\right), \tag{39}$$

where $G_{a,b,c}(t,d)$ is generalized G-Lorenzo-Hartley function [46] defined as

$$G_{a,b,c}(t,d)=L^{-1}\left\{\frac{s^b}{\left(s^a-d\right)^c}\right\}=\sum_{j=0}^{\infty}\frac{d^j\Gamma(j+1)}{\Gamma(c)\Gamma(j+c)}\frac{t^{(j+c)a-b-1}}{\Gamma[(j+c)a-b]}\ \text{for}\ \left|\frac{d}{s^a}\right|<1,\ \mathrm{Re}(s)>0,\ \mathrm{Re}(ac-b)>0, \tag{40}$$

$$\begin{aligned}b(t)=L^{-1}\{B(s)\}&=\left[\delta(t)+\left(M\sin^2\theta+\frac{1}{K}\right)H(t)\right]\times a(t)=a(t)+\left(M\sin^2\theta+\frac{1}{K}\right)\int_0^t a(\tau)d\tau\\&=a(t)+Gr\sum_{n=0}^{\infty}(-1)^n\left(M\sin^2\theta+\frac{1}{K}\right)^{n+1}G_{1-\beta,-(n+1)\beta+1,n+1}\left(t,\mathrm{P}_\beta\right).\end{aligned} \tag{41}$$

In the above-mentioned relation, the following property of G-functions is used:

$$\int G_{a,b,c}(t,d)dt=G_{a,b+1,c}(t,d). \tag{42}$$

Now, using Eqs. (39) and (41), the convolution theorem, and the following formula

$$L^{-1}\left\{\frac{e^{-y\sqrt{s+m}}}{s+m}\right\}=erfc\left(\frac{y}{2\sqrt{t}}\right)e^{-mt}, \tag{43}$$

the velocity field is obtained as

$$\begin{aligned} u(y,t) &= \left[f'(t) + \left(M\sin^2\theta + \frac{1}{K}\right)f(t)\right] \times \left[e^{-\left(M\sin^2\theta+\frac{1}{K}\right)t} erfc\left(\frac{y}{2\sqrt{t}}\right)\right] + \\ & a(t)\times\phi\left(1, -\frac{1}{2}, -y\sqrt{P_\beta} t^{-\frac{\beta}{2}}\right) - b(t)\times\left[e^{-\left(M\sin^2\theta+\frac{1}{K}\right)t} erfc\left(\frac{y}{2\sqrt{t}}\right)\right], \; 0<\beta<1. \end{aligned} \tag{44}$$

For $\beta = 1$, using the particular form of the Wright function given by Eq. (29), Eq. (44) becomes

$$\begin{aligned} u(y,t) &= \left[f'(t) + \left(M\sin^2\theta + \frac{1}{K}\right)f(t)\right] \times \left[e^{-\left(M\sin^2\theta+\frac{1}{K}\right)t} erfc\left(\frac{y}{2\sqrt{t}}\right)\right] + \\ & a(t)\times erfc\left(\frac{y\sqrt{P_1}}{2\sqrt{t}}\right) - b(t)\times\left[e^{-\left(M\sin^2\theta+\frac{1}{K}\right)t} erfc\left(\frac{y}{2\sqrt{t}}\right)\right], \; 0<\beta<1. \end{aligned} \tag{45}$$

The skin friction coefficient is given by

$$\begin{aligned} C_{f\beta} &= -\left.\frac{\partial u(y,t)}{\partial y}\right|_{y=0} = -L^{-1}\left\{\left.\frac{\partial \bar{u}(y,s)}{\partial y}\right|_{y=0}\right\} = L^{-1}\left\{\frac{\left(s+M\sin^2\theta+\frac{1}{K}\right)F(s) - B(s)}{\sqrt{s+M\sin^2\theta+\frac{1}{K}}} + \sqrt{P_\beta}\frac{A(s)}{s^{1-\frac{\beta}{2}}}\right\} \\ &= \left[f'(t) + \left(M\sin^2\theta+\frac{1}{K}\right)f(t) - b(t)\right]\times\frac{1}{\sqrt{\pi t}}e^{-\left(M\sin^2\theta+\frac{1}{K}\right)t} + \sqrt{P_\beta}\, a(t)\times\frac{t^{-\frac{\beta}{2}}}{\Gamma\left(1-\frac{\beta}{2}\right)}, \; 0<\beta<1, \end{aligned} \tag{46}$$

