J. Appl. Comput. Mech., 5(4) (2019) 627-638
DOI: 10.22055/JACM.2018.26729.1352

ISSN: 2383-4536
jacm.scu.ac.ir

M&MoCS
Shahid Chamran University of Ahvaz

Journal of
Applied and Computational Mechanics

Research Paper

Modified Multi-level Residue Harmonic Balance Method for Solving Nonlinear Vibration Problem of Beam Resting on Nonlinear Elastic Foundation

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Received August 07 2018; Revised September 19 2018; Accepted for publication October 14 2019.
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Abstract. Nonlinear vibration behavior of beam is an important issue of structural engineering. In this study, a mathematical modeling of a forced nonlinear vibration of Euler-Bernoulli beam resting on nonlinear elastic foundation is presented. The nonlinear vibration behavior of the beam is investigated by using a modified multi-level residue harmonic balance method. The main advantage of the method is that only one nonlinear algebraic equation is generated at each solution level. The computational time of using the new method is much less than that spent on solving the set nonlinear algebraic equations generated in the classical harmonic balance method. Besides the new method can generate higher-level nonlinear solutions neglected by previous multi-level residue harmonic balance methods. The results obtained from the proposed method compared with those obtained by a classical harmonic balance method to verify the accuracy of the method which shows good agreement between the proposed and classical harmonic balance method. Besides, the effect of various parameters such as excitation magnitude, linear and nonlinear foundation stiffness, shearing stiffness etc. on the nonlinear vibration behaviors are examined.

Keywords: Harmonic balance; Nonlinear dynamics; Nonlinear foundation; Large amplitude vibration.

1. Introduction

Over the past decades, the nonlinear vibration analysis of beam, plate, shell has been research interest of many researchers for its variety of engineering application such as civil, aeronautical, mechanical, etc. [1-8]. A significance research about the nonlinear vibration of beam resting on elastic foundation has been performed because of its various applications in engineering such as civil, mechanics, marine, aerospace engineering etc. These structures are often subjected to large amplitude vibration and its governing equation become nonlinear. In the recent year some researchers and engineers have focused on developing structural theories for beams, plates and shells in which various shear deformation theories are introduced such as higher order shear deformation theory with Green-Lagrange sense and nonlinearity in von Karman sense [9-13]. Chen and Wu [14] developed a new higher order shear deformation theory to study composite laminated beam. A generalization of the Euler–Bernoulli and Timoshenko beam theories using the simplified Green–Lagrange strain tensor is presented by Reddy and Mahaffey [15]. Panda and Singh [16, 17] presented a nonlinear finite element model for geometrically large amplitude free vibration analysis of doubly curved composite spherical and shallow shell panel using higher order shear deformation theory. Nonlinear free vibration response of functionally graded curved shell panels is computed based on higher order shear deformation theory and Green-Lagrange type nonlinearity is considered by Kar and Panda [18]. Besides, several research articles are found where Panda et al. [19-27] used higher order shear deformation theory and Green-Lagrange type nonlinearity. Kaci et al. [28] used a new higher order shear deformation theory for the Post-buckling analysis of composite beam. Mohamed

Published online: April 29 2019

et al. [29] developed a simple and refined trigonometric higher-order beam theory for bending and vibration of functionally graded beam. A nonlocal zeroth-order shear deformation theory for free vibration of functionally graded nanoplates resting on elastic foundation is presented by Bounouara et al. [30]. Further same theory is applied by Bellifa et al. [31] for the nonlinear post buckling behavior of nanobeams. Khadra et al. [32] investigated bending and flexural vibration behaviors of functionally graded nanobeams using a nonlocal quasi-3D theory. Mouffoki et al. [33] proposed a shear deformation theory to study the effect of moisture and temperature on free vibration behavior of functionally graded nanobeams resting on elastic foundation. Attia et al. [34] presented a refined four variable plate theory to analyze thermomechanical bending of functionally graded plates resting on elastic foundation. Generally, it is difficult to determine the exact or closed-form solutions of those equations. In that case, analytical approximation methods [35] and numerical methods [36, 37] are used to study these problems. Although, the numerical methods are straightforward, but it requires heavy computational effort and proper initial guess. Therefore, many researchers focused on analytical approximation methods to investigate these problems. There are several analytical approximation methods such as Perturbation methods [38-39], Homotopy Analysis Method (HAM) [40-41], Homotopy Perturbation Method (HPM) [42], Variational Iteration Method (VIM) [43], Harmonic Balance Method (HBM) [44-46] etc. have been developed for determination of the solution of that nonlinear problems. Foda [47] analyzed the large-amplitude free vibrations of beams by using KBM method. Coskun and Engin [48] used perturbation method to study the effect of nonlinear vibration of a flexible beam on elastic foundation. Peng et al. [49] introduced a semi-analytical method based on the Lindstedt-Poincare and differential quadrature approximation methods. They investigate the geometrically nonlinear vibration response of Euler-Bernoulli beams for various boundary conditions. Pirbodaghi et al. [50] used the homotopy analysis method to investigate the nonlinear vibration behaviour of an axially loaded Euler-Bernoulli beam. In their study, they examined the effect of vibration amplitude on the nonlinear frequency and buckling load. An analytic solution, namely homotopy analysis method was used by Sedighi et al. [51] of transversal oscillation of quintic non-linear beam. Also, Motallebi et al. [52] analyzed the free nonlinear vibration of a simply supported beam under axial force using the homotopy analysis method. Sedighi and Daneshmand [53] investigated the nonlinear transversely vibrating beams problem by using homotopy perturbation method. Baghani et al. [54] used variational iteration method to investigate the large amplitude free vibrations and post-buckling analysis of unsymmetrically laminated composite beams on nonlinear elastic foundation. An analysis of large amplitude vibration and post-buckling of functionally graded beams setting on non-linear elastic foundation using variational method is carried out by Fallah and Aghdam [55]. Yaghoobi and Torabi [56, 57] also used the same method to analysis the post-buckling and nonlinear free vibration of geometrically imperfect functionally graded beams resting on nonlinear elastic foundation. In addition, the effect of nonlinear elastic foundation on large amplitude free and forced vibration of functionally graded beam using variational iteration method is investigated by Kanani et al. [58]. Hasan et al. [59] used multi-level residue harmonic balance method to investigate nonlinearly vibrating beams on a linear elastic foundation. Rahman and Lee [60] introduced a new modified multi-level residue harmonic balance method for solving nonlinearly vibrating double-beam problem. In the above literature, most of the authors considered functionally graded beam resting on elastic foundation.

