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Fractional Thermoelasticity Model of a 2D Problem of Mode-I Crack in a Fibre-Reinforced Thermal Environment

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Abstract. A model of fractional-order of thermoelasticity is applied to study a 2D problem of mode-I crack in a fibre-reinforced thermal environment. The crack is under prescribed distributions of heat and pressure. The normal mode analysis is applied to deduce exact formulae for displacements, stresses, and temperature. Variations of field quantities with the axial direction are illustrated graphically. The results regarding the presence and absence of fiber reinforcement and fractional parameters are compared. Some particular cases are also investigated via the generalized thermoelastic theory. The presented results can be applied to design different fibre-reinforced isotropic thermoelastic elements subjected to the thermal load in order to meet special technical requirements.

Keywords: Mode-I crack; Fractional-order theory; Thermoelasticity; Fibre-reinforced; Normal mode analysis.

1. Introduction

The theory of thermoelasticity has been well embedded to include the effect of temperature change. Accordingly, the temperature should be coupled with the elastic strain. The classic heat transfer, Fourier equation, is widely applied in various engineering applications. The classical theory of thermoelasticity (see Nowacki [1, 2]) is based on the Fourier hypothesis of thermal conductivity. In this theory, the temperature distribution is regulated by a partial differential equation of a parabolic type. Conceptually, it predicts that the thermal signal is immediately felt in the body. This reveals an infinite speed of propagation of a thermal signal that is physically impractical, especially for a short time. Therefore, using the Fourier equation can lead to discrepancies in certain particular conditions, such as heat transfer at low temperature, high frequency heat transfer or very high heat flux, and so on.

Generalized thermoelasticity, which allows a finite speed of propagation of thermoelastic perturbations, has gained a lot of attention in recent years. The theories of Lord and Shulman [3] (LS) and Green and Lindsay [4] (GL) expanded the coupled one by inserting thermal relaxation times in their constitutive equations. These generalized thermoelasticity theories eliminated the paradox of infinite velocity of heat propagation. Green and Naghdi [5] (GN) formulated another generalized theory which is said to be the theory of thermoelasticity without energy dissipation. It contains "thermal displacement gradient" amidst its independent constitutive variables, and varies from LS and GL theories in that it does not allow the thermal energy to dissipate (Ignaczak and Ostoja-Starzewski [6]).

An idea of inserting a continuous self-reinforcement at each point of the elastic medium was presented by Belfield et al. [7] and this technique was used by Verma and Rana [8] to study rotation of circular cylindrical tubes. Different problems of surface

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waves in fiber-reinforced anisotropic elastic solids were discussed by many investigators (Sengupta and Nath [9]; Hashin and Rosen [10]; Singh and Singh [11]; Singh [12, 13]; Kumar and Gupta [14]; Abbas and Abd-Alla [15]; Ailawalia and Budhiraja [16]). The effects of anisotropy, hydrostatic initial stress, and magnetic field on the incompressible and thermoelastic fibre-reinforced solid were investigated. Prabhakar et al. [17] analyzed the influence of magneto-thermo-electromechanical effects on band structure calculations by using the fully coupled model. Recently, El-Naggar et al. [18] studied the effect of thermomagnetic field, rotation, initial stress, and voids on reflection of P-wave with one relaxation time. Abd-Alla et al. [19] investigated Love waves in an inhomogeneous orthotropic magneto-elastic layer subjected to an initial stress overlying a semi-infinite solid.

The fractional calculus is a natural extension of classical mathematics. Many definitions of fractional derivative have been presented and are demonstrated to be equivalent (Podlubny [20]). Recently, fractional calculus has been involved in field of thermoelasticity [21-27]. Othman et al. [21] presented the effect of fractional parameter on the plane waves of magneto-thermo-elastic diffusion with room temperature-dependent elastic solid. Povstenko [22] organized a quasi-static uncoupled thermoelasticity model based on the heat conduction equation with a fractional-order time derivative. In fact, he applied Caputo fractional derivative (Caputo [28]) to obtain the stresses corresponding to fundamental solution of Cauchy problem for the fractional-order heat conduction equation in both 1D and 2D situations. Youssef [23] presented a new generalized thermoelasticity model by considering the heat conduction with a fractional-order. Sherief et al. [24] organized another theory in generalized thermoelasticity via the use of fractional time derivatives. Ezzat and El Karamany [25] constructed a new mathematical two-temperature magneto-thermoelasticity theory by considering the fractional-order heat conduction law. Abouelregal [26] proposed the generalized thermoelasticity theory based on a fractional-order model to solve a 1D boundary value problem of a semi-infinite piezoelectric solid. Recently, Zenkour and Abouelregal [27] presented the state-space approach for an infinite solid with a spherical cavity based on the two-temperature generalized thermoelasticity theory and the fractional heat conduction. In addition, Abbas and Zenkour [29] presented the two-temperature generalized thermoelastic interaction in an infinite fiber-reinforced anisotropic plate with a circular hole and two relaxation times.

The aim of this study is to determine normal distributions of displacement, stress, and temperature in a fibre-reinforced generalized thermoelastic solid via the effect of fractional-order thermoelasticity. The normal mode analysis is used to get exact formulae of displacements, stresses, and temperatures. The variations of the considered field quantities with the axial direction are illustrated graphically. The results with the inclusion and absence of fiber-reinforcing and fractional parameter are compared. The results are reduced to the corresponding classical ones when the reinforced elastic parameters tend to zero and the medium became isotropic.