which for $\beta = 1$, it yields

$$C_f = \left[f'(t) + \left(M\sin^2\theta+\frac{1}{K}\right)f(t) - b(t)\right]\times\frac{1}{\sqrt{\pi t}}e^{-\left(M\sin^2\theta+\frac{1}{K}\right)t} + \sqrt{P_1}\, a(t)\times\frac{1}{\sqrt{\pi t}}. \tag{47}$$

4. Numerical results and discussions

In this study, the convective flows of a linearly viscous fluid near a vertical isothermal plate in the presence of Darcy's resistance of the porous medium and the magnetic field of a variable direction were investigated. The heat transfer process was modeled with a generalized fractional constitutive equation which provides a damped thermal flux. To do this, the time-fractional derivative with a power-law singular kernel introduced by Caputo was required. Using the Laplace transform, the closed forms of the non-dimensional temperature and velocity fields, expressed with the Wright functions and the generalized G-Lorenzo-Hartley functions were obtained. The Nusselt number and the skin friction coefficient were also determined.

The solutions for the ordinary fluid were obtained as a particular case of the solutions for the fractional model when the memory parameter (the fractional derivative order) tended to 1. The behavior of the fluid with a thermal memory described by the fractional model and the behavior of the ordinary fluid described by the ordinary differential equations was analyzed by numerical simulations and graphical representations.

It was found that the effects of thermal memory are stronger for the small values of the time t. This is in line with the variation in time of the kernel of Caputo time-fractional derivative which tends to infinity for t tending to zero and is decreasing to zero for the large values of the time t. In order to get information about the heat transfer and the fluid motion, variations of the temperature and velocity were studied. Specifically, the influence of the memory (fractional) parameter β on the heat and momentum transport was analyzed. Moreover, the influence of the inclination angle of the magnetic field direction on the fluid velocity was studied. The results are presented in Figs. 2-10. For numerical calculations and to draw graphs, Mathcad software was employed. To do this analysis, the plate motion described by the function $f(t) = \sin(\omega t)$ and the following values of the material parameters were considered: $\rho = 1.205\,(kg/m^3)$, $\nu = 1.511\times10^{-5}(m/s)$, $k = 0.606(Kgm/s^3K)$, $c_p = 1.005\times10^3(m^2/s^2K)$, $U_0 = 0.01(m/s)$ respectively, non-dimensional parameters $Gr = 4$, $M = 0.1$, $\alpha = \pi/6$, $K = 3.5$, $\omega = \pi/4$. For the generalized thermal conductivity and the generalized Prandtl number, the expressions $k_\beta = (2-\beta)(0.0257)$, $P_\beta = \left(\left(1.511\times10^{-5}\right)^{2-\beta}\rho c_p\right)/\left(U_0^{2(1-\beta)}k_\beta\right)$. were used. When the fractional parameter β belongs the interval $[0,1]$, the generalized parameter P_β varies in the interval $[P_0, P_1]$, $P_0 = 0.054$, $P_1 = \Pr = 0.712$.

In Fig. 2, the sketched profiles of the dimensionless temperature for different values of the fractional parameter β, versus *t*, and versus y are presented. It is observed that for small values of the time, the temperature is decreasing with the increasing of β and the temperature shows an opposite behavior for large values of the time t. In order to verify the accuracy of this solution (26) for the temperature field, a comparison with the temperature field given by Ref. [14], Eq. (3.6) was made, where the effects of the thermal radiation are considered and the temperature solution is expressed with the repeated integral of the complementary error function. In the particular case of $\beta = 1$ in our solution and the parameter of thermal boundary condition $r = 1$ and the thermal radiation parameter $N \to \infty$ in the solution of Ref. [14], Eq. (3.6), both solutions are identical as can be seen in Fig. 3.