In this paper, the forced vibration of a nonlinear isotopic and symmetric beam resting on nonlinear elastic foundation is investigated. Although, Euler-Bernoulli beam theory has neglected the shear deformation and inertia rotary the theoretical formulation of the governing equation is presented based on Euler-Bernoulli beam theory. In the present work, the mid plane stretching is considered, therefore the governing equation is formed the cubic nonlinearity. In addition, linear and nonlinear foundation stiffness and shearing stiffness are considered with the main governing equation. A modified multi-level residue harmonic balance method is applied to determine the approximate solution of the equation. The results obtained by the proposed method are verified by comparing those from classical harmonic balance method. In addition, the effect of various parameters which include excitation magnitude, linear and nonlinear foundation stiffness, shearing stiffness etc. are presented herein.

2. Theoretical Model of Beam

In Fig. 1, a beam of length L, width b and thickness h, setting on a nonlinear elastic foundation is considered. Young's modulus and material density of the beam are E and ρ , respectively. The normal strain and normal stress (details in Appendix A) are given by [58]

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + z\,\frac{\partial^2 w}{\partial x^2} \tag{1}$$

$$\sigma_x = E\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + z\,\frac{\partial^2 w}{\partial x^2}\right] \tag{2}$$

The force and moment resultants (details in Appendix B) are defined by

$$N = EA\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] \tag{3}$$

$$M = EI\frac{\partial^2 w}{\partial x^2} \tag{4}$$

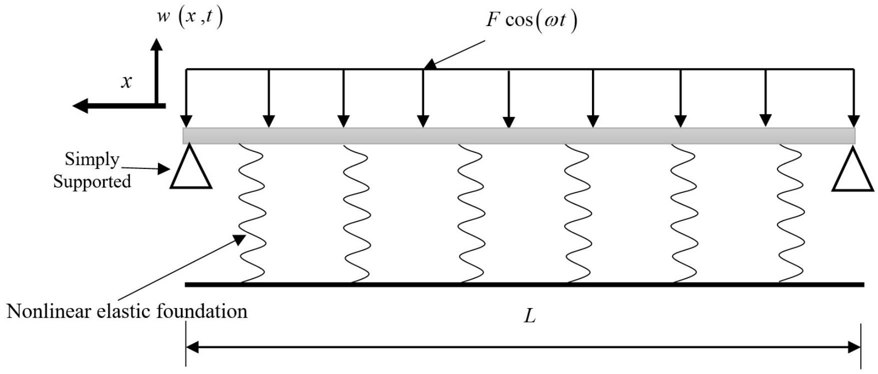


Fig. 1. Model of nonlinear beam resting on nonlinear foundation

The equations of motion governing the axial and transverse vibration of beam can be derived as [58] by applying Hamilton's principle

$$m\frac{\partial^2 u}{\partial t^2} - \frac{\partial N}{\partial x} = f_u \tag{5}$$

$$m\frac{\partial^2 w}{\partial t^2} + \frac{\partial^2 M}{\partial x^2} - N\frac{\partial^2 w}{\partial x^2} = f_w \tag{6}$$

where m is the mass per unit length of the unreformed beam, f_u and f_w are the distributed axial and transverse forces, respectively. Substituting Eqs. (3-4) into Eqs. (5-6), we obtain

$$m\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left[EA\left(\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right)\right] = f_u \tag{7}$$

$$m\frac{\partial^2 w}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 w}{\partial x^2}\right) - EA\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]\frac{\partial^2 w}{\partial x^2} = f_w \tag{8}$$

We assume that in-plane inertia is negligible and distributed axial force is zero, then from Eq. (5) we obtain (details in Appendix C)

$$\frac{\partial u}{\partial x} = -\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{1}{2L}\int_0^L \left(\frac{\partial w}{\partial x}\right)^2 dx \tag{9}$$