2. Basic Equations

Consider a homogeneous transversely isotropic fiber-reinforced solid without heat sources. The linear equations governing thermoelastic interactions via generalized thermoelasticity with fractional-order may be expressed in the following forms:

1. The equations of motion

$$\sigma_{ij,j} + F_i = \rho \ddot{u}_i, \tag{1}$$

where σ_{ij} represent components of stress tensor, F_i denote body forces, ρ represents material density, and u_i denote displacements.

2. The modified fractional order heat conduction equation has the following form [24]:

$$KT_{,ii} = \rho C_E \left(\delta + t_0 \frac{\partial^\nu}{\partial t^\nu}\right) \dot{T} + \gamma T_0 \left(1 + t_0 \frac{\partial^\nu}{\partial t^\nu}\right) \dot{u}_{i,i}, \;\; 0 < \nu \le 1, \tag{2}$$

where K represents thethermal conductivity, t_0 denotes a constant with dimension of time that acts as a relaxation time, ν denotes the fractional parameter, C_E represents the specific heat at a uniform strain, T_0 represents temperature of the medium in its natural state, supposed to be such as $|(T - T_0)/T| = 1$ and $\gamma = (3\lambda + 2\mu)\alpha_t$ in which α_t denotes thermal expansion coefficient, and λ, μ are Lamé constants.

In the case that any function $f(t)$ is absolutely continuous, then

$$\lim_{\nu \to 1} \frac{\mathrm{d}^\nu}{\mathrm{d}t^\nu} f(t) = f'(t). \tag{3}$$

As stated by Kimmich [30], Eq. (2) depicts different cases of diffusion as follows:

- weak diffusion (sub-diffusion): $0 < \nu < 1$,
- normal diffusion: $\nu = 1$,
- strong diffusion (super-diffusion): $0 < \nu < 2$,
- ballistic diffusion: $\nu = 2$.

3. The constitutive relations for fibre-reinforced linearly elastic anisotropic medium with respect to the reinforcement direction $\vec{b} = (b_1, b_2, b_3)$, with $b_1^2 + b_2^2 + b_3^2 = 1$, are

$$\sigma_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu_T e_{ij} + \alpha(b_k b_m e_{km}\delta_{ij} + b_i b_j e_{kk}) + 2(\mu_L - \mu_T)(b_k b_i e_{kj} + b_k b_j e_{ki}) + \beta b_k b_m e_{km} b_i b_j - \gamma(T - T_0), \tag{4}$$

where $e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ denote strain components, μ_T and μ_L denote elastic constants, α, β, and $\xi = \mu_L - \mu_T$ represents reinforcement parameters.

The theories of coupled thermoelasticity, generalized thermoelasticity with one relaxation time, and the generalized theory without energy dissipation follow as limited cases depending on the value of δ, t_0, and ν.

The heat conduction equation, given in Eq. (2), in limiting case $\nu \to 1$ and $\delta = 1$ transforms to:

$$KT_{,ii} = \left(\frac{\partial}{\partial t} + t_0\frac{\partial^2}{\partial t^2}\right)(\rho C_E T + \gamma T_0 e_{k,k}), \tag{5}$$

which is the same equation derived by the generalized theory with one relaxation time. While in a limiting case, when $\nu \to 1$, $t_0 = 1$, and $\delta = 0$, it transforms to:

$$KT_{,ii} = \frac{\partial^2}{\partial t^2}(\rho C_E T + \gamma T_0 e_{k,k}), \tag{6}$$

which is the same equation of generalized theory without energy dissipation introduced by Green and Naghdi. The coupled theory of thermoelasticity can be obtained from Eq. (2) in the limiting case $\nu \to 0$, $t_0 \to 0$, and $\delta = 1$, as

$$KT_{,ii} = \frac{\partial}{\partial t}(\rho C_E T + \gamma T_0 e_{k,k}). \tag{7}$$

3. Theoretical Formulation

In the present study, a problem of a fibre-reinforced anisotropic half-space ($x \geq 0$) with a mode-I crack which is defined by $|x| \leq 2l$ is considered. All functions depend on time t and coordinates (x, y, z). In this problem, it is assumed that the plane boundary of the crack is subjected to a prescribed normal point load, and a thermal source as shown in Fig. 1.

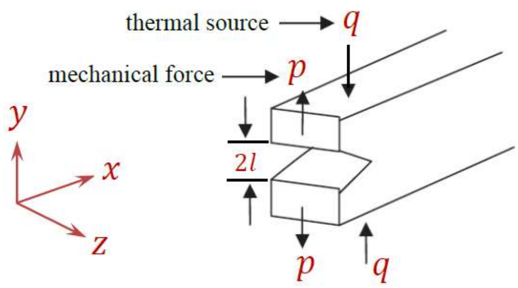

Fig. 1. Displacement of an external mode-I crack.

For the 2D problem, the displacements are considered in the following form:

$$u_1 = u(x, y, t), \quad u_2 = v(x, y, t), \quad u_3 = 0, \tag{8}$$

and all solutions are supposed to be independent of z i.e., $\partial(\ \ )/\partial z = 0$.

