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Research Paper

Residual Power Series Method for Solving Time-fractional Model of Vibration Equation of Large Membranes

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Abstract. The primary aim of this manuscript is to present the approximate analytical solutions of the time fractional order $\alpha\,(1<\alpha\leq 2)$ Vibration Equation (VE) of large membranes with the use of an iterative technique namely Residual Power Series Method (RPSM). The fractional derivative is defined in the Caputo sense. Example problems have been solved to demonstrate the efficacy of the present method and the results obtained are verified graphically. The convergence analysis of the proposed method has also been included in this article. It is seen that the present method is found to be reliable, very effective and easy to implement for various kinds of fractional differential equations used in science and engineering.

Keywords: Fractional vibration equation, Caputo derivative, Residual Power Series, Mittag-Leffler function

1. Introduction

The investigation of Vibration Equation (VE) is challenging and needful for the application. Components of microphones, audio system, and other gadgets are formed by membranes. Moreover, membranes might be utilized to research on wave mechanics and propagation. Various tissues of the human body are foreseen as membranes in bioengineering. The physical highlights of the VE of an eardrum are utilized in accepting the hearing mechanism in our body. The design of deaf aid devices requires comprehension of vibration activities of membranes. The governing equation of large membrane includes the time-fractional derivatives. In present-day fractional calculus has been taken into consideration as a useful tool to define other real-life problems to a very useful and precise way. In this association, various researchers have performed innovative work in which one may see the references [1-7].

The VE of the drum head is the fundamental equation of wave propagation which may be written as

$$\left.\begin{array}{c}\dfrac{\partial^2 w}{\partial t^2}=c^2\left(\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}\right)\\ w=0\;on\;\;\partial\Omega\end{array}\right\}\;\text{for}\;\;(x,y)\in\Omega\,. \tag{1}$$

where Ω is the open disk. At any time t, the height of the drum head shape at a point (x,y) and is denoted by $w(x,y,t)$. Let $\partial\Omega$ denote the boundary of Ω that is the circle of radius a centered at the origin. Due to the circular shape of Ω it is appropriate to use cylindrical coordinates (r,t,θ), then Eq. (1) can be rewritten as

$$\left.\begin{array}{c}\dfrac{\partial^2 w}{\partial t^2}=c^2\left(\dfrac{\partial^2 w}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial w}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2 w}{\partial \theta^2}\right)\\ w=0 \; for \; r=a\end{array}\right\}\text{ for } \; 0\le r<a,\, 0\le\theta\le 2\pi\,. \tag{2}$$

Here c gives the speed at which transverse vibration waves will propagate in the membrane. If the vibration of a circular drum head is axisymmetric then w does not depend upon the angle θ. So Eq. (2) reduces to Standard Vibration Equation (SVE)

$$\frac{\partial^2 w}{\partial t^2}=c^2\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right). \tag{3}$$

As such, in this article, time-fractional ordered vibration equation of large membranes is considered as [8]

$$\frac{\partial^\alpha w}{\partial t^\alpha}=c^2\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right),\quad r>0,\; t>0,\, 1<\alpha\le 2\,. \tag{4}$$

with initial conditions (IC)

$$w(r,0)=\xi(r) \;\text{ and }\; \frac{\partial}{\partial t}w(r,0)=c\psi(r), \tag{5}$$

where ξ,ψ are two functions of r.

The fractional VE has been examined with the help of various techniques such as Variational iteration method (VIM), Adomian decomposition method (ADM), q-Homotopy analysis transform method (q-HATM) [8] and so on. These methods have their particular limits and inadequacies, tremendous computational work and take high running time. RPSM is an efficient and effective method firstly proposed by the Jordan mathematician Abu Arqub [9] for evaluating the values of the coefficients of the power series solution of the differential equations. The RPSM has been well implemented in the generalized Lane-Emden equation [10], the nonlinear fractional KdV-Burgers equation [11], and the solution of the fractional foam drainage equation [12]. This method has also been implemented for finding the exact solution of time-fractional Schrodinger equations [13]. Moreover, the RPSM has been successfully applied for finding the approximate solution of time-space-fractional Benny-Lin equation [14].

In this article, the RPSM has been implemented for solving the fractional order VE for large membranes. Some main characteristics of this method are: firstly, this technique obtains expansions of the solutions in the form of polynomials, the solutions, and all their derivatives are applicable for each arbitrary point in the given interval. It does not require any modification while switching from the first order to the higher order. So this technique can be applied directly to the given system by choosing an appropriate value for the initial guesses approximations. This technique needs small computational requirements with high precision and less time. The performance and precision of the proposed method are examined by comparing the solution of the titled problem solved by RPSM with the same fractional problem solved by q-HATM and Laplace decomposition method (LDM). In recent times, fractional order VE for large membranes has been solved by taking a particular case when $\xi(r)=r^2$ and $\psi(r)=r$ through the usage of the VIM and the modified decomposition method (MDM). However, to the best of the authors' information, time-fractional order VE for more general cases has not yet been explained by RPSM. Also for $\alpha=2$, that is, for SVE, $w(r,t)$ has been calculated for various cases which are in good agreement with Srivastava et al. [8].

2. Preliminaries on Fractional Calculus and RPSM

Definition 2.1:

The Riemann-Liouville (R-L) fractional derivative operator D^α of order α is described as [2,4]

$$D^\alpha u(x)=\begin{cases}\dfrac{d^m}{dx^m}u(x), & \alpha=m,\\ \dfrac{1}{\Gamma(m-\alpha)}\dfrac{d}{dx^m}\displaystyle\int_0^x\dfrac{u(t)}{(x-t)^{\alpha-m+1}}dt, & m-1<\alpha<m.\end{cases} \tag{6}$$

where $m\in Z^+$, $\alpha\in R^+$ and

$$D^{-\alpha}u(x)=\frac{1}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}u(t)dt,\; 0<\alpha\le 1\,. \tag{7}$$

Definition 2.2:

The R-L fractional order integration operator J^α is described as [2,4]

$$J^{\alpha}u(x)=\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}u(t)dt,\quad t>0,\quad \alpha>0. \tag{8}$$

Following Podlubny [2] we may have

$$J^{\alpha}t^{n}=\frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)}t^{n+\alpha}. \tag{9}$$

$$D^{\alpha}t^{n}=\frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)}t^{n-\alpha}. \tag{10}$$