Taking into account Eq. (9) in Eq. (8) and Eq. (8) can be written as

$$m\frac{\partial^2 w}{\partial t^2} + EI\frac{\partial^4 w}{\partial x^4} - \frac{EA}{2L}\frac{\partial^2 w}{\partial x^2}\int_0^L \left(\frac{\partial w}{\partial x}\right)^2 dx = f_w \tag{10}$$

where f_w is the transverse force component, which consists of both the reaction force of the elastic foundation and the distributed harmonic force acting on the beam can be expressed as [58]:

$$f_w = -k_L w - k_N w^3 + k_S\frac{\partial^2 w}{\partial x^2} + F\cos(\omega t) \tag{11}$$

where k_L and k_N are linear and nonlinear coefficients of elastic foundation, respectively and k_S is the coefficient of shear stiffness of the elastic foundation. F is the excitation amplitude ($F = \rho A\kappa g$); ω is the excitation frequency; ρ is the material density; κ is the dimensionless excitation magnitude; and g is the gravity acceleration (9.81 ms^{-2}). Substitute Eq. (11) into Eq. (10) one can arrive at:

$$EI\frac{\partial^4 w}{\partial x^4} + m\frac{\partial^2 w}{\partial t^2} + k_L w + k_N w^3 - k_s\frac{\partial^2 w}{\partial x^2} - \frac{EA}{2L}\frac{\partial^2 w}{\partial x^2}\int_0^L \left(\frac{\partial w}{\partial x}\right)^2 dx = F\cos(\omega t) \tag{12}$$

Eq. (12) represents the nonlinear transverse forced vibration of Euler-Bernoulli beam resting on nonlinear foundation. The boundary conditions of simply supported beam is

$$w(x,t) = w''(x,t) = 0, \quad \text{at} \quad x = 0, L \tag{13}$$

Substitute of $w(x,t) = q(t)\varphi(x)$ in Eq. (12) and apply the Galerkin discretization method to obtain the following equation:

$$\ddot{q} + \omega_1^2 q + \alpha q^3 = F_1 \cos(\omega t) \tag{14a}$$

where

$$\omega_1^2 = (EI\int_0^L \varphi\varphi''''dx)/(m\int_0^L \varphi^2 dx) + k_L/m - (k_S\int_0^L \varphi\varphi''dx)/(m\int_0^L \varphi^2 dx)$$

$$\alpha = (k_N\int_0^L \varphi^4 dx)/(m\int_0^L \varphi^2 dx) - (EA\int_0^L \varphi\varphi''dx\int_0^L \varphi'^2 dx)/(2mL\int_0^L \varphi^2 dx) \tag{14b}$$

$$F_1 = (F\int_0^L \varphi dx)/(m\int_0^L \varphi^2 dx) = \kappa g(\int_0^L \varphi dx)/(m\int_0^L \varphi^2 dx)$$

and q is the modal amplitude, φ is the structural mode shape, $\varphi(x) = \sin(\pi x/L)$; for simply supported beam.

3. Solution Procedure

Consider the solution of Eq. (14) as follows

$$q(t) \cong q_0(t) + q_1(t) + q_2(t) + \cdots \tag{15}$$

where $q_0(t)$, $q_1(t)$ and $q_2(t)$ are zero, first and second level solutions, respectively.

3.1. Zero Level Solution

Substitute Eq. (15) into Eq (14), the following zero level governing equation is obtained

$$\ddot{q}_0 + \omega_1^2 q_0 + \alpha q_0^3 - F_1\cos(\omega t) = \Delta_0(t) \tag{16}$$

The zero-level solution of Eq. (16) is considered in the following form

$$q_0(t) = A0_1 \cos(\omega t) \tag{17}$$

where $A0_1$ is the 1st order harmonic component of the zero-level amplitude; $\Delta_0(t)$ is the unbalanced higher order harmonic term at the zero level. Putting Eq. (17) into Eq. (16), equate the coefficient of $\cos(\omega t)$ to obtain

$$\frac{3}{4}\alpha A0_1^3 + \left(-\omega^2 + \omega_1^2\right)A0_1 - F_1 = 0 \tag{18}$$

For damped vibration, Eq. (18) can be written in the following form

$$A0_1 = \frac{F_1}{-\omega^2 + \omega_1^2 + \frac{3}{4}\alpha A0_1^2 + 2j\xi\omega\omega_p} \tag{19}$$

where $j = \sqrt{-1}$ and ω_p is the zero-level nonlinear peak frequency (in the case of linear vibration $\omega_p = \omega_1$).