The fibre-direction as $\vec{b} = (1,0,0)$ are chosen, so that the prioritized direction is x-axis, then Eq. (4) yields

$$\sigma_{xx} = A_{11}\frac{\partial u}{\partial x} + A_{12}\frac{\partial v}{\partial y} - \gamma(T - T_0), \tag{9}$$

$$\sigma_{yy} = A_{12}\frac{\partial u}{\partial x} + A_{22}\frac{\partial v}{\partial y} - \gamma(T - T_0), \tag{10}$$

$$\sigma_{zz} = A_{12}\frac{\partial u}{\partial x} + \lambda\frac{\partial v}{\partial y} - \gamma(T - T_0), \tag{11}$$

$$\sigma_{xy} = \mu_L\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \quad \sigma_{xz} = \sigma_{yz} = 0, \tag{12}$$

where

$$A_{11} = \lambda + 2(\alpha + \mu_T) + 4(\mu_L - \mu_T) + \beta, \quad A_{12} = \alpha + \lambda, \quad A_{22} = \lambda + 2\mu_T. \tag{13}$$

Using the summation convection, from Eqs. (9)-(12), the equations of motion, with neglecting the body forces, become

$$A_{11}\frac{\partial^2 u}{\partial x^2}+B_1\frac{\partial^2 u}{\partial y^2}+B_2\frac{\partial^2 v}{\partial x\partial y}-\beta_1\frac{\partial T}{\partial x}=\rho\frac{\partial^2 u}{\partial t^2}, \tag{14}$$

$$A_{22}\frac{\partial^2 u}{\partial y^2}+B_1\frac{\partial^2 u}{\partial x^2}+B_2\frac{\partial^2 u}{\partial x\partial y}-\beta_2\frac{\partial T}{\partial y}=\rho\frac{\partial^2 v}{\partial t^2}, \tag{15}$$

where

$$B_1=\mu_L,\quad B_2=\alpha+\lambda+\mu_L. \tag{16}$$

For further considerations, it is convenient to introduce the dimensionless variables defined by

$$\begin{gathered}\{x',y',u',v'\}=c_0\eta\{x,y,u,v\},\quad \{t',t_0'\}=c_0^2\eta\{t,t_0\},\\ \theta=\frac{\gamma(T-T_0)}{A_{22}},\quad \sigma_{ij}'=\frac{\sigma_{ij}}{\gamma T_0},\quad \eta=\frac{\rho C_E}{K},\quad c_0^2=\frac{A_{22}}{\rho}.\end{gathered} \tag{17}$$

The above-mentioned governing equations, with the help of Eq. (17), can be transformed into the dimensionless form after deleting the primes as:

$$l_1\frac{\partial^2 u}{\partial x^2}+h_1\frac{\partial^2 u}{\partial y^2}+h_2\frac{\partial^2 v}{\partial x\partial y}-\frac{\partial\theta}{\partial x}=\frac{\partial^2 u}{\partial t^2}, \tag{18}$$

$$l_2\frac{\partial^2 u}{\partial y^2}+h_1\frac{\partial^2 u}{\partial x^2}+h_2\frac{\partial^2 u}{\partial x\partial y}-\frac{\partial\theta}{\partial y}=\frac{\partial^2 v}{\partial t^2}, \tag{19}$$

$$\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}=\left(\delta+t_0\frac{\partial^\nu}{\partial t^\nu}\right)\frac{\partial\theta}{\partial t}+\varepsilon\frac{\partial}{\partial t}\left(1+t_0\frac{\partial^\nu}{\partial t^\nu}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right), \tag{20}$$

$$\mu_T\sigma_{xx}=A_{11}\frac{\partial u}{\partial x}+A_{12}\frac{\partial v}{\partial y}-A_{22}\theta, \tag{21}$$

$$\mu_T\sigma_{yy}=A_{12}\frac{\partial u}{\partial x}+A_{22}\left(\frac{\partial v}{\partial y}-\theta\right), \tag{22}$$

$$\mu_T\sigma_{zz}=A_{12}\frac{\partial u}{\partial x}+\lambda\frac{\partial v}{\partial y}-A_{22}\theta, \tag{23}$$

$$\mu_T\sigma_{xy}=\mu_L\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right),\quad \sigma_{xz}=\sigma_{yz}=0, \tag{24}$$

where

$$l_1=\frac{A_{11}}{A_{22}},\quad l_2=\frac{\mu_L}{A_{22}},\quad h_1=\frac{B_1}{A_{22}},\quad \varepsilon=\frac{\gamma^2 T_0}{\rho C_E A_{22}}. \tag{25}$$

4. Normal Mode Analysis

The normal mode expansion method was supposed by Cheng and Zhang [31] to model the thermoelastic process of generation of elastic waveforms in an isotropic plate. Allam et al. [32] applied the normal mode analysis to discuss the 2D problem of the electromagnetic-thermal elasticity for a homogeneously isotropic perfectly conductive thick plate subjected to a time-dependent heat source in the context of Green and Naghdi thermoelasticity theory.