Definition 2.3:

The Caputo fractional derivative operator D^{α} of order α is defined as [2,3]

$$^{C}D^{\alpha}u(x)=\begin{cases}\dfrac{1}{\Gamma(m-\alpha)}\displaystyle\int_0^x \frac{u^{m}(t)}{(x-t)^{\alpha-m+1}}dt, & m-1<\alpha<m,\\[2ex] \dfrac{d^{m}}{dt^{m}}u(x), & \alpha=m.\end{cases} \tag{11}$$

Definition 2.4 [2-4]:

(a) $D_t^{\alpha}J_t^{\alpha}f(t)=f(t).$ (12)

(b) $J_t^{\alpha}D_t^{\alpha}f(t)=f(t)-\sum_{k=0}^{m}f^{(k)}(0^{+})\frac{t^{k}}{k!}$, for $t>0$.

Definition 2.5 [16-19]:

A series of the form

$$\sum_{i=0}^{\infty}\sum_{j=0}^{n-1}f_{ij}(x)(t-t_0)^{i\alpha+j},\quad \text{for}\quad 0\le n-1<\alpha\le n,\ t\ge t_0. \tag{13}$$

is called multiple Fractional power series (FPS) about $t=t_0$, where f_{ij} is the coefficient of series which is the function of x.

Lemma 2.1 [15,19]:

Let $w(x,t)$ and $D_t^{i\alpha}$ are continuous in the interval $(\Re,(t_0,t_0+R))$ for $i=0,1,2,...n+1$ then

$$J_t^{(n+1)\alpha}D_t^{(n+1)\alpha}w(x,t)=\frac{D_t^{(n+1)\alpha}w(x,\xi)}{\Gamma((n+1)\alpha+1)}(t-t_0)^{(n+1)\alpha},\ t_0\le\xi\le t<t_0+R, 0\le n-1<\alpha\le n, \tag{14}$$

where R and $\Re$ are the radius of convergence of FPS and real number respectively.

Lemma 2.2 [15,19]:

Suppose $w(x,t)$ and $D_t^{i\alpha}$ are continuous in the interval $(\Re,(t_0,t_0+R))$ and $D_t^{i\alpha}$ is differentiable $(n-1)$ times with respect to t on (t_0,t_0+R) for $i=0,1,2,...n+1$ then

$$w(x,t)=\sum_{j=0}^{n}\sum_{k=0}^{n-1}\frac{D_t^{j\alpha+k}w(x,t_0)}{\Gamma(j\alpha+k+1)}(t-t_0)^{k+j\alpha}+\frac{D_t^{(n+1)\alpha}w(x,\tau)}{\Gamma((n+1)\alpha+1)}(t-t_0)^{(n+1)\alpha}, \tag{15}$$

for $t_0\le\tau\le t<t_0+R, 0\le n-1<\alpha\le n$.

Proof: One can see the reference [15,19] for the proofs of Lemma 2.1 and Lemma 2.2.

3. Construction of RPSM

In order to illustrate this method, we have considered the linear fractional partial differential equation of the form

$$D_t^{\alpha}w(x,t)=L(w(x,t)),\quad x\in\Re, t\ge t_0, 0\le n-1<\alpha\le n, n\in N. \tag{16}$$

with IC

$$\frac{\partial^j w(x,t_0)}{\partial t^j} = \phi_j(x), \quad x \in \Re, j = 0,1,2,....,n-1, \tag{17}$$

where L is a linear operator and ϕ is a function.

Step 1: Let us suppose the infinite power series solution of Eq. (16) with Eq. (17) as

$$w(x,t) = \sum_{i=0}^{\infty}\sum_{j=0}^{n-1} f_{ij}(x)\frac{(t-t_0)^{i\alpha+j}}{\Gamma(i\alpha+j+1)}, \quad x \in \Re,\ n-1<\alpha\le n,\ t_0 \le t < t_0 + R. \tag{18}$$

As Eq. (18) is assumed to be the solution of Eq. (16) with Eq. (17), so the solution $w(x,t)$ must satisfy Eq. (17). Hence substituting Eq. (18) in Eq. (17), we have

$$\frac{\partial^j w(x,t_0)}{\partial t^j} = j!\, f_{0j}(x) = \phi_j(x), \quad j = 0,1,....,n-1. \tag{19}$$

So,

$$f_{0j}(x) = \frac{\phi_j(x)}{j!}, \ j = 0,1,....,n-1. \tag{20}$$

Approximating IC, we obtain

$$w_{(0,n-1)}(x,t) = \sum_{j=0}^{n-1}\frac{\phi_j(x)}{j!}(t-t_0)^j, \quad t_0 \le t < t_0 + R. \tag{21}$$

Using Eq. (21), we can modify Eq. (18) as follows

$$w(x,t) = \sum_{j=0}^{n-1}\frac{\phi_j(x)}{j!}(t-t_0)^j + \sum_{i=1}^{\infty}\sum_{j=0}^{n-1} f_{ij}(x)\frac{(t-t_0)^{i\alpha+j}}{\Gamma(i\alpha+j+1)}, \quad t_0 \le t < t_0 + R. \tag{22}$$

The Residual Power Series (RPS) provides an approximate solution in term of infinite series. However for obtaining the numerical value of the series, Eq. (22) will be truncated as follows

$$w_{(k,l)}(x,t) = \sum_{j=0}^{n-1}\frac{\phi_j(x)}{j!}(t-t_0)^j + \sum_{i=1}^{k}\sum_{j=0}^{l} f_{ij}(x)\frac{(t-t_0)^{i\alpha+j}}{\Gamma(i\alpha+j+1)}, \quad t_0 \le t. \tag{23}$$

for $k = 1,2,3,....$ and $l = 0,1,2,....,n-1$, where $w_{(k,l)}(x,t)$ represents (k,l)- Truncated Series (TS) of $w(x,t)$.