3.2. First Level Solution

Similarly, substitute Eq. (15) into Eq (14), to obtain the following the 1st level governing equation

$$\ddot{q}_1 + \omega_1^2 q_1 + \alpha(3q_0^2 q_1 + 3q_0 q_1^2 + q_1^3) + \Delta_0(t) = \Delta_1(t) \tag{20}$$

The solution of Eq. (20) is chosen as follows

$$q_1(t) = A1_1\cos(\omega t) + A1_3\cos(3\omega t) \tag{21}$$

where $A1_1$ and $A1_3$ are the 1st and 3rd harmonic components of the 1st level response; $\Delta_1(t)$ is the unbalanced higher order harmonic term at the 1st level. Substitute Eq. (21) into Eq. (20), equate the coefficient of $\cos(\omega t)$ and neglect all the nonlinear terms associated with $A1_1$ and $A1_3$ to obtain Eq. (22a). Similarly, equating the coefficient of $\cos(3\omega t)$ and considering both the linear and nonlinear terms associated with $A1_1$ and $A1_3$, Eq. (22b) is obtained.

$$A1_1 = l1_1 A1_3 \tag{22a}$$

$$a1_1 A1_3^3 + a1_2 A1_3^2 + a1_3 A1_3 + a1_4 = 0 \tag{22b}$$

where $l1_1 = (-3/4\times\alpha A0_1^2)/(-\omega^2+\omega_1^2+9/4\times\alpha A0_1^2)$, $a1_1 = 1/4\times\alpha(3+6l1_1^2+l1_1^3)$, $a1_2 = 3/4\times\alpha(4l1_1+l1_1^2)A0_1$, $a1_3 = -9\omega^2+\omega_1^2+3/4\times\alpha(2+l1_1)A0_1^2$ and $a1_4 = 1/4\times\alpha A0_1^3$.

The complex forms for damped vibration can be obtained by replacing $-\omega^2+\omega_1^2$, $-9\omega^2+\omega_1^2$ with $-\omega^2+\omega_1^2+2j\xi\omega\omega_p$, $-9\omega^2+\omega_1^2+6j\xi\omega\omega_{p1}$, respectively; ω_{p1} is the first-level nonlinear peak frequency. Solving Eqs. (22a-b), the unknown coefficients $A1_1$ and $A1_3$ are obtained. Then, the first-level overall amplitude is defined by:

$$A1 = \sqrt{(A0_1 + A1_1)^2 + A1_3^2} \tag{23}$$

3.3. Second Level Solution

Similarly, substitute Eq. (17) into Eq. (16), to obtain the following the 2nd level governing equation

$$\ddot{q}_2 + \omega_1^2 q_2 + \alpha\left(3q_0^2q_2 + 6q_0q_1q_2 + 3q_1^2q_2 + 3q_0q_2^2 + 3q_1q_2^2 + q_2^3\right) + \Delta_1(t) = \Delta_2(t) \tag{24}$$

Consider the following solution form of Eq. (24)

$$q_2(t) = A2_1\cos(\omega t) + A2_3\cos(3\omega t) + A2_5\cos(5\omega t) \tag{25}$$

where $A2_1$, $A2_3$ and $A2_5$ are the 1st, 3rd and 5th harmonic components of the 2nd-level response; $\Delta_2(t)$ is the unbalanced higher order harmonic term at the 2nd-level. Substitute Eq. (25) into Eq. (24), equate the coefficients $\cos(\omega t)$ and $\cos(3\omega t)$, and ignore all the nonlinear terms associated with $A2_1$, $A2_3$ and $A2_5$ to obtain Eqs. (26a–b). Similarly, equating the coefficient of $\cos(5\omega t)$ and keeping all the linear and nonlinear terms associated with $A2_1$, $A2_3$ and $A2_5$, Eq. (26c) is obtained.

$$A2_1 = l2_1 A2_5 + c2_1 \tag{26a}$$

$$A2_3 = l2_3 A2_5 + c2_3 \tag{26b}$$

$$a2_1 A2_5^3 + a2_2 A2_5^2 + a2_3 A2_5 + a2_4 = 0 \tag{26c}$$

where $l2_1$, $l2_3$, $c2_1$, $c2_3$, $a2_1$, $a2_2$, $a2_3$ and $a2_4$ are given in Appendix D. The complex forms for damped vibration can be obtained by replacing $-\omega^2+\omega_1^2$, $-9\omega^2+\omega_1^2$, $-25\omega^2+\omega_1^2$ with $-\omega^2+\omega_1^2+2j\xi\omega\omega_p$, $-9\omega^2+\omega_1^2+6j\xi\omega\omega_{p1}$, $-25\omega^2+\omega_1^2+10j\xi\omega\omega_{p2}$ respectively; ω_{p2} is the second-level nonlinear peak frequency. Solving Eqs. (26a–c), the values of unknown coefficients $A2_1$, $A2_3$ and $A2_5$ can be obtained. Then, the second-level overall amplitude is defined by:

$$A2 = \sqrt{(A0_1 + A1_1 + A2_1)^2 + (A1_3 + A2_3)^2 + A2_5^2} \tag{27}$$

4. Results and Discussion

In this section, the dimension and material properties of the beam is considered as follows: beam dimension is $0.5m \times 0.2m \times 1mm$; Young's modulus of the beam, $E = 71\times10^9 N/m^2$; mass per unit volume of the beam $\rho = 2700kg/m^3$; radius of gyration, $r = \sqrt{I/A}$; I is the moment of inertia and A is the cross-sectional area; non-dimensional linear, nonlinear and shearing foundation coefficients are $\bar{K}_L = L^4 k_L / EI$, $\bar{K}_N = L^4 r^2 k_N / EI$ and $\bar{K}_S = L^2 k_S / EI$, respectively.

In Figure 2, a comparison of the solution curves obtained from the proposed method and a classical harmonic balance method are presented. The first-level solution is plotted in this figure. In the comparison of results, it can be seen that the result obtained from the proposed method agree reasonably well with those from the classical harmonic balance method.