The solution of the temperature, displacements, and stresses can be become spoiled in terms of normal modes by

$$\{u,v,\theta,\sigma_{ij}\}(x,y,t)=\{u^*,v^*,\theta^*,\sigma_{ij}^*\}(x)\mathrm{e}^{\omega t+iay}, \tag{26}$$

where ω represents the (complex) frequency constant, $i=\sqrt{-1}$, a denotes the wave number in y direction, and $u^*(x)$, $v^*(x)$, $\theta^*(x)$, and $\sigma_{ij}^*(x)$ denote the amplitudes of the field quantities. Using Eq. (26), Eqs. (18)-(24) take the forms

$$[l_1D^2-A_1]u^*+iah_2Dv^*=D\theta^*, \tag{27}$$

$$[h_1D^2-A_2]v^*+iah_2Du^*=ia\theta^*, \tag{28}$$

$$[D^2 - A_3]\theta^* = A_4(Du^* + iav^*), \tag{29}$$

$$\mu_T \sigma_{xx}^* = A_{11}Du^* + iaA_{12}v^* - A_{22}\theta^*, \tag{30}$$

$$\mu_T \sigma_{yy}^* = A_{12}Du^* + iaA_{22}v^* - A_{22}\theta^*, \tag{31}$$

$$\mu_T \sigma_{zz}^* = A_{12}Du^* + ia\lambda v^* - A_{22}\theta^*, \tag{32}$$

$$\mu_T \sigma_{xy}^* = \mu_L(iau^* + Dv^*), \tag{33}$$

where

$$A_1 = \omega^2 - a^2 h_1, \quad A_2 = \omega^2 - a^2 l_2, \\ A_3 = a^2 + \omega(\delta + t_0\omega^\nu), \quad A_4 = \varepsilon\omega(\delta + t_0\omega^\nu). \tag{34}$$

Eliminating $\theta^*(x)$ and $v^*(x)$ in Eqs. (27)-(29), one gets

$$[D^6 - AD^4 + BD^2 - C]u^*(x) = 0, \tag{35}$$

in which

$$A = \frac{A_2}{h_1} + \frac{A_1 + A_4}{l_1} - \frac{h_2^2 a^2}{h_1 l_1} + A_3, \quad C = \frac{A_1A_2A_3 + A_1A_4a^2}{h_1 l_1}, \\ B = \frac{A_1A_2 + A_3(l_1A_2 + h_1A_1 - h_2^2a^2) + A_4(l_1a^2 + A_2 - 2h_2^2a^2)}{h_1 l_1}. \tag{36}$$

Equation (35) can be factorized as

$$(D^2 - k_1^2)(D^2 - k_2^2)(D^2 - k_3^2)u^*(x) = 0, \tag{37}$$

where k_i^2, $i = 1,2,3$ represent the roots of the characteristic equation

$$k^6 - Ak^4 + Bk^2 - C = 0. \tag{38}$$

The solution of Eq. (37), which is bounded at $x \to \infty$, is represented by

$$u^*(x) = \sum_{n=1}^{3} M_n(a,\omega)\mathrm{e}^{-k_n x}. \tag{39}$$

In a similar manner, we get

$$[D^6 - AD^4 + BD^2 - C]\{v^*, \theta^*\}(x) = 0. \tag{40}$$

Similarly

$$\{v^*, \theta^*\}(x) = \sum_{n=1}^{3} \{M_n', M_n''\}(a,\omega)\mathrm{e}^{-k_n x}, \tag{41}$$

where M_n, M_n', and M_n'' denote some parameters depending on a and ω. By substituting Eqs. (29)-(41) into Eqs. (27)-(29), the following relations are obtaind:

$$M_n'(a,\omega) = H_{1n}M_n(a,\omega), \quad M_n''(a,\omega) = H_{2n}M_n(a,\omega), \ \ n = 1,2,3, \tag{42}$$

where

$$H_{1n} = \frac{ia(h_2 - l_1)k_n^2 + iaA_1 + 2\omega\Omega k_n}{k_n^2 - A_3}, \quad H_{2n} = \frac{A_4(-k_n + iaH_{1n})}{k_n^2 - A_3}. \tag{43}$$

Thus, we have

$$\{v^*, \theta^*\}(x) = \sum_{n=1}^{3} \{H_{1n}, H_{2n}\}M_n(a,\omega)\mathrm{e}^{-k_n x}. \tag{44}$$

Substituting Eqs. (39) and (44) into Eqs. (30)-(33), one obtains

$$\{\sigma_{xx}^*, \sigma_{yy}^*, \sigma_{zz}^*, \sigma_{xy}^*\}(x) = \sum_{n=1}^{3} \{H_{3n}, H_{4n}, H_{5n}, H_{6n}\}M_n(a,\omega)\mathrm{e}^{-k_n x}. \tag{45}$$

where

$$\mu_T \begin{Bmatrix} H_{3n} \\ H_{4n} \\ H_{5n} \end{Bmatrix} = \begin{bmatrix} -A_{11} & iaA_{12} & -A_{22} \\ -A_{12} & iaA_{22} & -A_{22} \\ -A_{12} & ia\lambda & -A_{22} \end{bmatrix} \begin{Bmatrix} k_n \\ H_{1n} \\ H_{2n} \end{Bmatrix}, \quad \mu_T H_{6n} = \mu_L (ia - k_n H_{1n}). \tag{46}$$

To get M_n, the following boundary conditions at the vertical plane $y = 0$ and in the beginning of the crack at $x = 0$ are considered:

$$\sigma_{xx}(x, y, t) = -p, \quad \theta(x, y, t) = q, \quad \sigma_{xy}(x, y, t) = 0, \quad \frac{\partial \theta(x, y, t)}{\partial y} = 0, \tag{47}$$

where p and q denote the magnitudes of mechanical force and thermal source. Using Eqs. (44) and (45) in the above-mentioned boundary condition, three equations with three unknown parameters M_n are obtained as

$$\sum_{n=1}^{3} \{H_{3n}, H_{2n}, H_{6n}\} M_n(a, \omega) = \{-p^*, q^*, 0\}. \tag{48}$$