Step 2: Now applying RPS for determining the coefficient of f_{ij} of Eq. (23), we must introduce the Residual function (RF) of Eq. (16) as follows

$$res(x,t) = D_t^{\alpha} w(x,t) - L(w(x,t)), \quad x \in \Re,\ t_0 \le t. \tag{24}$$

Eq. (24) can be written in (k,l)-TS form as

$$res_{(k,l)}(x,t) = D_t^{\alpha} w_{(k,l)}(x,t) - L(w_{(k,l)}(x,t)), \quad t_0 \le t. \tag{25}$$

From [12-14], we get

$$res(x,t) = 0 \ \text{ for } \ t \in [t_0, t_0 + R), x \in \Re. \tag{26}$$

and

$$D_t^{(i-1)\alpha} D_t^j res(x,t) = 0 \text{ for each } i = 1,2,...,k \ \text{ and } \ j = 0,1,2,...,l \tag{27}$$

Considering Eqs. (26) and (27), we can write

$$D_t^{(i-1)\alpha} D_t^j res(x,t_0) = D_t^{(i-1)\alpha} D_t^j res_{(i,j)}(x,t_0) = 0 \ \text{ for } \ x \in \Re \tag{28}$$

Step 3: Summarizing all the steps of RPSM, we now proceed with the following steps as:
Fix $i=1$ run j from 0 to l calculate $(1,j)$-TS of the solution, next fix $i=2$ run j from 0 to l to calculate $(2,j)$-TS, and so on. To find $(1,0)$-TS expansion of Eqs. (16) and (17), Eq. (23) is used as following

$$w_{(1,0)}(x,t)=\phi_0(x)+\phi_1(x)(t-t_0)+\ldots+\phi_{n-1}\frac{(t-t_0)^{(n-1)}}{(n-1)!}+f_{10}(x)\frac{(t-t_0)^{\alpha}}{\Gamma(\alpha+1)}. \tag{29}$$

In order to determine $f_{10}(x)$ in Eq. (29), substitute Eq. (29) into Eq. (25), we get

$$res_{(1,0)}(x,t)=f_{10}(x)-L\left(\phi_0(x)+\phi_1(x)(t-t_0)+\ldots+\phi_{n-1}\frac{(t-t_0)^{(n-1)}}{(n-1)!}\right). \tag{30}$$

Using Eq. (28), Eq. (30) reduces to

$$f_{10}(x)=L(\phi_0(x))=p_0(x)\phi_0^{k_0}+q_{00}(x), \tag{31}$$

where $\phi_0^{k_0}(x)$ represents k_0-th derivatives of $\phi_0(x)$ and q_{00}, p_0 are the functions of x.
Using $(1,0)$-TS of Eq. (29), $(1,0)$-truncated RPS approximate solution of Eqs. (16) and (17) become

$$w_{(1,0)}(x,t)=\phi_0(x)+\phi_1(x)(t-t_0)+\ldots+\phi_{n-1}\frac{(t-t_0)^{(n-1)}}{(n-1)!}+L(\phi_0(x))\frac{(t-t_0)^{\alpha}}{\Gamma(\alpha+1)}. \tag{32}$$

Step 4: Similarly in order to find $(1,1)$-TS expansion of Eqs. (16) and (17), we use Eq. (23) and we obtain

$$w_{(1,1)}(x,t)=\phi_0(x)+\phi_1(x)(t-t_0)+\ldots+\phi_{n-1}\frac{(t-t_0)^{(n-1)}}{(n-1)!}+L(\phi_0(x))\frac{(t-t_0)^{\alpha}}{\Gamma(\alpha+1)}+f_{11}(x)\frac{(t-t_0)^{(\alpha+1)}}{\Gamma(\alpha+2)}. \tag{33}$$

To find the coefficient $f_{11}(x)$ in Eq. (33), formulate $(1,1)$-RF using Eq. (25) and substitute Eq. (33) in $(1,1)$-RF founded in Eq. (25), we have

$$\begin{aligned}&res_{(1,1)}(x,t)=f_{10}(x)+f_{11}(x)(t-t_0)-\\&L\left(\phi_0(x)+\phi_1(x)(t-t_0)+\ldots+\phi_{n-1}\frac{(t-t_0)^{(n-1)}}{(n-1)!}+L(\phi_0(x))\frac{(t-t_0)^{\alpha}}{\Gamma(\alpha+1)}+f_{11}(x)\frac{(t-t_0)^{(\alpha+1)}}{\Gamma(\alpha+2)}\right).\end{aligned} \tag{34}$$

By taking Eq. (28) for $(i,j)=(1,1)$ into consideration and applying the operator D_t on both sides of Eq. (34), we get

$$\begin{aligned}&D_t res_{(1,1)}(x,t_0)=f_{11}-\\&D_tL\left(\phi_0(x)+\phi_1(x)(t-t_0)+\ldots+\phi_{n-1}\frac{(t-t_0)^{(n-1)}}{(n-1)!}+L(\phi_0(x))\frac{(t-t_0)^{\alpha}}{\Gamma(\alpha+1)}+f_{11}(x)\frac{(t-t_0)^{(\alpha+1)}}{\Gamma(\alpha+2)}\right)_{t=t_0}\end{aligned} \tag{35}$$

By using Eq. (28) for $(i,j)=(1,1)$, the coefficient $f_{11}(x)$ of Eq. (35) is evaluated as

$$f_{11}(x)=p_1(x)\phi_0^{k_0}+p_0(x)\phi_1^{k_1}+q_{01}(x). \tag{36}$$

where $\phi_0^{k_0}(x)$, $\phi_1^{k_1}(x)$ represent k_0-the and k_1-th derivatives of $\phi_0(x)$ and $\phi_1(x)$ respectively and q_{01}, p_0, p_1 are the functions of x.
Substituting Eqs. (31) and (36) in Eq. (33), RPS approximate solution of Eq. (16) and Eq. (17) for $(i,j)=(1,1)$ reduces to:

$$\begin{aligned}&w_{(1,1)}(x,t)=\phi_0(x)+\phi_1(x)(t-t_0)+\ldots+\phi_{n-1}\frac{(t-t_0)^{(n-1)}}{(n-1)!}+\\&\left(p_0(x)\phi_0^{k_0}+q_{00}(x)\right)\frac{(t-t_0)^{\alpha}}{\Gamma(\alpha+1)}+\left(p_1(x)\phi_0^{k_0}+p_0(x)\phi_1^{k_1}+q_{01}(x)\right)\frac{(t-t_0)^{(\alpha+1)}}{\Gamma(\alpha+2)}.\end{aligned} \tag{37}$$

This process will continue until all the value of $f_{ij}, i=0,1,2,\ldots k$ and $j=1,2,\ldots,l$ are evaluated. Also if a pattern in series occurs then evaluating few coefficient terms will lead to the solution.

