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Research Paper

Axial and Torsional Free Vibrations of Elastic Nano-Beams by Stress-Driven Two-Phase Elasticity

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Abstract. Size-dependent longitudinal and torsional vibrations of nano-beams are examined by two-phase mixture integral elasticity. A new and efficient elastodynamic model is conceived by convexly combining the local phase with strain- and stress-driven purely nonlocal phases. The proposed stress-driven nonlocal integral mixture leads to well-posed structural problems for any value of the scale parameter. Effectiveness of stress-driven mixture is illustrated by analyzing axial and torsional free vibrations of cantilever and doubly clamped nano-beams. The local/nonlocal integral mixture is conveniently replaced with an equivalent differential law equipped with constitutive boundary conditions. Exact solutions of fundamental natural frequencies associated with strain- and stress-driven mixtures are evaluated and compared with counterpart results obtained by strain gradient elasticity theory. The provided numerical benchmarks can be effectively employed for modelling and design of Nano-Electro-Mechanical-Systems (NEMS).

Keywords: Free vibrations; Nonlocal integral elasticity; Mixtures; Size effects; Hellinger-Reissner variational principle; Analytical modelling.

1. Introduction

Due to superior mechanical, chemical and electrical properties, nano-structured materials are extensively employed to build innovative nano-electro-mechanical systems (NEMS) [1, 2]. Since experimental tests at the nano-scales are difficult to perform, methodologies of classical continuum mechanics based on gradient, couple-stress and nonlocal laws can be conveniently adopted to design nano-structured sensors and actuators. Recent contributions are addressed in [3-23] and review contributions can be found in [24-27].

Eringen integral model (EIM) [28, 29] is one of the most noteworthy theories of nonlocal elasticity. In the framework of EIM, the stress field is defined as the output of integral convolution between the elastic strain field and a suitable averaging kernel. In nonlocal problems involving unbounded domains, Eringen proposed to substitute the EIM with the equivalent differential law, namely Eringen differential model (EDM). EDM is widely employed in the literature to analyze the mechanical behavior of nano-beams [30-37]. However, equivalence between EIM and EDM for bounded domains requires the satisfaction of suitable constitutive boundary conditions. It may be shown that several structural problems formulated according to EDM are ill-posed as a result of incompatibility between nonlocal equilibrium conditions [38, 39]. To partially resolve the ill-posedness of the EIM, a two-phase local-nonlocal mixture, namely mixture Eringen integral model (MEIM), is defined by a convex combination of local and nonlocal phases wherein the nonlocal fraction is governed by the EIM [40-49]. Nevertheless, a singular behavior emerges in the limit of a vanishing local fraction, and consequently, the ill-posedness of the EIM is not completely eliminated [50, 51].

An effective stress-driven nonlocal integral model (SDM) has been recently proposed by G. Romano and R. Barretta [52, 53]. Unlike EIM, SDM leads to well-posed structural problems and provides results which are technically significant. The SDM is efficiently employed for flexural and torsional analysis [54-56], nonisothermal structural problems [57], linear and nonlinear analysis of transverse free vibrations [58, 59] and longitudinal free vibrations of Bernoulli-Euler nano-beams [60] in addition to examining the flexural response of Timoshenko nano-beams [61, 62]. More recently, a two phase constitutive mixture, namely mixture stress-driven nonlocal integral model (MSDM), is introduced by Barretta et al. [63] to examine size-dependent static behavior of functionally graded nano-beams. The MSDM is then exploited to investigate the static torsional behavior of nano-beams [64]. MSDM is defined by a convex combination of local and nonlocal phases wherein the nonlocal fraction is modeled according to SDM. MSDM provides a viable approach for the design of nano-structures.