Solving the above-mentioned equations after applying the inverse of the matrix method, the parameters M_n are derived as $M_n = \Delta_n / \Delta$ where

$$\begin{aligned} \Delta_1 &= p^*(H_{22}H_{63} - H_{23}H_{62}) + q^*(H_{42}H_{63} - H_{43}H_{62}), \\ \Delta_2 &= p^*(H_{23}H_{61} - H_{21}H_{63}) + q^*(H_{42}H_{61} - H_{41}H_{63}), \\ \Delta_3 &= p^*(H_{21}H_{62} - H_{22}H_{61}) + q^*(H_{41}H_{62} - H_{42}H_{61}), \\ \Delta &= H_{21}(H_{42}H_{63} - H_{43}H_{62}) + H_{22}(H_{43}H_{61} - H_{41}H_{63}) + H_{23}(H_{41}H_{62} - H_{42}H_{61}). \end{aligned} \tag{49}$$

Therefore, all expressions for displacements, temperature, and another physical quantities of the plate muscles are obtained.

5. Particular and Special Cases

- The equations of coupled thermoelasticity (CTE) theory are derived when $\nu \to 0$, $t_0 = 1$, and $\delta = 0$.
- The equations of Lord-Shulman (LS) theory are retrieved when $\nu \to 1$ and $\delta = 1$.
- The equations of the generalized thermoelasticity without energy dissipation (the linearized GN theory of type II) are derived when $\nu \to 1$, $t_0 \to 0$, and $\delta = 1$.

In addition, all results are reduced to classical isotropic results when the anisotropic parameters for fibre-reinforced medium tend to zero (if necessary writing $\alpha = 0$, $\beta = 0$, and setting $|\xi| \to 0$).

6. Numerical Results and Discussions

The composite reinforced materials are used in fibers in a variety of structures because of their high strength and low weight. The continuous models are used to explain the thermal and mechanical properties of this materials. Therefore, characterization of their thermal and mechanical behavior is particularly important in the structural designs utilizing these materials.

The aim of this study is to support the information about the propagation of thermal waves in a layer of fiber-reinforced isotopic and transversely thermo-elastic material. This study has many applications in several fields of technology and science, namely, industrial engineering, atomic physics, thermal power plants, pressure vessel, submarine structures, chemical pipes, aerospace, and metallurgy.

To discuss the effect of reinforcement and fractional on the wave propagation, the physical constants for generalized fibre-reinforced thermoelastic material at $T_0 = 293$ K are applied:

$$\begin{gathered} \lambda = 7.59 \times 10^9 \ \mathrm{N/m^2}, \quad \mu = 3.86 \times 10^{10} \ \mathrm{N/m^2}, \\ \mu_T = 1.89 \times 10^9 \ \mathrm{N/m^2}, \quad \mu_L = 2.45 \times 10^9 \ \mathrm{N/m^2}, \\ \alpha = -1.28 \times 10^9 \ \mathrm{N/m^2}, \quad \beta = 0.32 \times 10^9 \ \mathrm{N/m^2}, \quad \rho = 7800 \ \mathrm{kg/m^3}, \\ \alpha_t = 1.78 \times 10^{-5} (1/\mathrm{K}), \quad C_E = 383.1 \ \mathrm{J/(kg\,K)}, \quad K = 386 \ \mathrm{J/(msK)}, \\ t_0 = 0.02, \quad a = 1, \quad p^* = 2, \quad q^* = 1, \quad \omega = \omega_0 + i\zeta, \quad \omega_0 = 2, \quad \zeta = 1. \end{gathered}$$

The calculations are carried out for $t = 0.15$, and all variables are taken in the non-dimensional forms. Figures 2-7 describe the variation of the values of real part of temperature θ, displacement component components u, v, and the stresses against the thickness x for the absence ($\nu = 1$) and the presence ($\nu = 0.25, 0.75$) of the fractional at $y = 1$. The different values of the parameter ν with a wide range ($0 < \nu \leq 1$) cover the two cases of the conductivity; $0 < \nu < 1$ for the weak conductivity and $\nu = 1$ for the normal conductivity (ordinary heat conduction equation). The field quantities including temperature θ, displacements u and v, and stresses σ_{xx}, σ_{yy}, and σ_{xy} depend not only on space x and time t, but also on fractional order ν.

Figure 2 shows the variation of temperature θ along the distance x. It is indicated that θ has the maximum value at the boundary $x = 0$ and then it is directly decreasing to get vanish. Therefore, the temperature θ begins with its maximum value at the beginning of the crack, and it is sooth reducing to fall just near the crack. The amplitude of temperature θ is affected by the variation of fractional order parameter ν.

As shown in Fig. 3, the horizontal displacement u starts with negative values and terminates at the zero value. The fraction theory with $\nu = 0.25$ gives the smallest horizontal displacement while LS theory gives the greatest one. Figure 4 shows that vertical displacement v always starts from a negative value and also terminates at the zero value. The vertical displacement v is no longer increasing and reaches the maximum value at different position according the value of ν.