4. Convergence Analysis

Lemma 1 [2].
If $f(x)$ is a continuous function and $\alpha,\beta>0$ then

$$I_c^{\alpha}I_c^{\beta}f(x)=I_c^{\beta}I_c^{\alpha}f(x)=I_c^{\alpha+\beta}f(x). \tag{38}$$

where I denotes integral operator.

Theorem 4.1.

(a) If the FPS of the form $\sum_{n=0}^{\infty} a_n x^{n\alpha}, x \geq 0$ converges at $x = x_1$, then it converges absolutely $\forall x$ satisfying $|x| < |x_1|$.

(b) If the FPS diverges at $x = x_1$, then it will diverge $\forall$ real x satisfying $|x| > |x_1|$.

Theorem 4.2 [14].

For $0 \leq n-1 < \alpha \leq n$, suppose $D_t^{r+k\alpha}, D_t^{r+(k+1)\alpha} \in C[R, t_0] \times [R, t_0 + R]$, then

$$\left(I_t^{r+k\alpha} D_t^{r+k\alpha} u\right)(x,t) - \left(I_t^{r+(k+1)\alpha} D_t^{r+(k+1)\alpha} u\right)(x,t) = \frac{(t-t_0)^{r+k\alpha}}{\Gamma(r+k\alpha+1)} D_t^{r+k\alpha} u(x, t_0), \tag{39}$$

where $D_t^{r+k\alpha} = \underbrace{(D_t . D_t . D_t ......)}_{r-times} \underbrace{(D_t^{\alpha} . D_t^{\alpha} . D_t^{\alpha} ....)}_{k-times}$.

Proof: Proof of the above theorem is given in [14], but for the sake of completeness, it is reproduced here. Using Lemma 1,

$$\begin{aligned}
&\left(I_t^{r+k\alpha} D_t^{r+k\alpha} u\right)(x,t) - \left(I_t^{r+(k+1)\alpha} D_t^{r+(k+1)\alpha} u\right)(x,t) \\
&= I_t^{r+k\alpha}\left(\left(D_t^{r+k\alpha} u\right)(x,t) - \left(I_t^{\alpha} D_t^{r+(k+1)\alpha} u\right)(x,t)\right) \\
&= I_t^{r+k\alpha}\left(\left(D_t^{r+k\alpha} u\right)(x,t) - \left(I_t^{\alpha} D_t^{\alpha}\right)\left(D_t^{r+k\alpha} u\right)(x,t)\right) \\
&= I_t^{r+k\alpha}\left(\left(D_t^{r+k\alpha} u\right)(x,t_0)\right) = \frac{(t-t_0)^{r+k\alpha}}{\Gamma(r+k\alpha+1)} D_t^{r+k\alpha} u(x,t_0).
\end{aligned} \tag{40}$$

Theorem 4.3 [14].

Let $w(x,t)$, $D_t^{k\alpha} w(x,t) \in C[R, t_0] \times [R, t_0 + R]$ where $k = 0,1,2,..., p+1$ and $j = 0,1,2,....,n-1$. Also $D_t^{k\alpha} w(x,t)$ can be differentiated n-1 times with respect to "t" on $(t_0, t_0 + R)$. Then

$$w(x,t) \cong \sum_{j=0}^{n-1} \sum_{i=0}^{p} W_{j+i\alpha}(x)(t-t_0)^{j+i\alpha}, \tag{41}$$

where $W_{j+i\alpha}(x) = \frac{D_t^{j+i\alpha} w(x,t_0)}{\Gamma(j+i\alpha+1)}$. Moreover, $\exists$ a value ε, $0 \leq \varepsilon \leq t$, the error term has the term as follows

$$\left\| E_p(x,t) \right\| = \sup_{t \in [0,T]} \left| \sum_{j=0}^{n-1} \left[\frac{D^{j+(p+1)\alpha} w(x,\varepsilon)}{\Gamma((p+1)\alpha + j + 1)} t^{(p+1)\alpha + j} \right] \right|. \tag{42}$$

Proof. From Eq. (40),

$$\begin{aligned}
&\sum_{j=0}^{n-1} \sum_{i=0}^{p} \left(\left(I_t^{j+i\alpha} D_t^{j+i\alpha} w \right)(x,t) - \left(I_t^{j+(i+1)\alpha} D_t^{j+(i+1)\alpha} w \right)(x,t) \right) \\
&= \sum_{j=0}^{n-1} \sum_{i=0}^{p} \frac{(t-t_0)^{j+i\alpha}}{\Gamma(j+i\alpha+1)} D_t^{j+i\alpha} w(x,t_0) \\
&= \sum_{j=0}^{n-1} \sum_{i=0}^{p} W_{j+i\alpha}(x)(t-t_0)^{j+i\alpha}.
\end{aligned} \tag{43}$$

That is,

$$w(x,t) - \sum_{j=0}^{n-1} \left[\left(I_t^{j+(p+1)\alpha} D_t^{j+(p+1)\alpha} w \right)(x,t) \right] = \sum_{j=0}^{n-1} \sum_{i=0}^{p} W_{j+i\alpha}(x)(t-t_0)^{j+i\alpha}. \tag{44}$$

Considering the second term of Eq. (44), we have

$$\sum_{j=0}^{n-1}\left[\left(I_t^{j+(p+1)\alpha}D_t^{j+(p+1)\alpha}w\right)(x,t)\right]=\sum_{j=0}^{n-1}\left[\frac{1}{\Gamma((p+1)\alpha+j)}\int_0^t\frac{D^{j+(p+1)\alpha}w(x,\varepsilon)}{(t-\tau)^{1-(j+(p+1)\alpha)}}d\tau\right]$$

$$=\sum_{j=0}^{n-1}\left[\frac{D^{j+(p+1)\alpha}w(x,\varepsilon)}{\Gamma((p+1)\alpha+j+1)}t^{(j+(p+1)\alpha)}\right]\quad\text{(Mean value theorem for integral)}\tag{45}$$