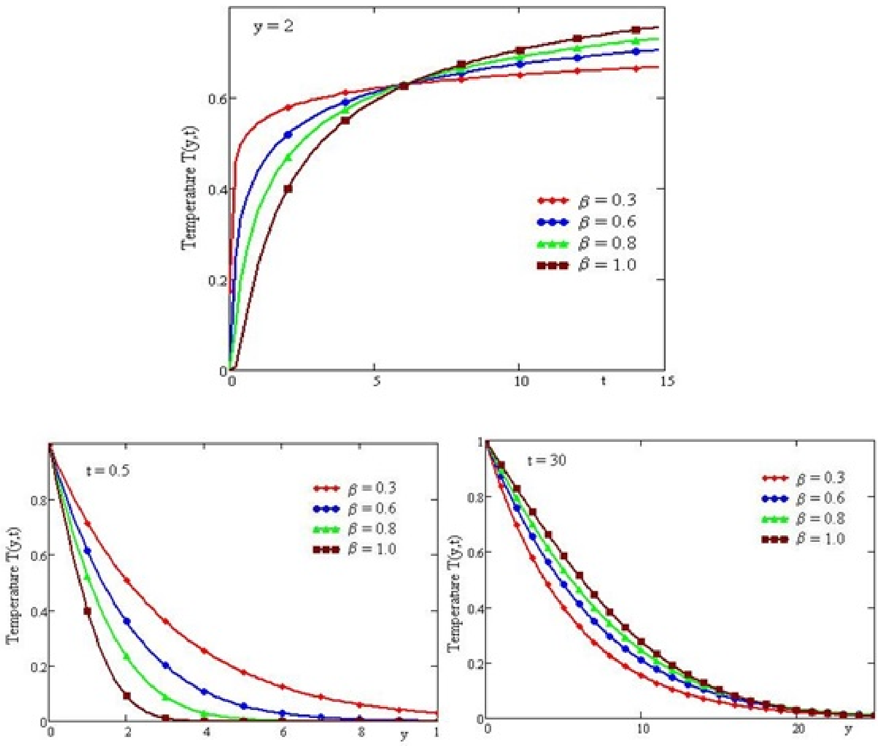

Fig. 2. Profiles of dimensionless temperature $T(y,t)$ versus y for β variation and two values of the time t.

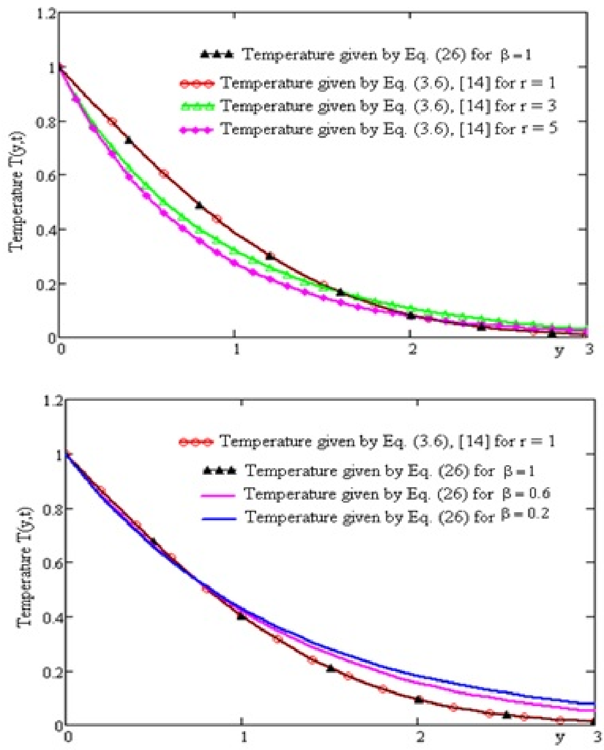

Fig. 3. Comparison between solutions given by Eq. (26) and Ref. [14], Eq. (3.6) for t = 0.5 and Pr = 0.7.