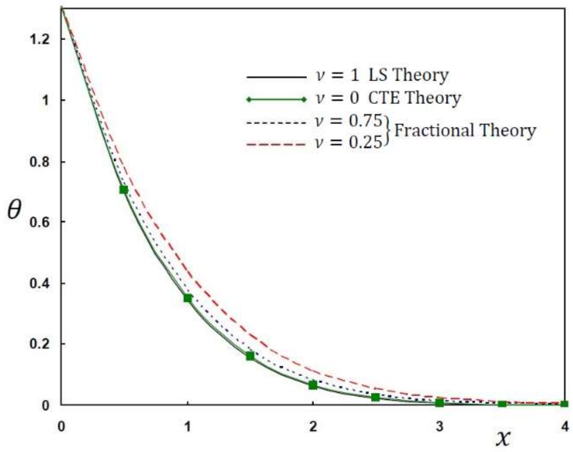

Fig. 2. Dependence of temperature θ on distance for different values of fractional order parameter.

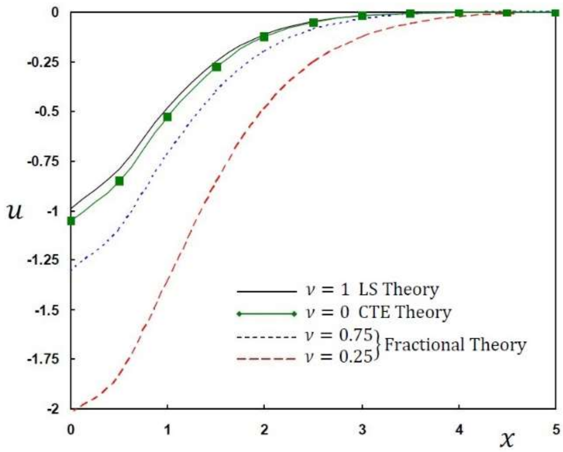

Fig. 3. Dependence of horizontal displacement distribution u on distance for different values of fractional order parameter.

Figure 5 shows that the stress σ_{xx} increases in the domain $0 \leq x \leq 1$ and ultimately goes to zero for $x > 1$. Figure 6 shows the same behavior of the stress σ_{yy} as that found in Figure 5. Both stresses reach the minimum values near the end of the crack and converge to zero with the increasing distance x. The fractional order ν decreases the amplitudes of the stresses. The fraction theory with $\nu = 0.25$ gives the greatest stress while LS theory gives the smallest ones.

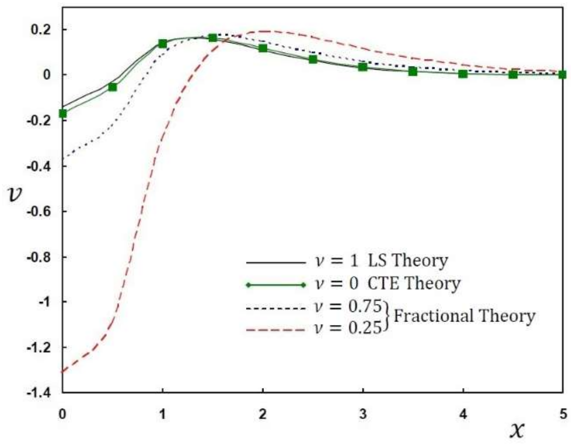

Fig. 4. Dependence of vertical displacement v on distance for different values of fractional order parameter

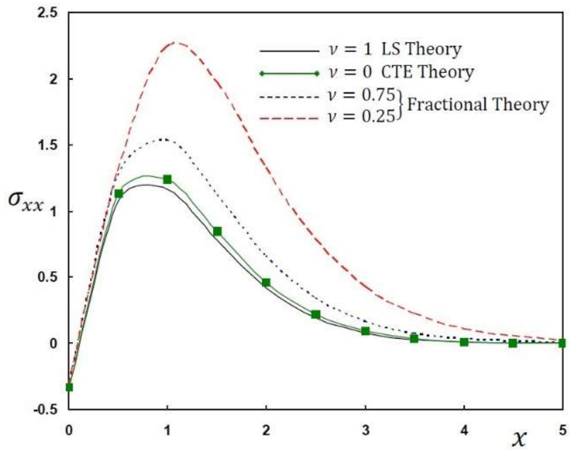

Fig. 5. Dependence of stress σ_{xx} on distance for different values of fractional order parameter

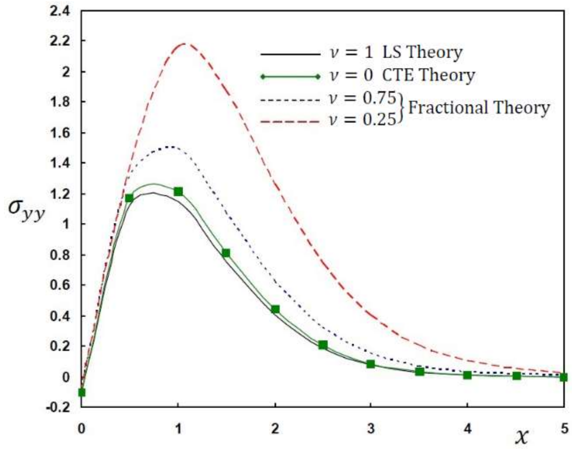

Fig. 6. Dependence of stress σ_{yy} on distance for different values of fractional order parameter.