From Eqs. (44) and (45),

$$w(x,t)-\sum_{j=0}^{n-1}\sum_{i=0}^{p}W_{j+i\alpha}(x)(t-t_0)^{j+i\alpha}=\sum_{j=0}^{n-1}\left[\frac{D^{j+(p+1)\alpha}w(x,\varepsilon)}{\Gamma((p+1)\alpha+j+1)}t^{(j+(p+1)\alpha)}\right].$$

Now the error term is

$$\left\|E_p(x,t)\right\|=\left\|w(x,t)-\sum_{j=0}^{n-1}\sum_{i=0}^{p}W_{j+i\alpha}(x)(t-t_0)^{j+i\alpha}\right\|=\left\|\sum_{j=0}^{n-1}\left[\frac{D^{j+(p+1)\alpha}w(x,\varepsilon)}{\Gamma((p+1)\alpha+j+1)}t^{(j+(p+1)\alpha)}\right]\right\|$$

$$\left\|E_p(x,t)\right\|=\sup_{t\in[0,T]}\left|\sum_{j=0}^{n-1}\left[\frac{D^{j+(p+1)\alpha}w(x,\varepsilon)}{\Gamma((p+1)\alpha+j+1)}t^{(j+(p+1)\alpha)}\right]\right|$$

As $p\to\infty$, $\left\|E_p(x,t)\right\|\to 0$. Hence $w(x,t)$ may be approximated as $w(x,t)\cong\sum_{j=0}^{n-1}\sum_{i=0}^{p}W_{j+i\alpha}(x)(t-t_0)^{j+i\alpha}$ with error term defined in Eq. (42).

5. Implementation of RPSM to Time-fractional Vibration Equation

Firstly the vibration Eqs. (4) and (5) can be written in operator form as

$$D_t^{\alpha}w=c^2\left(D_{rr}w+\frac{1}{r}D_rw\right).\tag{46}$$

with IC

$$w(r,0)=\xi(r)\ \text{ and }\ w_t(r,0)=c\psi(r),\tag{47}$$

for $1<\alpha\le 2,\ r>0,\ t>0$.

Step 1: From Eq. (22), we have $\phi_0=\xi(r),\phi_1=c\psi(r)$, taking $L(w(x,t))=c^2(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r})$ and using Eq. (22), the FPS solution of Eqs. (46) and (47) can be written as

$$w(r,t)=\xi(r)+c\psi(r)t+\sum_{i=1}^{\infty}\sum_{j=0}^{1}f_{ij}\frac{t^{i\alpha+j}}{\Gamma(1+j+i\alpha)}.\tag{48}$$

According to Eq. (21), $w_{(0,1)}(r,t)=\xi(r)+c\psi(r)t$ is the initial approximation. The $(k.l)$-TS of $w(r,t)$ and the $(k.l)$-truncated RF of Eq. (46) can be defined as

$$w_{(k,l)}(r,t)=\xi(r)+c\psi(r)t+\sum_{i=1}^{k}\sum_{j=0}^{l}f_{ij}\frac{t^{i\alpha+j}}{\Gamma(1+j+i\alpha)}\quad\text{for }\ k=1,2,3,...\ \ l=0,1\tag{49}$$

and

$$res_{(k,l)}(r,t)=D_t^{\alpha}w_{(k,l)}(r,t)-c^2\left(D_{rr}w_{(k,l)}(r,t)+\frac{1}{r}D_rw_{(k,l)}(r,t)\right)\ \text{ respectively.}\tag{50}$$

Step 2: To calculate f_{10}, we have to formulate the $(1,0)$-TS of $w(r,t)$ and $(1,0)$-truncated RF of $w(r,t)$ as

$$w_{(1,0)}(r,t)=\xi(r)+c\psi(r)t+f_{10}(r)\frac{t^{\alpha}}{\Gamma(\alpha+1)}.\tag{51}$$

and

$$res_{(1,0)}(r,t)=D_t^{\alpha}w_{(1,0)}(r,t)-c^2\left(D_{rr}w_{(1,0)}(r,t)+\frac{1}{r}D_r w_{(1,0)}(r,t)\right)\ \text{respectively.} \tag{52}$$

Simplifying $res_{(1,0)}(r,t)$, we obtain

$$res_{(1,0)}(r,t)=f_{10}(r)-c^2\left[\begin{array}{l}\xi^{(2)}(r)+ct\psi^{(2)}(r)+f_{10}{}^{(2)}(r)\dfrac{t^{\alpha}}{\Gamma(1+\alpha)}+\\ \qquad\dfrac{1}{r}\left(\xi^{(1)}(r)+ct\psi^{(1)}(r)+f_{10}{}^{(1)}(r)\dfrac{t^{\alpha}}{\Gamma(1+\alpha)}\right)\end{array}\right]. \tag{53}$$

By using Eq. (26) for $(i,j)=(1,0)$ and setting $t=0$, from Eq. (53) we have

$$f_{10}(r)=c^2\left[\xi^{(2)}(r)++\frac{1}{r}\left(\xi^{(1)}(r)\right)\right]. \tag{54}$$

Step 3: Similarly, in order to find $f_{11}(r)$, we have to evaluate $(1,1)$-TS of $w(r,t)$ as

$$w_{(1,1)}(r,t)=\xi(r)+c\psi(r)t+c^2\left(\xi^{(2)}(r)++\frac{1}{r}\left(\xi^{(1)}(r)\right)\right)\frac{t^{\alpha}}{\Gamma(1+\alpha)}+f_{11}(r)\frac{t^{\alpha+1}}{\Gamma(2+\alpha)}. \tag{55}$$

and $(1,1)$-truncated RF of $w(r,t)$ is written as

$$res_{(1,1)}(r,t)=D_t^{\alpha}w_{(1,1)}(r,t)-c^2\left(D_{rr}w_{(1,1)}(r,t)+\frac{1}{r}D_r w_{(1,1)}(r,t)\right). \tag{56}$$