The aim of the present study is to formulate for the first time a local- nonlocal mixture stress-driven integral model for axial and torsional vibrations of nano-beams. The plan is the following. Axial and torsional analysis and modelling of nano-beams in the framework of Eringen's strain-driven local/nonlocal two-phase mixture is preliminarily recalled in Sect.2. The corresponding equivalent differential law, equipped with constitutive boundary conditions, is also determined. The stress-driven two-phase mixture model is then formulated in Sect.3. For comparison sake, strain gradient elasticity for vibrations of elastic nano-beams is presented in in Sect.4 by utilizing Hellinger-Reissner's variational principle. The stress-driven two-phase mixture illustrated in Sect.3 is adopted in Sect.5 to establish exact natural frequencies of cantilever and doubly-clamped nano-beams.

2. Eringen local/nonlocal two-phase mixture of elasticity

A straight beam of length L with cross-section Ξ and material density ρ, subjected to a distribution of torsional couples per unit length f_t, axial forces per unit length f_a in the interval $[0,L]$ and concentrated torsional couple $\bar{T}$ and axial force $\bar{N}$ is considered and schematically illustrated in Fig. 1. The abscissa along the beam axis will be denoted by x and the position vector of a cross-sectional point with respect to the centroid O will be denoted by $\mathbf{r}$. The displacement field of the beam, up to an inessential additional rigid body motion, can be expressed as

$$\mathbf{u}(x,\mathbf{r},t) = u(x,t) + \varphi(x,t)\mathbf{Rr} \tag{1}$$

where u and φ denote cross-sectional axial displacement and torsional rotation at the abscissa x and the vector $\mathbf{Rr}$ is the $\pi/2$ counterclockwise rotated of the position vector $\mathbf{r}$. The non-vanishing components of linearized strain tensor field, kinematically compatible with the displacement **u**, write as [65]

$$\varepsilon(x,t) = \kappa_a(x,t), \qquad \boldsymbol{\gamma}(x,\mathbf{r},t) = \kappa_t(x,t)\mathbf{Rr} \tag{2}$$

where the geometric axial strain $\kappa_a = \partial u/\partial x$ and torsional curvature $\kappa_t = \partial\varphi/\partial x$ are kinematically compatible with the axial displacement u and torsional rotation field φ, respectively. The differential and classical boundary conditions of dynamic equilibrium are given by [65]

$$\begin{aligned} &\frac{\partial T(x,t)}{\partial x} + f_t = 0, && \frac{\partial N(x,t)}{\partial x} + f_a = 0 \\ &\left(T + \bar{T}\right)\delta\varphi\big|_{x=0} = 0, && \left(N + \bar{N}\right)\delta u\big|_{x=0} = 0 \\ &\left(T - \bar{T}\right)\delta\varphi\big|_{x=L} = 0, && \left(N - \bar{N}\right)\delta u\big|_{x=L} = 0 \end{aligned} \tag{3}$$

where T and N stand for twisting moment and axial force fields. Moreover, A and J are the area of the cross-section and the polar moment of inertia about the center of the cross-section, accordingly. The torsional couples f_t and axial forces f_a per unit length can be conveniently replaced with the corresponding d'Alembert inertial couples $(-\rho J\partial^2\varphi/\partial t^2)$ and forces $(-\rho A\partial^2 u/\partial t^2)$ [66].

The local-nonlocal mixture constitutive model can then be introduced as a convex combination of local and nonlocal phases [67, 68]. In the framework of MEIM, the torsional and axial nonlocal elastic law for nano-beams can be expressed as

$$\begin{aligned} T(x,t) &= \alpha C_t\kappa_t(x,t) + (1-\alpha)\int_0^L \psi(x-y,l_c)C_t\kappa_t(y,t)dy \\ N(x,t) &= \alpha C_a\kappa_a(x,t) + (1-\alpha)\int_0^L \psi(x-y,l_c)C_a\kappa_a(y,t)dy \end{aligned} \tag{4}$$