Figures 3a–c show the amplitude ratios of the beam plotted against the normalised excitation frequency for various non-dimensional foundation stiffness coefficients. Figure 3a~~,~~ shows that the normalised frequency ratio decreases with the increase of linear foundation stiffness, however, it is seen that the amplitude is nearly to be constant with the change of linear foundation stiffness. Figure 3b shows that the zero-level and first-level nonlinear resonant peaks values become lower when the nonlinear foundation stiffness increases. It means that the nonlinear foundation stiffness significantly increases the nonlinearity of the beam. From figure 3c, it can be seen that when the shearing foundation stiffness increases, the normalised zero-level and first-level peak frequencies decreases, and the peak amplitude slowly increases. It can be said in other words that the shearing foundation stiffness reduce the nonlinearity of the beam.

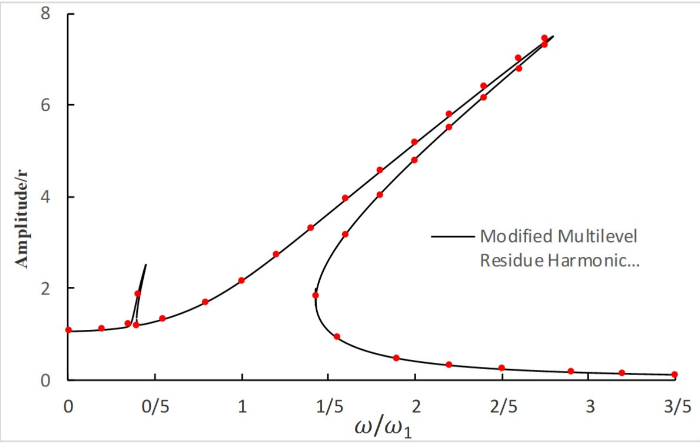


Fig. 2. Comparison between the results obtained by proposed method and classical harmonic balance method

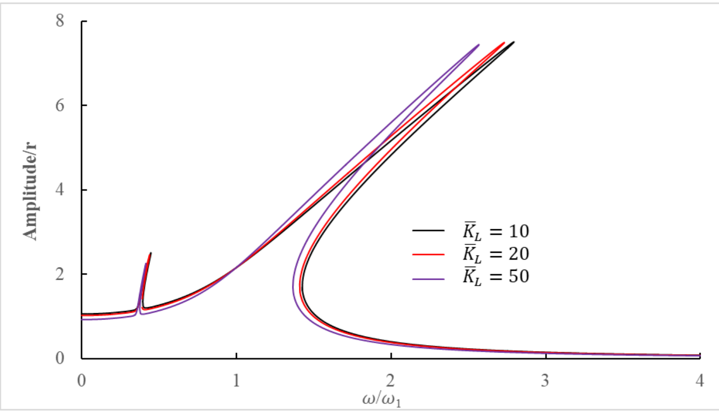


Fig. 3(a). Amplitude ratio versus excitation frequency for various linear foundation coefficients ($\xi = 0.01, \kappa = 0.2, \bar{K}_N = 10, \bar{K}_S = 10$)

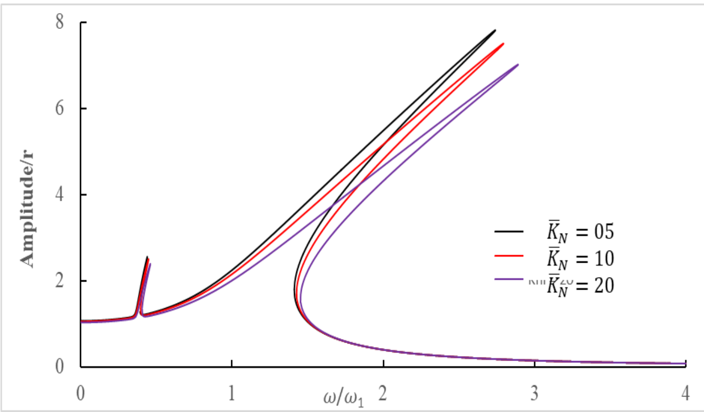


Fig. 3(b). Amplitude ratio versus excitation frequency for various non-linear foundation coefficients ($\xi = 0.01, \kappa = 0.2, \bar{K}_L = 10, \bar{K}_S = 10$)