Figure 7 shows that stress σ_{xy} satisfies the boundary condition at $x = 0$ and has a different behaviour compared to that of σ_{xx} or σ_{yy}. Once again, the fractional order ν decreases the amplitudes of the stresses. The fraction theory with $\nu = 0.25$ gives the greatest stress σ_{xy} while LS theory gives the smallest ones in the domain $0 \leq x \leq 2$ and vice versa for $x > 2$.

It is concluded from Figs.2-7 that all variables like temperature, displacements, and stresses depend on time and space as well as the characteristic parameter ν of the fractional order thermoelasticity theory. It is noticed that the results of all variables in terms of the fractional order parameter included in the heat equation are distinctly different from those without the fractional order parameter. The important phenomenon of finite speeds of propagation is manifested in all these figures.

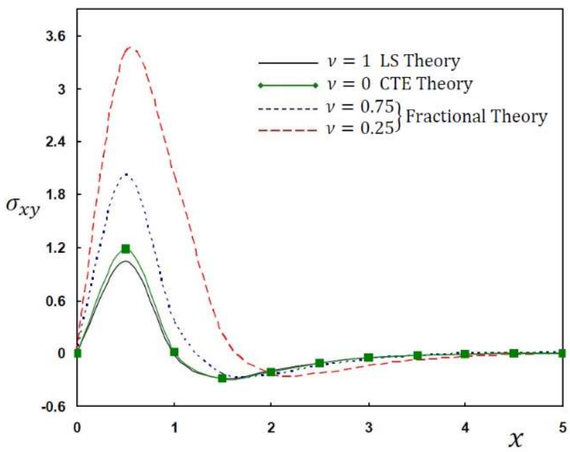

Fig. 7. Dependence of stress σ_{xy} on distance for different values of fractional order parameter.

Additional results are illustrated in Figs. 8-13 to show the variation of all variables with space x under two types with ($\alpha \neq 0$, $\beta \neq 0$ and $\xi \neq 0$) and without ($\alpha = 0$, $\beta = 0$ and $\xi = 0$) the reinforcement.

Fibre-reinforced materials have wide applications in the fields of automotive and aerospace as well as in schooners, especially in modern motorcycles and bicycles where their strength to weight is of great importance. Materials reinforced by strong symmetry fibers exhibit extremely elastic behavior in that their elastic coefficient for extension in the fiber direction is often more than of 50 times or more of their elastic coefficients in the transverse or shear extension. Therefore, the study of stresses and the displacements in addition to temperature is very important in such designs

The values of temperature θ, horizontal displacement u, and stresses σ_{xx} and σ_{yy} are evidently smaller with the inclusion of the reinforcement when compared to those without the reinforcement. It is obvious that θ, u, σ_{xx}, and σ_{yy} increase as the fraction parameter ν decreases. This is not the same for the vertical displacement v and tangential stress σ_{xy}. In general, all variables are more sensitive to the variation of fractional parameter, especially with the inclusion of the reinforcement. Therefore, the surface waves in the fibre-reinforced medium are affected by the reinforced and fraction parameters.

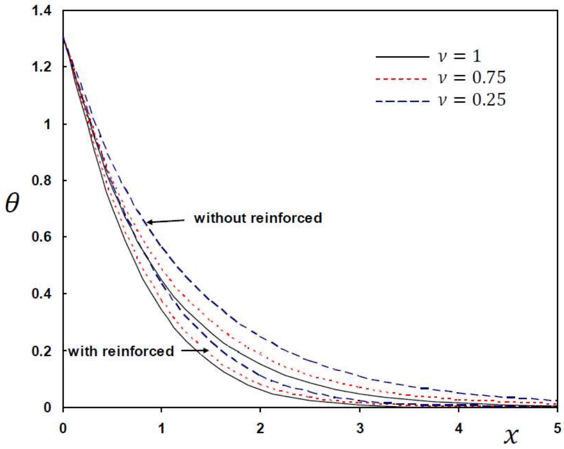

Fig. 8. Effect of fiber-reinforcement on the distribution of temperature θ for different values of fractional order parameter.

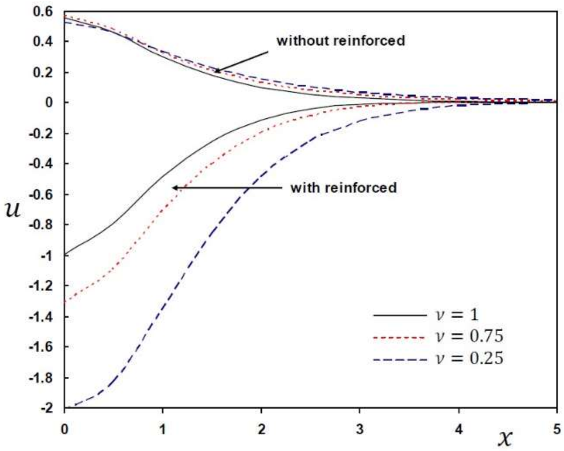

Fig. 9. Effect of fiber-reinforcement on the distribution of displacement u for different values of fractional order parameter.