where $0 \le \alpha \le 1$ designates the mixture parameter so that the fully nonlocal constitutive law corresponds to $\alpha = 0$ and the local constitutive model can be recovered by setting $\alpha = 1$. Also, $l_c = e_0 L$ is the characteristic length depicting the mutual long-range elastic interactions along with e_0 representing a material constant. The scalar kernel ψ represents the space weight function fulfilling the symmetry, positivity and limit impulsivity properties [52, 53]. Furthermore, the elastic torsional and axial stiffness are given by [56]

$$C_t = \iint_{\Xi} G\mathbf{r}\bullet\mathbf{r}dA, \qquad C_a = \iint_{\Xi} EdA \tag{5}$$

where the dot into the integral denotes the inner product between vectors and G and E are the shear and Euler-Young moduli. The bi-exponential function is commonly adopted in nonlocal elasticity for the scalar kernel ψ as [51]

$$\psi(x, l_c) = \frac{1}{2l_c}\exp\left(-\frac{|x|}{l_c}\right) \tag{6}$$

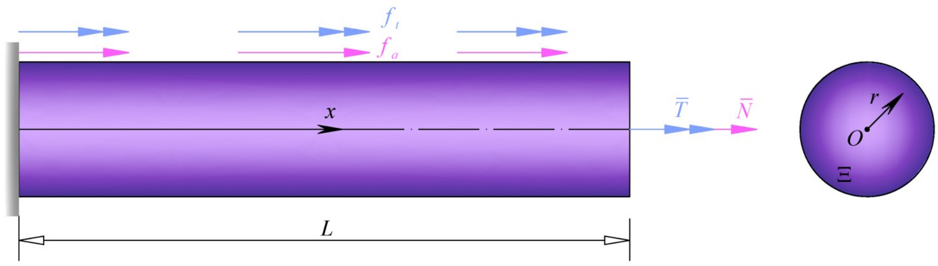

Fig. 1. Coordinate system and configuration of a nano-beam

As discussed by Romano et al. [50, 51], the nonlocal MEIM model Eq. (4) endowed with the bi-exponential kernel Eq. (6) is equivalent to the subsequent mixture Eringen differential model (MEDM)

$$\begin{aligned} \frac{T(x,t)}{l_c^2} - \frac{\partial^2 T(x,t)}{\partial x^2} &= \frac{C_t}{l_c^2}\kappa_t(x,t) - \alpha C_t \frac{\partial^2 \kappa_t(x,t)}{\partial x^2} \\ \frac{N(x,t)}{l_c^2} - \frac{\partial^2 N(x,t)}{\partial x^2} &= \frac{C_a}{l_c^2}\kappa_a(x,t) - \alpha C_a \frac{\partial^2 \kappa_a(x,t)}{\partial x^2} \end{aligned} \tag{7}$$

equipped with the torsional and axial constitutive boundary conditions, at $x = 0$ and $x = L$ as

$$\begin{aligned} &\frac{\partial T(0,t)}{\partial x} - \frac{T(0,t)}{l_c} = \alpha C_t\left(\frac{\partial \kappa_t(0,t)}{\partial x} - \frac{\kappa_t(0,t)}{l_c}\right), \quad \frac{\partial T(L,t)}{\partial x} + \frac{T(L,t)}{l_c} = \alpha C_t\left(\frac{\partial \kappa_t(L,t)}{\partial x} + \frac{\kappa_t(L,t)}{l_c}\right) \\ &\frac{\partial N(0,t)}{\partial x} - \frac{N(0,t)}{l_c} = \alpha C_a\left(\frac{\partial \kappa_a(0,t)}{\partial x} - \frac{\kappa_a(0,t)}{l_c}\right), \quad \frac{\partial N(L,t)}{\partial x} + \frac{N(L,t)}{l_c} = \alpha C_a\left(\frac{\partial \kappa_a(L,t)}{\partial x} + \frac{\kappa_a(L,t)}{l_c}\right) \end{aligned} \tag{8}$$

Setting the mixture parameter $\alpha = 0$, the fully nonlocal Eringen strain-driven integral model and the equivalent differential and constitutive boundary conditions, are correspondingly recovered from Eq. (4) and Eq. (7-8). Since the elastodynamic problem based on the EDM admits no solution for bounded continuous nano-beams, as a result the MEDM tends to an ill-posed elastodynamic problem as the mixture parameter $\alpha \to 0$ [39, 50-51].