Substituting Eq. (55) in Eq. (56), we get

$$\begin{aligned}res_{(1,1)}(r,t)&=c^2\left(\xi^{(2)}(r)++\frac{1}{r}\left(\xi^{(1)}(r)\right)\right)+f_{11}(r)t-\\ &c^2\left[\begin{array}{l}\xi^{(2)}(r)+c\psi^{(2)}(r)t+c^2\left\{\xi^{(4)}(r)+\dfrac{2\xi^{(3)}(r)}{r}-\dfrac{\xi^{(2)}(r)}{r^2}+\dfrac{\xi^{(1)}(r)}{r^3}\right\}\dfrac{t^{\alpha}}{\Gamma(1+\alpha)}+f_{11}{}^{(2)}(r)\dfrac{t^{\alpha+1}}{\Gamma(\alpha+2)}\\ \dfrac{1}{r}\left\{\xi^{(1)}(r)+c\psi^{(1)}(r)t+c^2\left\{\xi^{(3)}(r)+\dfrac{\xi^{(2)}(r)}{r}-\dfrac{\xi^{(1)}(r)}{r^2}\right\}\dfrac{t^{\alpha}}{\Gamma(1+\alpha)}+f_{11}{}^{(1)}(r)\dfrac{t^{\alpha+1}}{\Gamma(\alpha+2)}\right\}\end{array}\right].\end{aligned} \tag{57}$$

Using Eq. (28) for $(i,j)=(1,1)$ and putting $t=0$, we have from Eq. (57) as follows

$$f_{11}(r)=c^2\left[c\psi^{(2)}(r)+\frac{1}{r}c\psi^{(1)}(r)\right]=c^3\left(\psi^{(2)}(r)+\frac{1}{r}\psi^{(1)}(r)\right). \tag{58}$$

Substituting Eq. (58) into Eq. (55) we obtain

$$w_{(1,1)}(r,t)=\xi(r)+c\psi(r)t+c^2\left(\xi^{(2)}(r)++\frac{1}{r}\left(\xi^{(1)}(r)\right)\right)\frac{t^{\alpha}}{\Gamma(1+\alpha)}+c^3\left(\psi^{(2)}(r)+\frac{1}{r}\psi^{(1)}(r)\right)\frac{t^{\alpha+1}}{\Gamma(2+\alpha)}. \tag{59}$$

Step 4: Continuing the same way, we can evaluate $f_{20}(r), f_{21}(r), f_{30}(r), f_{31}(r),.....$, and the values of these are as follows

$$f_{20}(r)=c^4\left[\xi^{(4)}(r)+\frac{2}{r}\xi^{(3)}(r)-\frac{1}{r^2}\xi^{(2)}(r)+\frac{1}{r^3}\xi^{(1)}(r)\right]. \tag{60}$$

$$f_{21}(r)=c^5\left[\psi^{(4)}(r)+\frac{2}{r}\psi^{(3)}(r)-\frac{1}{r^2}\psi^{(2)}(r)+\frac{1}{r^3}\psi^{(1)}(r)\right]. \tag{61}$$

$$f_{30}(r)=c^6\left[\xi^{(6)}(r)+\frac{3}{r}\xi^{(5)}(r)-\frac{3}{r^2}\xi^{(4)}(r)+\frac{6}{r^3}\xi^{(3)}(r)-\frac{9}{r^4}\xi^{(2)}(r)+\frac{9}{r^5}\xi^{(1)}(r)\right]. \tag{62}$$

$$f_{31}(r)=c^7\left[\psi^{(6)}(r)+\frac{3}{r}\psi^{(5)}(r)-\frac{3}{r^2}\psi^{(4)}(r)+\frac{6}{r^3}\psi^{(3)}(r)-\frac{9}{r^4}\psi^{(2)}(r)+\frac{9}{r^5}\psi^{(1)}(r)\right]. \tag{63}$$

Substituting Eqs. (60)-(63) in Eq. (48), we get the infinite series solution as

$$\begin{aligned} w(r,t) &= \xi(r) + c\psi(r)t + c^2\left(\xi^{(2)}(r) + +\frac{1}{r}\left(\xi^{(1)}(r)\right)\right)\frac{t^{\alpha}}{\Gamma(1+\alpha)} + c^3\left(\psi^{(2)}(r) + \frac{1}{r}\psi^{(1)}(r)\right)\frac{t^{\alpha+1}}{\Gamma(2+\alpha)} + \\ & c^4\left[\xi^{(4)}(r) + \frac{2}{r}\xi^{(3)}(r) - \frac{1}{r^2}\xi^{(2)}(r) + \frac{1}{r^3}\xi^{(1)}(r)\right]\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + c^5\left[\psi^{(4)}(r) + \frac{2}{r}\psi^{(3)}(r) - \frac{1}{r^2}\psi^{(2)}(r) + \frac{1}{r^3}\psi^{(1)}(r)\right] \\ & \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + c^6\left[\xi^{(6)}(r) + \frac{3}{r}\xi^{(5)}(r) - \frac{3}{r^2}\xi^{(4)}(r) + \frac{6}{r^3}\xi^{(3)}(r) - \frac{9}{r^4}\xi^{(2)}(r) + \frac{9}{r^5}\xi^{(1)}(r)\right]\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \\ & c^7\left[\psi^{(6)}(r) + \frac{3}{r}\psi^{(5)}(r) - \frac{3}{r^2}\psi^{(4)}(r) + \frac{6}{r^3}\psi^{(3)}(r) - \frac{9}{r^4}\psi^{(2)}(r) + \frac{9}{r^5}\psi^{(1)}(r)\right]\frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)} + \ldots\ldots\ldots\ldots\ldots \end{aligned} \tag{64}$$

6. Particular Cases

Case 1: If $\xi(r) = r^2$ and $\psi(r) = r$ the Eq. (64) reduces to

$$w(r,t) = r^2 + \frac{4c^2 t^{\alpha}}{\Gamma(\alpha+1)} + ctrE_{\alpha,2}\left(\frac{c^2}{r^2}Mt^{\alpha}\right), \tag{65}$$

where $E_{\alpha}(t) = \sum_{n=0}^{\infty}\frac{t^n}{\Gamma(n\alpha+1)}$, $E_{\alpha,a}(t) = \sum_{n=0}^{\infty}\frac{t^n}{\Gamma(n\alpha+a)}$ are Mittag-Leffler function and generalized Mittag-Leffler function respectively and $M = [1.3.5\ldots\ldots\ldots\ldots(2n-1)]^2$.