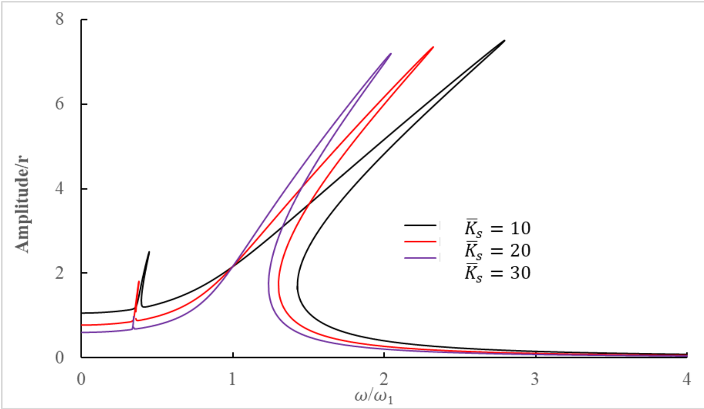
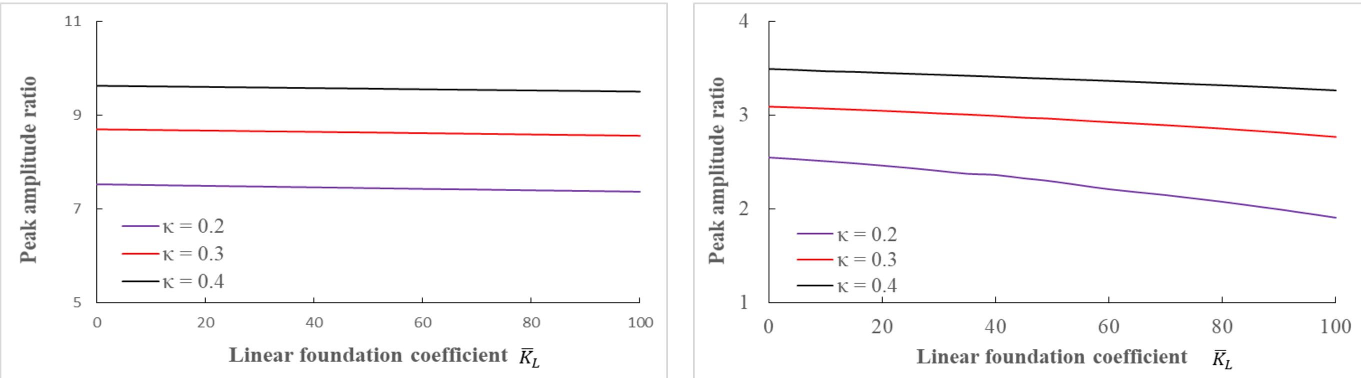


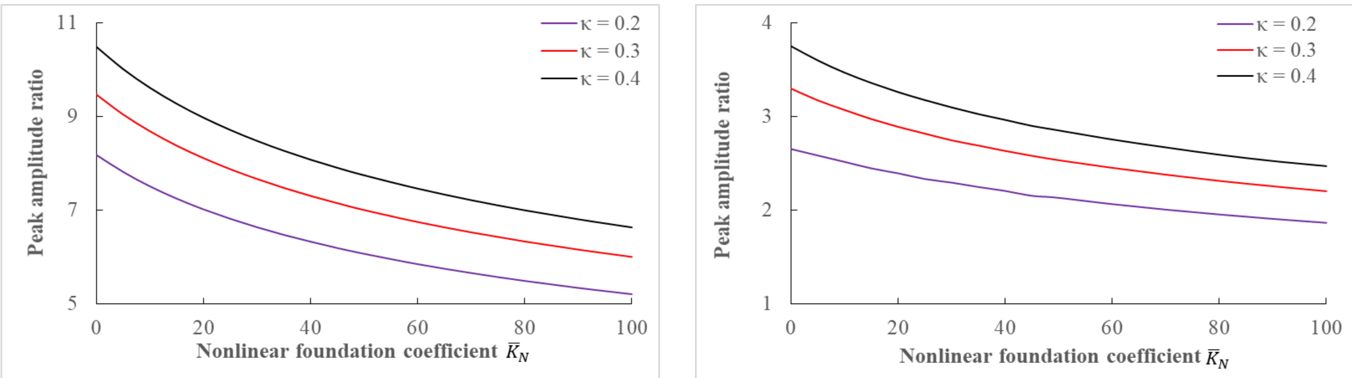
Fig. 3(c). Amplitude ratio versus excitation frequency for various shearing foundation coefficients ($\xi = 0.01, \kappa = 0.2, \bar{K}_L = 10, \bar{K}_N = 10$)

Figures 4a–b, 5a–b, and 6a–b show the zero-level and 1st-level peak amplitude ratios of the beam, which are plotted against the normalised linear foundation stiffness, nonlinear foundation stiffness, and shearing stiffness, respectively for various excitation magnitude κ . Generally, it is evident from these figures that as the excitation magnitude increases, both the zero-level and first-level peak values increase; while the nonlinear and shearing stiffness of foundation increase, both the zero-level and first-level peak values decrease. From figure 4a-b, it is seen that as the linear foundation coefficient increases, only the first-level peak values decrease monotonically, and zero-level peak values are not sensitive. From figure 6b, it is also seen that when the shearing coefficient increases first level nonlinear resonant peak values abruptly decreases and after the certain values of $\bar{K}_S$ first-level peak does not appear (for example in the case of $\kappa = 0.2$, when $\bar{K}_S > 40$).