3. Stress-driven two-phase mixture of elasticity

MSDM was recently introduced by Barretta et al. [63] and successfully utilized for bending and torsional analyses of Bernoulli-Euler nano-beams [64]. A convex combination of local and nonlocal stress–driven phases, based on the nonlocal stress-driven integral model introduced by Romano and Barretta [52, 53], is employed to define MSDM. In this framework, the elastic axial and torsional curvatures κ_a and κ_t are provided in terms of stress-driven two-phase laws as

$$\begin{aligned} \kappa_t(x,t) &= \alpha\frac{T(x,t)}{C_t} + (1-\alpha)\int_0^L \psi(x-y, l_c)\frac{T(y,t)}{C_t}dy \\ \kappa_a(x,t) &= \alpha\frac{N(x,t)}{C_a} + (1-\alpha)\int_0^L \psi(x-y, l_c)\frac{N(y,t)}{C_a}dy \end{aligned} \tag{9}$$

where the twisting moment T and axial force N fulfill the equilibrium condition Eq. (3) and $0 \le \alpha \le 1$ is the mixture parameter. While the stress–driven purely nonlocal integral law can be recovered by setting $\alpha = 0$, the local constitutive model corresponds to $\alpha = 1$. Utilizing the Helmholtz bi-exponential kernel Eq. (6), the MSDM formulation Eq. (9) is equivalent to the subsequent mixture differential model [50, 51]

$$
\begin{aligned}
&\frac{\kappa_t(x,t)}{l_c^2}-\frac{\partial^2\kappa_t(x,t)}{\partial x^2}=\frac{T(x,t)}{l_c^2 C_t}-\alpha\frac{\partial^2}{\partial x^2}\frac{T(x,t)}{C_t}\\
&\frac{\kappa_a(x,t)}{l_c^2}-\frac{\partial^2\kappa_a(x,t)}{\partial x^2}=\frac{N(x,t)}{l_c^2 C_a}-\alpha\frac{\partial^2}{\partial x^2}\frac{N(x,t)}{C_a}
\end{aligned}
\tag{10}
$$

subjected to the torsional and axial constitutive boundary conditions, at $x=0$ and $x=L$ as

$$
\begin{aligned}
&\frac{\partial\kappa_t(0,t)}{\partial x}-\frac{\kappa_t(0,t)}{l_c}=\frac{\alpha}{C_t}\left(\frac{\partial T(0,t)}{\partial x}-\frac{T(0,t)}{l_c}\right),\quad \frac{\partial\kappa_t(L,t)}{\partial x}+\frac{\kappa_t(L,t)}{l_c}=\frac{\alpha}{C_t}\left(\frac{\partial T(L,t)}{\partial x}+\frac{T(L,t)}{l_c}\right)\\
&\frac{\partial\kappa_a(0,t)}{\partial x}-\frac{\kappa_a(0,t)}{l_c}=\frac{\alpha}{C_a}\left(\frac{\partial N(0,t)}{\partial x}-\frac{N(0,t)}{l_c}\right),\quad \frac{\partial\kappa_a(L,t)}{\partial x}+\frac{\kappa_a(L,t)}{l_c}=\frac{\alpha}{C_a}\left(\frac{\partial N(L,t)}{\partial x}+\frac{N(L,t)}{l_c}\right)
\end{aligned}
\tag{11}
$$

Eringen strain-driven mixture leads to ill-posed structural problems and reveals a singular behavior when the local part vanishes [50, 51], while the stress-driven mixture Eq. (9) does not yield inconsistencies.