Case 2: If $\xi(r) = r$ and $\psi(r) = r$ the Eq. (64) gives rise to

$$w(r,t) = rE_{\alpha}\left(\frac{c^2}{r^2}Mt^{\alpha}\right) + ctrE_{\alpha,2}\left(\frac{c^2}{r^2}Mt^{\alpha}\right), \tag{66}$$

Case 3: If $\xi(r) = r^2$ and $\psi(r) = r^2$ the Eq. (64) leads to

$$w(r,t) = r^2 + ctr^2 + \frac{4c^2 t^{\alpha}}{\Gamma(\alpha+1)} + \frac{4c^3 t^{\alpha+1}}{\Gamma(\alpha+2)}. \tag{67}$$

The above-obtained results from Eqs. (65)-(67) are a good agreement with the results obtained by Srivastava et al. [8].

7. Standard Vibration Equation (SVE)

The SVE is found by putting $\alpha = 2$ in the fractional VE which is as follows

$$D_t^2 w = c^2\left(D_{rr}w + \frac{1}{r}D_r w\right). \tag{68}$$

The approximate solutions of Eq. (7.1) can be obtained by putting $\alpha = 2$ in the Eqs. (6.1)-(6.3), we have

(a) For Case 1

$$w(r,t) = r^2\left[1 + c\left(\frac{t}{r}\right) + 2c^2\left(\frac{t}{r}\right)^2 + \frac{c^3}{6}\left(\frac{t}{r}\right)^3 + \frac{c^5}{120}\left(\frac{t}{r}\right)^5 + \ldots\ldots\ldots\right]. \tag{69}$$

(b) For Case 2

$$w(r,t) = \left[r + c\,t\,r + + \frac{c^2t^2}{2r} + \frac{c^3t^3}{6r^2} + \frac{c^4t^4}{24r^3} + \frac{c^5t^5}{120r^4} + \ldots\ldots\ldots\ldots\right]. \tag{70}$$

(c) For Case 3

$$w(r,t) = \left[r^2 + ctr^2 + +2c^2t^2 + \frac{2}{3}c^3t^3\right]. \tag{71}$$

The above-obtained results for SVE are same as Srivastava et al. [8]

8. Numerical Results and Discussion

In this segment, $w(r,t)$ is computed for different values of t and α in three different cases that have been discussed above and depicted in Tables 1-3 and Figures 1-6. It is observed from Figures 1, 3 and 5 that $w(r,t)$ increases with the increase of r and t for a fixed c. From figures 2, 4 and 6, it is noticed that $w(r,t)$ increases with the increase of t for fixed r and c but $w(r,t)$ decreases with the increase of α. It is also observed that the rate of change of $w(r,t)$ is more in case 3 than the other two cases which confirms that the rate of change of $w(r,t)$ is more when $\xi(r)$ and $\psi(r)$ are nonlinear than $\xi(r)$ and $\psi(r)$ are linear. From Tables 1, 2 and 3 it is concluded that the results solved by the present method are an excellent agreement with the result of Srivastava et al. [8]. All the calculations and plots have been performed by Maple 2016 software package.

Table 1. Comparative analysis among q-HATM [8], LDM [8] and RPSM for $w(r,t)$ at different values of α and t when $c=5$ and $r=6$ for Case 1.

t	α	w_{LDM}		w_{q-HATM}		w_{RPSM}
		$n=1, h=-1$		$n=2, h=-1.9$		$n=3, h=-2.9$
0.2	1.5	48.84069915	48.840699	48.840737	48.840725	48.8415
	2	44.02781659	44.027816	44.027815	44.027816	44.0278
0.4	1.5	67.67015655	67.670156	67.670776	67.670575	67.6820
	2	56.22348737	56.223487	56.223484	56.223486	56.2235
0.6	1.5	90.74714285	90.747142	90.750151	90.749177	90.7980
	2	72.75992569	72.759925	72.759916	72.759923	72.7598
0.8	1.5	117.590204	117.590204	117.597202	117.594995	117.6890
	2	93.82174202	93.821742	93.821686	93.821715	93.8205
1	1.5	148.1551003	148.155100	148.152828	148.153869	148.0990
	2	119.6163907	119.616390	119.615949	119.616127	119.6060

Table 1. Comparative analysis among q-HATM [8], LDM [8] and RPSM for $w(r,t)$ at different values of α and t when $c=5$ and $r=6$ for Case 2.

t	α	w_{LDM}		w_{q-HATM}		w_{RPSM}
		$n=1, h=-1$		$n=2, h=-1.9$		$n=3, h=-2.9$
0.2	1.5	12.39363752	12.393637	12.393836	12.393770	12.3976
	2	12.11134445	12.111344	12.111344	12.111344	12.1113
0.4	1.5	19.44466400	19.444664	19.446532	19.445917	19.4789
	2	18.56001541	18.560015	18.560014	18.560015	18.5600
0.6	1.5	27.30617809	27.306178	27.312877	27.310713	27.4116
	2	25.52687115	25.526871	25.526858	25.526865	25.5265
0.8	1.5	36.25325862	36.253258	36.262771	36.259864	36.3641
	2	33.21266497	33.212664	33.212509	33.212573	33.2085
1	1.5	46.89642831	46.896428	46.862463	46.874131	46.4003
	2	41.85613567	41.856135	41.854834	41.855325	41.8176

Table 1. Comparative analysis among q-HATM [8], LDM [8] and RPSM for $w(r,t)$ at different values of α and t when $c=5$ and $r=6$ for Case 3.

t	α	w_{LDM}		w_{q-HATM}		w_{RPSM}
		$n=1, h=-1$		$n=2, h=-1.9$		$n=3, h=-2.9$
0.2	1.5	81.419694	81.419694	81.419691	81.419694	81.419694
	2	74.666666	74.666666	74.666665	74.666666	74.666666
0.4	1.5	142.255183	142.255183	142.255172	142.255181	142.255183
	2	121.333333	121.333333	121.333329	121.333332	121.333333
0.6	1.5	220.915409	220.915409	220.915385	220.915406	220.915409
	2	180.000000	180.0000	179.999988	179.999998	180.000000
0.8	1.5	319.949750	319.949750	319.949706	319.949744	319.949750
	2	254.666666	254.666666	254.666643	254.666663	254.666666
1	1.5	441.675833	441.675833	441.675762	441.675824	441.675833
	2	349.333333	349.333333	349.333291	349.333327	349.333333