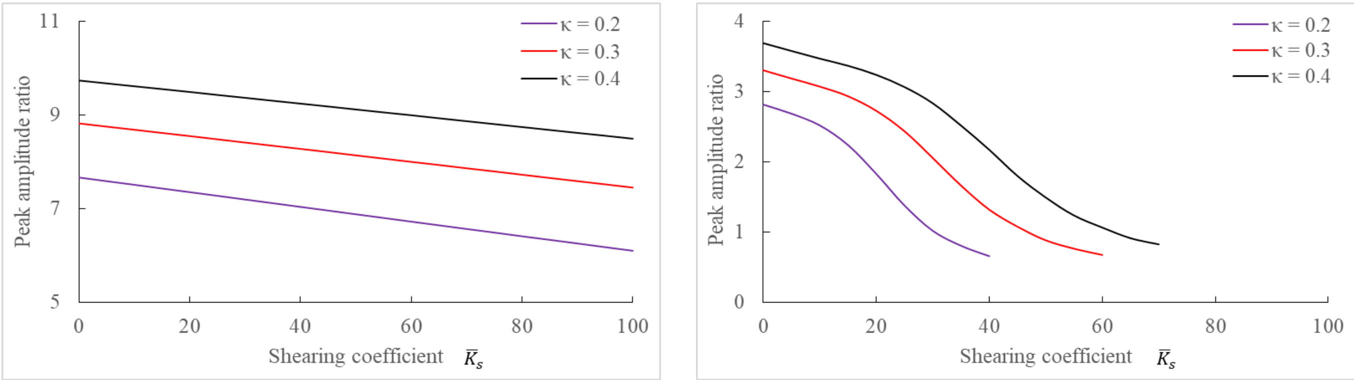
(a) (b)

Fig. 4. (a) Zero-level and (b) first-level peak amplitude ratio versus linear foundation coefficient for various excitation levels ($\xi = 0.01, \bar{K}_N = 10, \bar{K}_S = 10$)



(a) (b)

Fig. 5. (a) Zero-level and (b) first-level peak amplitude ratio versus linear foundation coefficient for various excitation levels ($\xi = 0.01, \bar{K}_L = 10, \bar{K}_S = 10$)



(a) (b)

Fig. 6. (a) Zero-level and (b) first-level peak amplitude ratio versus linear foundation coefficient for various excitation levels ($\xi = 0.01, \bar{K}_L = 10, \bar{K}_N = 10$)

5. Conclusion

In this study, the mathematical modelling of a nonlinear beam resting on nonlinear foundation is presented and a new modified multi-level residue harmonic balance method has been successfully applied to solve a nonlinear beam resting on nonlinear foundation considering nonlinear foundation stiffness and shearing stiffness form cubic governing equation of beam. The main focus of the method is that only one nonlinear algebraic equation is generated at each solution level. This reduces the computational effort compared with solving the coupled nonlinear algebraic equations generated in the classical harmonic balance method. This method can able to generate the higher-level nonlinear solutions that are neglected by the previous modified harmonic balance method. The accuracy of results of the proposed method shows good agreement between the proposed and the classical harmonic balance method. The effects of various foundation stiffness and excitation magnitude on the nonlinear vibration behaviour have been examined. From this study it can be concluded that:
1) The nonlinear and shearing foundation stiffness imposes a large effect on the nonlinear beam vibration;
2) Linear foundation stiffness is less sensitive to the nonlinear vibration compared with nonlinear and shearing foundation stiffness.

Conflict of Interest

The author(s) declared no potential conflicts of interest with respect to the research, authorship and publication of this article.

Funding

The author(s) received no financial support for the research, authorship and publication of this article.

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Appendix A

According to the Euler–Bernoulli beam theory, axial displacement U and transvers displacement W along the x and z coordinates of an arbitrary point in the beam, can be expressed as [58]:

$$U(x,z,t)=u(x,t)+z\,\frac{\partial w(x,t)}{\partial x} \tag{A1}$$

$$W(x,z,t)=w(x,t) \tag{A2}$$

where $u(x,t)$ an $w(x,t)$ are displacement components in the mid-plane and t is the time. The beam is assumed to be elastic and isotropic, with small uniform thickness h. The effects of transverse shear deformations and rotatory inertia are neglected. The transverse deflection, w, is large compared with the beam thickness h leading to geometrical non-linearity but is very small compared with the beam length L $(w << L)$. With these conditions the beam analogue of von Karman plate theory. According to von Karman strain-displacement relation normal strain can be expressed in the form

$$\varepsilon_x = \frac{\partial U}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 = \frac{\partial u}{\partial x} + z\frac{\partial^2 w}{\partial x^2} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \tag{A3}$$

Stress-strain relation can be written in the following form

$$\sigma_x = E\,\varepsilon_x = E\left[\frac{\partial u}{\partial x} + z\frac{\partial^2 w}{\partial x^2} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] \tag{A4}$$

Appendix B

The force and moment resultants (per unit width) are defined by:

$$\begin{aligned} N &= b\int_z \sigma_x dz = bE\int_{-h/2}^{h/2}\left[\frac{\partial u}{\partial x} + z\frac{\partial^2 w}{\partial x^2} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]dz \\ &= bE\left[\frac{\partial u}{\partial x}z + \frac{1}{2}z^2\frac{\partial^2 w}{\partial x^2} + \frac{1}{2}z\left(\frac{\partial w}{\partial x}\right)^2\right] \\ &= Ebh\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] = EA\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] \end{aligned} \tag{B1}$$

$$\begin{aligned} M &= b\int_z \sigma_x z dz = bE\int_{-h/2}^{h/2}\left[\frac{\partial u}{\partial x} + z\frac{\partial^2 w}{\partial x^2} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]zdz \\ &= bE\left[\frac{\partial u}{\partial x}\frac{z^2}{2} + \frac{z^3}{3}\frac{\partial^2 w}{\partial x^2} + \frac{z^2}{4}\left(\frac{\partial w}{\partial x}\right)^2\right] \\ &= \frac{1}{12}Ebh^3\frac{\partial^2 w}{\partial x^2} = EI\frac{\partial^2 w}{\partial x^2} \end{aligned} \tag{B2}$$