In Figs. 4 and 5, the profiles of dimensionless velocity for different values of the fractional parameter β at two small/large values of the time t, respectively, are drawn. At small time, the velocity is decreasing by increasing the value of β, respectively, and the opposite behavior appears at large values of the time t. The observed behaviors are explained by the properties of the Caputo kernel. For small values of the time t and small values of the fractional parameter β, the power-law kernel increases, therefore, the thermal memory is strongly affected by the weight function of the thermal gradient. The similar results are obtained from Figs. 6 and 7 in which the temperature and velocity fields are presented when both parameters (y, t) are simultaneous variables. The grid points for the plotting are $(y_i = 0.015i, t_j = 0.01j), i, j = 1, 2...100.$

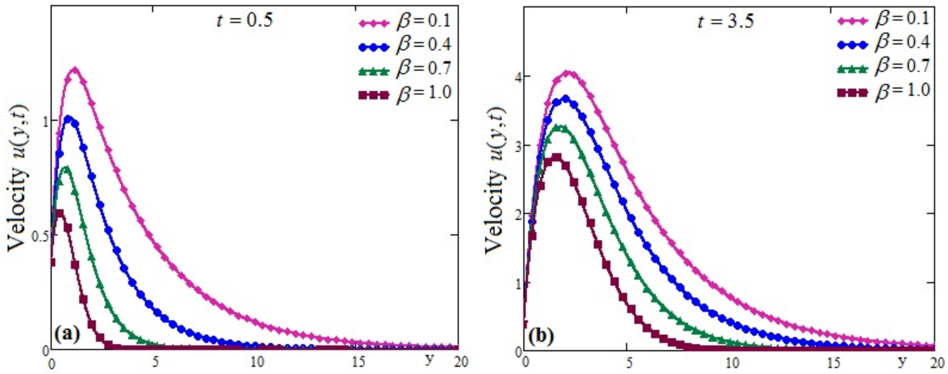

Fig. 4. Profiles of the dimensionless velocity $u(y,t)$ versus y for β variation and two small values of the time t.

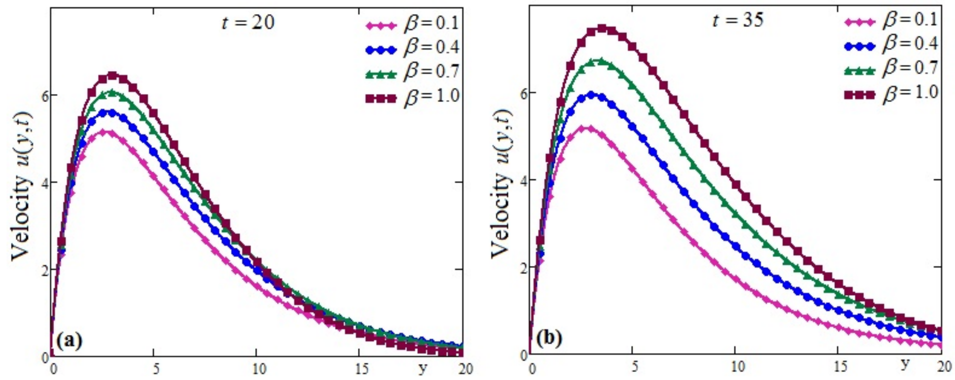

Fig. 5. Profiles of dimensionless velocity $u(y,t)$ versus y for β variation and two large values of the time t.