4. Strain gradient theory of elasticity

The size-dependent axial and torsional elastic model of nano-beams is formulated employing the Hellinger-Reissner variational principle [21, 22], with the aim of comparing the results of the two-phase constitutive integral models with the counterpart results of the classical strain gradient theory of elasticity (SGT). Hellinger-Reissner's functional $\mathbb{F}_{\mathrm{HR}}$ is introduced as [22]

$$
\mathbb{F}_{\mathrm{HR}}=\int_0^L\left[T^{(0)}\frac{\partial\varphi}{\partial x}+T^{(1)}\frac{\partial^2\varphi}{\partial x^2}-f_t\varphi-\frac{\left(T^{(0)}\right)^2}{2C_t}+\frac{\left(T^{(1)}\right)^2}{2C_t l_s^2}\right]dx-\bar{T}\varphi\Big|_0^L+\int_0^L\left[N^{(0)}\frac{\partial u}{\partial x}+N^{(1)}\frac{\partial^2 u}{\partial x^2}-f_a\varphi-\frac{\left(N^{(0)}\right)^2}{2C_a}+\frac{\left(N^{(1)}\right)^2}{2C_a l_s^2}\right]dx-\bar{N}u\Big|_0^L \tag{12}
$$

while the fields $T^{(0)},T^{(1)}$ are defined as the dual mathematical objects of the elastic torsional curvature and of its derivative $\kappa_t^{(0)}=\partial\varphi/\partial x,\kappa_t^{(1)}=\partial\kappa_t^{(0)}/\partial x$, the fields $N^{(0)},N^{(1)}$ are introduced as the dual mathematical objects of the elastic axial curvature and of its derivative $\kappa_a^{(0)}=\partial u/\partial x,\kappa_a^{(1)}=\partial\kappa_a^{(0)}/\partial x$, respectively. Also, l_s is a length-scale parameter.

While the axial displacement u and torsional rotation of cross-sections φ along with the fields $N^{(0)},T^{(0)},N^{(1)},T^{(1)}$ are chosen as the primary variables subject to variation, performing the first-order variation of the Hellinger-Reissner functional, followed by integration by parts, yields

$$
\begin{aligned}
\delta\mathbb{F}_{\mathrm{HR}}=\int_0^L\Bigg[&\delta T^{(0)}\left(\frac{\partial\varphi}{\partial x}-\frac{T^{(0)}}{C_t}\right)+\delta T^{(1)}\left(\frac{\partial^2\varphi}{\partial x^2}-\frac{T^{(1)}}{C_t l_s^2}\right)-\left(\frac{\partial T^{(0)}}{\partial x}-\frac{\partial^2 T^{(1)}}{\partial x^2}+f_t\right)\delta\varphi\\
&+\delta N^{(0)}\left(\frac{\partial u}{\partial x}-\frac{N^{(0)}}{C_a}\right)+\delta N^{(1)}\left(\frac{\partial^2 u}{\partial x^2}-\frac{N^{(1)}}{C_a l_s^2}\right)-\left(\frac{\partial N^{(0)}}{\partial x}-\frac{\partial^2 N^{(1)}}{\partial x^2}+f_a\right)\delta u\\
&+\left(T^{(0)}-\frac{\partial T^{(1)}}{\partial x}-\bar{T}\right)\delta\varphi\Bigg|_0^L+T^{(1)}\delta\frac{\partial\varphi}{\partial x}\Bigg|_0^L+\left(N^{(0)}-\frac{\partial N^{(1)}}{\partial x}-\bar{N}\right)\delta u\Bigg|_0^L+N^{(1)}\delta\frac{\partial u}{\partial x}\Bigg|_0^L
\end{aligned}
\tag{13}
$$