Appendix C

We assume that in-plane inertia is negligible and distributed axial force is zero then the Eq. (5) can be expressed as $\partial N / \partial x = 0$.

$$\begin{aligned} &\text{i.e.}\quad N = c_1 \\ &\text{i.e.}\quad EA\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] = c_1 \\ &\text{i.e.}\quad \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 = \frac{c_1}{EA} \\ &\text{i.e.}\quad \frac{\partial u}{\partial x} = -\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{c_1}{EA} \end{aligned} \tag{C1}$$

Integrating Eq. (C1) with respect to x over the beam length, one can obtain:

$$u(L,t) - u(0,t) = -\frac{1}{2}\int_0^L \left(\frac{\partial w}{\partial x}\right)^2 dx + \frac{c_1}{EA}L \tag{C2}$$

For immovable beam condition $u(L,t) = u(0,t) = 0$, Hence, from Eq. (C2) it can be obtained:

$$c_1 = \frac{EA}{2L}\int_0^L \left(\frac{\partial w}{\partial x}\right)^2 dx \tag{C3}$$

Putting Eq. (C3) into Eq. (C1), one can find:

$$\frac{\partial u}{\partial x} = -\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{1}{2L}\int_0^L \left(\frac{\partial w}{\partial x}\right)^2 dx \tag{C4}$$

Appendix D

$$l2_1 = \frac{p_2 q_3 - p_3 q_2}{p_1 q_2 - p_2 q_1}, l2_3 = \frac{p_3 q_1 - p_1 q_3}{p_1 q_2 - p_2 q_1} \tag{D1}$$

$$c2_1 = -\frac{p_4 q_2}{p_1 q_2 - p_2 q_1}, c2_3 = \frac{p_4 q_1}{p_1 q_2 - p_2 q_1} \tag{D2}$$

$$p_1 = -\omega^2 + \omega_1^2 + \frac{3}{4}\alpha\left(3A0_1^2 + 6A0_1A1_1 + 3A1_1^2 + 2A0_1A1_3 + 2A1_1A1_3 + 2A1_3^2\right) \tag{D3}$$

$$p_2 = \frac{3}{4}\alpha\left(A0_1^2 + 2A0_1A1_1 + A1_1^2 + 4A0_1A1_3 + 4A1_1A1_3\right) \tag{D4}$$

$$p_3 = \frac{3}{4}\alpha\left(2A0_1A1_3 + 2A1_1A1_3 + A1_3^2\right) \tag{D5}$$

$$p_4 = \frac{3}{4}\alpha\left(3A0_1A1_1^2 + A1_1^3 + 2A0_1A1_1A1_3 + A1_1^2A1_3 + 2A0_1A1_3^2 + 2A1_1A1_3^2\right) \tag{D6}$$

$$q_1 = \frac{3}{4}\alpha\left(A0_1^2 + 2A0_1A1_1 + A1_1^2 + 4A0_1A1_3 + 4A1_1A1_3\right) \tag{D7}$$

$$q_2 = -9\omega^2 + \omega_1^2 + \frac{3}{4}\alpha\left(2A0_1^2 + 4A0_1A1_1 + 2A1_1^2 + 3A1_3^2\right) \tag{D8}$$

$$q_3 = \frac{3}{4}\alpha\left(A0_1^2 + 2A0_1A1_1 + A1_1^2 + 2A0_1A1_3 + 2A1_1A1_3\right) \tag{D9}$$

$$a2_1 = \frac{3}{4}\alpha\left(1 + 2l2_1^2 + l2_1^2 l2_3 + 2l2_3^2 + l2_1 l2_3^2\right) \tag{D10}$$

$$a2_2 = \frac{3}{4}\alpha\left[\left(l2_3^2 + 4l2_1 + 2l2_1l2_3\right)\left(A0_1 + A1_1 + c2_1\right) + \left(l2_1^2 + 4l2_3 + 2l2_1l2_3\right)\left(A1_3 + c2_3\right)\right] \tag{D11}$$

$$\begin{aligned} a2_3 = &-25\omega^2 + \omega_1^2 + \frac{3}{4}\alpha[\left(2 + l2_1\right)\left(c2_3 + A1_3\right)^2 + \left(2 + l2_3\right)\left(c2_1 + A1_1\right)^2 + \\ &\left(2 + l2_3\right)\left(A1_1^2 + 2A0_1c2_1 + 2A0_1A1_1\right) + 2\left(c2_1 + A0_1 + A1_1\right)\left(l2_1 + l2_3\right)\left(c2_3 + A1_3\right)] \end{aligned} \tag{D12}$$

$$a2_4 = \frac{3}{4}\alpha\left(c2_1 + A0_1 + A1_1\right)\left(c2_3 + A1_3\right)\left(c2_1 + c2_3 + A0_1 + A1_1 + A1_3\right) \tag{D13}$$