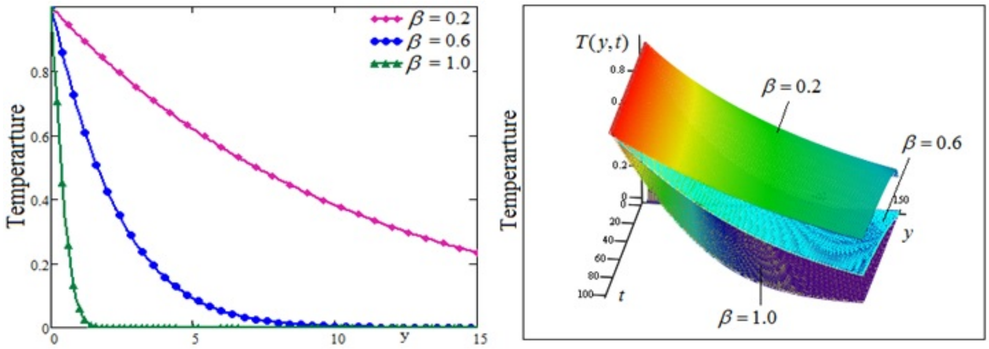

Fig. 6. Variation with *y, t*, and β of the non-dimensional temperature $T(y,t)$.

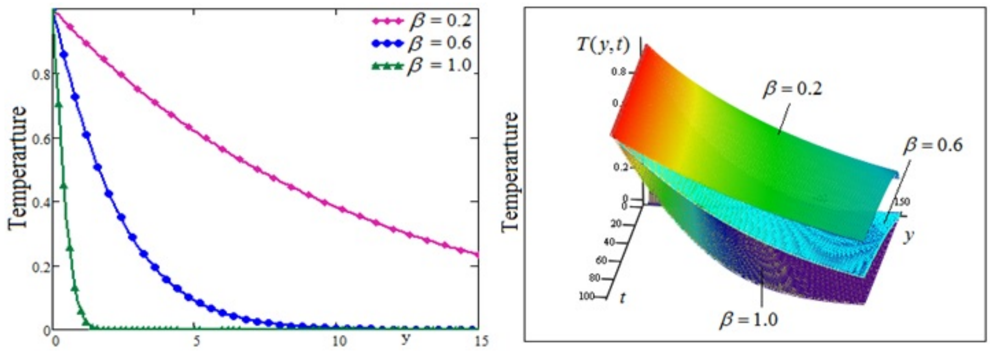

Fig. 7. Variation with *y, t*, and β of the non-dimensional velocity $u(y,t)$.

The influence of the magnetic parameter M on the fluid motion is presented in Fig 7. By increasing the value of magnetic parameter M, the Lorentz force increases and fluid motion is slowed because this force opposes the fluid velocity. The influence of the direction of the applied magnetic field, that is the influence of the angle θ on the fluid velocity, is presented in Fig. 9. By increasing the value of θ, the velocity decreases. The magnetic field has the maximum effect on the fluid motion when it is applied perpendicular to the flow direction. This result is in accordance with the Lorentz force $(\sigma B_0^2 / \rho)\sin^2(\theta)$ which has the maximum value for $\theta = \pi / 2$. The time-variation of the fluid velocity for different values of the memory parameter β is shown in Fig. 10. It is pointed out that for small values of time, the velocity is decreasing by increasing the fractional parameter. In case of a critical value of the time t, the fluid velocity has an opposite behavior.

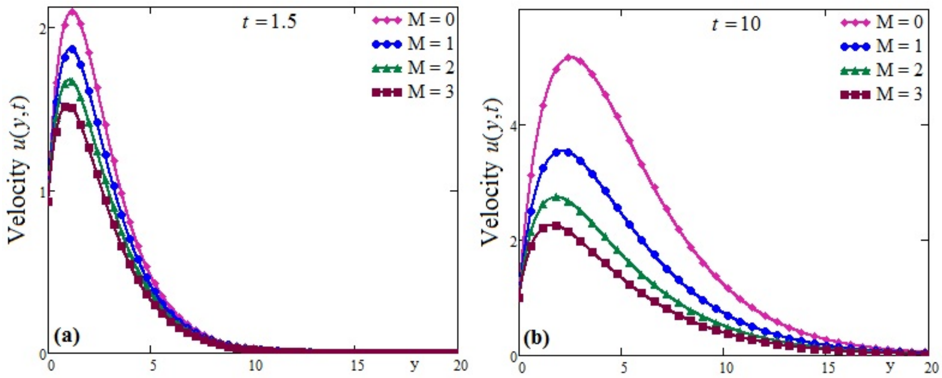

Fig. 8. Profiles of the dimensionless velocity $u(y,t)$ versus y for M variation and two values of the time t.