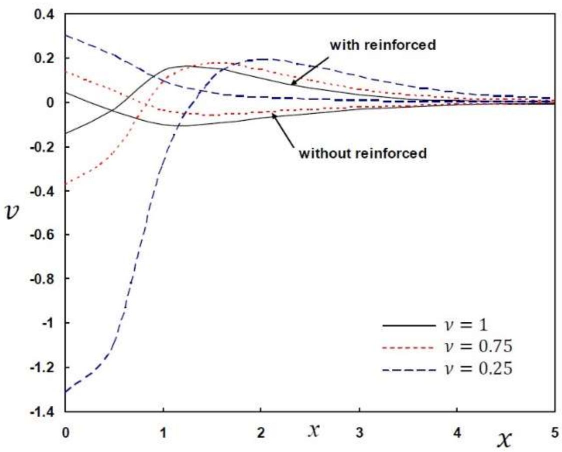

Fig. 10. Effect of fiber-reinforcement on the distribution of displacement v for different values of fractional order parameter.

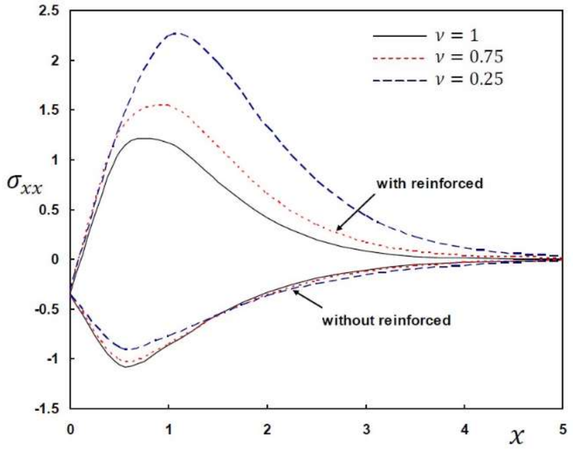

Fig. 11. Effect of fiber-reinforcement on the distribution of stress σ_{xx} for different values of fractional order parameter.

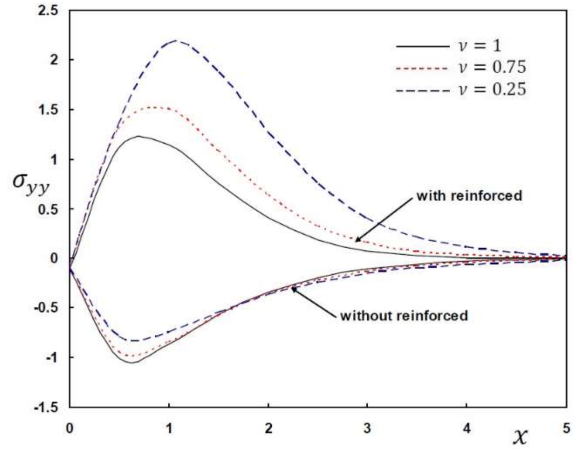

Fig. 12. Effect of fiber-reinforcement on the distribution of stress σ_{yy} for different values of fractional order parameter.

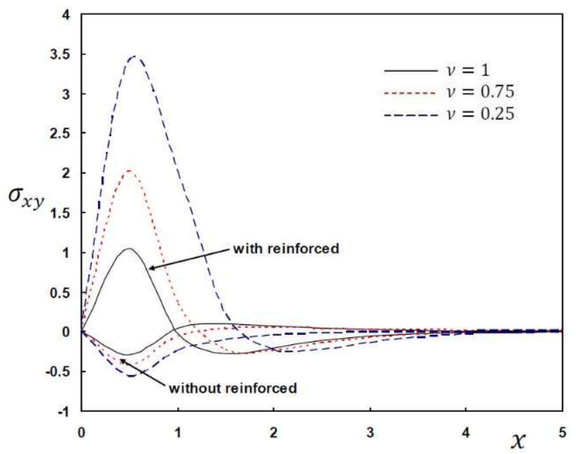

Fig. 13. Effect of fiber-reinforcement on the distribution of stress σ_{xy} for different values of fractional order parameter.

7. Conclusions

In this study, the analytical solutions for the themoelastic problem in solids are developed and utilized. A linear opening mode-I crack is discussed for copper solid. The mechanical behavior of various fibrous composite materials is carefully designed by the linear elasticity theory of transverse isotropic materials, with the preferred direction that coincides with the direction of the fibers. The displacement and stress distributions as well as temperature are evaluated as the functions of distance from crack edge. The presence and absence of reinforcement cases are addressed. The method of normal mode analysis is proposed in the field of generalized thermoelasticity and applied to two specific problems in which displacements and stresses are coupled.

The following remarks based on the above-mentioned analysis are proposed:

- A comparison with the effort of [33] indicates that the fibre-reinforcement plays an important role on distributions of all variables. It is also clear that the theories of coupled thermoelasticity and generalized thermoelasticity can be derived as limited cases. Therefore, the fiber-reinforcement of an anisotropic thermoelastic has a great effect on the field quantitiesn.

- The analytical solutions based on the normal mode method for a thermoelastic problem in solids are developed and utilized.
- The method used here is applicable to a wide range of problems in thermodynamics and thermoelasticity.
- The values of all field quantities converge to zero with increasing the distance x, and all functions are continuous.
- The results carried out here can be applied to design different fibre-reinforced thermoelastic elements under the thermal load to meet special engineering requirements.
- The field quantities are very sensitive to the variation of the fractional order parameter.
- Physical applications are established in various applications in the fields of mechanics, geophysics, and industry.

Conflict of Interest

The authors declare no conflict of interest.

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