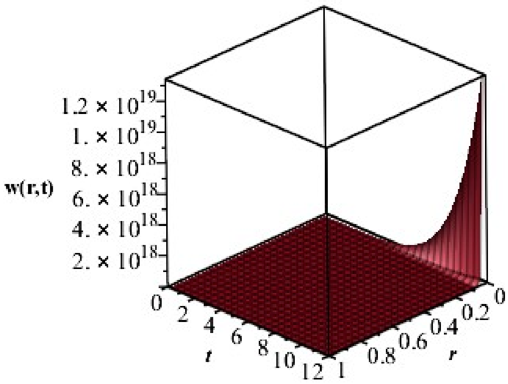

Fig. 1. Solution plot of $w\left(r,t\right)$ with respect to r and t obtained by RPSM when $\alpha=1.5$ and $c=5$ for case 1.

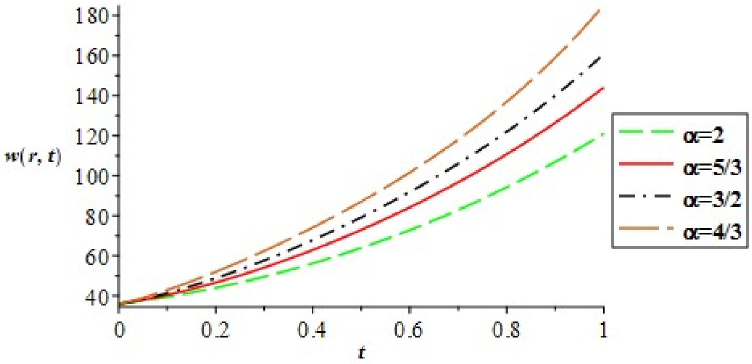

Fig. 2. Solution plot of $w\left(r,t\right)$ vs. t obtained by RPSM for different values of α when $r=6$ and $c=5$ for case 1.

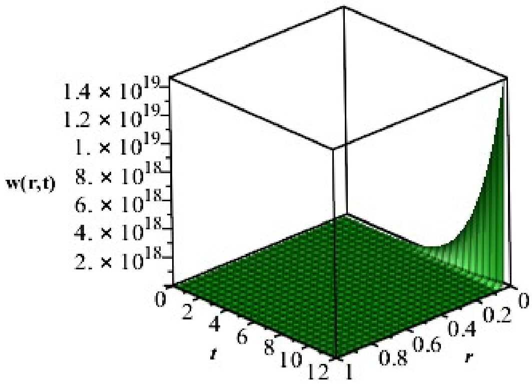

Fig. 3. Solution plot of $w\left(r,t\right)$ with respect to r and t obtained by RPSM when $\alpha=1.5$ and $c=5$ for case 2.

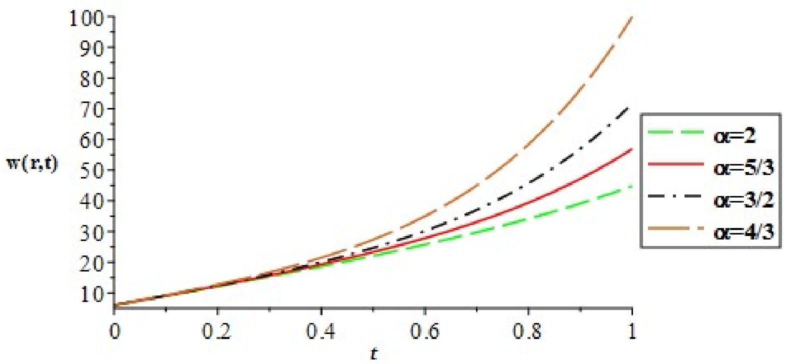

Fig. 4. Solution plot of $w\left(r,t\right)$ vs. t obtained by RPSM for different values of α when $r=6$ and $c=5$ for case 2.

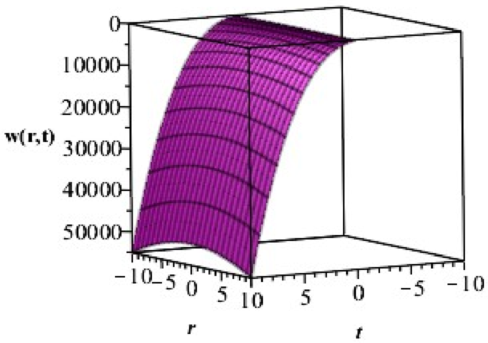

Fig. 5. Solution plot of $w(r,t)$ with respect to r and t obtained by RPSM when $\alpha = 1.5$ and $c = 5$ for case 3.

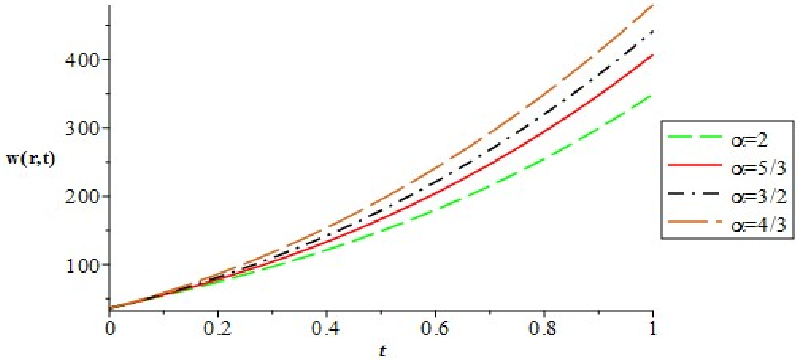

Fig. 6. Solution plot of $w(r,t)$ vs. t obtained by RPSM for different values of α when $r = 6$ and $c = 5$ for case 3.

9. Conclusion

In this article, RPSM is successfully implemented to examine the fractional VE for the large membrane. It has been noted that the solutions of fractional VE for large membrane achieved by the methods described in the literature are an excellent agreement with the proposed method. Thus, it is concluded that RPSM is a quite useful, convenient and powerful computational method to find the solution of the fractional problem arising in mathematics and engineering.

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Conflict of Interest

The author(s) declared no potential conflicts of interest with respect to the research, authorship and publication of this article.

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