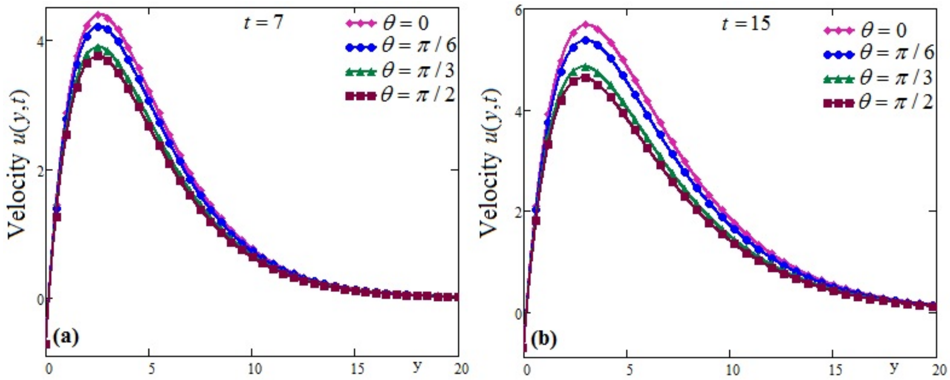

Fig. 9. Profiles of the dimensionless velocity $u(y,t)$ versus y for θ variation and two values of the time t.

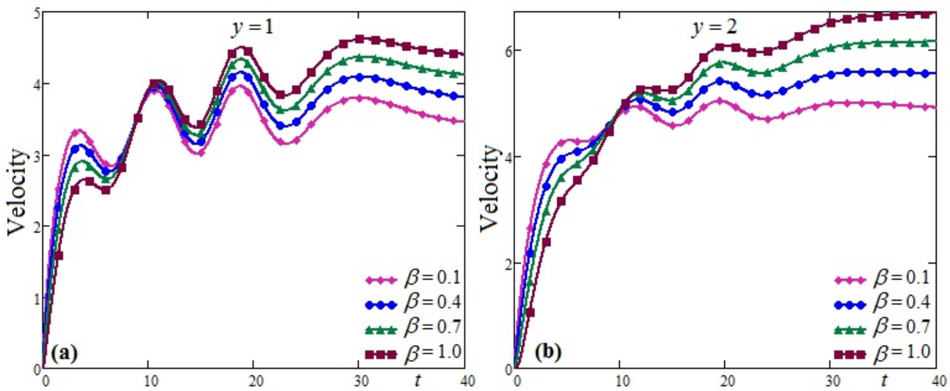

Fig. 10. Profiles of the dimensionless velocity $u(y,t)$ versus t for β variation and two values of the spatial coordinate y.

5. Conclusions

In the present study, the magneto-hydrodynamic convection flows with a thermal memory are investigated. Using Caputo time-fractional derivative with a power-law kernel, the thermal process with a weakly memory is defined by introducing the generalized fractional Fourier's constitutive equation thermal flux-temperature gradient. This fractional constitutive equation provides a weighted average of the temperature gradient; therefore, in this model the thermal flux is damped.

The fluid flow is influenced by the Darcy's resistance of the porous medium as well as an external magnetic field with a variable direction. The analytical solutions for the temperature and velocity fields are obtained using the Laplace transform method and are expressed through the Wright functions along with the generalized G-Lorenzo-Hartley functions. Using the properties of Wright functions, the solutions corresponding to the ordinary fluid (fluid without memory effects) are obtained as a particular case of the fractional case by making the fractional parameter tend to 1.

The influence of the fractional parameter and the magnetic field direction on the fluid behavior is studied by numerical simulations and graphical illustrations. It is found that the influence of the thermal damping is stronger for the small values of the time *t*. This fact is in accordance with the evolution in time of the Caputo power-law kernel which is decreasing with time *t* and tends to zero for large values of the time t. Furthermore, by varying the inclination angle of the magnetic field, the fluid flow is accelerated / retarded.

Conflict of Interest

The authors declare no conflict of interest.

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