Imposing stationarity of the Hellinger-Reissner functional, differential and boundary conditions of axial and torsional elastic model of strain gradient nano-beam take the form

$$
\begin{aligned}
&\frac{\partial T^{(0)}(x,t)}{\partial x}-\frac{\partial^2 T^{(1)}(x,t)}{\partial x^2}+f_t=0,\quad \frac{\partial N^{(0)}(x,t)}{\partial x}-\frac{\partial^2 N^{(1)}(x,t)}{\partial x^2}+f_a=0,\quad \left(T^{(0)}-\frac{\partial T^{(1)}}{\partial x}+\bar{T}\right)\delta\varphi\Bigg|_{x=0}=0,\\
&\left(N^{(0)}-\frac{\partial N^{(1)}}{\partial x}+\bar{N}\right)\delta u\Bigg|_{x=0}=0,\quad \left(T^{(0)}-\frac{\partial T^{(1)}}{\partial x}-\bar{T}\right)\delta\varphi\Bigg|_{x=L}=0,\quad \left(N^{(0)}-\frac{\partial N^{(1)}}{\partial x}-\bar{N}\right)\delta u\Bigg|_{x=L}=0,\\
&\left(T^{(1)}\delta\frac{\partial\varphi}{\partial x}\right)\Bigg|_{x=0,L}=0,\quad \left(N^{(1)}\delta\frac{\partial u}{\partial x}\right)\Bigg|_{x=0,L}=0
\end{aligned}
\tag{14}
$$

The differential and boundary conditions of dynamic equilibrium can be more simplified introducing the total twisting moment $T=T^{(0)}-\partial T^{(1)}/\partial x$ and total axial force $N=N^{(0)}-\partial N^{(1)}/\partial x$ while assuming arbitrary variations of torsional and axial curvatures

$$\begin{aligned}
&\frac{\partial T(x,t)}{\partial x}=\rho J\frac{\partial^2\varphi(x,t)}{\partial t^2}, && \frac{\partial N(x,t)}{\partial x}=\rho A\frac{\partial^2 u(x,t)}{\partial t^2}\\
&\left(T+\bar{T}\right)\delta\varphi\Big|_{x=0}=\left(T-\bar{T}\right)\delta\varphi\Big|_{x=L}=0, && \left(N+\bar{N}\right)\delta u\Big|_{x=0}=\left(N-\bar{N}\right)\delta u\Big|_{x=L}=0\\
&T^{(1)}\Big|_{x=0,L}=0, && N^{(1)}\Big|_{x=0,L}=0
\end{aligned} \tag{15}$$

where the d'Alembert inertial body force and couple fields [66] are correspondingly employed to derive the differential conditions in dynamic form. As predictable from the Hellinger-Reissner stationary principle, the desired constitutive laws cast as ordinary differential equations. While the torsional fields $T, T^{(0)}, T^{(1)}$ are given by

$$T^{(0)}(x,t)=C_t\frac{\partial\varphi(x,t)}{\partial x},\quad T^{(1)}(x,t)=C_t l_s^2\frac{\partial^2\varphi(x,t)}{\partial x^2},\quad T(x,t)=C_t\left(\frac{\partial\varphi(x,t)}{\partial x}-l_s^2\frac{\partial^3\varphi(x,t)}{\partial x^3}\right) \tag{16}$$

the axial fields $N, N^{(0)}, N^{(1)}$ write as

$$N^{(0)}(x,t)=C_a\frac{\partial u(x,t)}{\partial x},\quad N^{(1)}(x,t)=C_a l_s^2\frac{\partial^2 u(x,t)}{\partial x^2},\quad N(x,t)=C_a\left(\frac{\partial u(x,t)}{\partial x}-l_s^2\frac{\partial^3 u(x,t)}{\partial x^3}\right) \tag{17}$$

The MSDM is employed to derive exact solutions for longitudinal and torsional vibrations of nano-beams and the results are subsequently compared with those obtained by MEDM and SGT.

5. Axial and torsional vibration frequencies of nano-beams

The size-dependent nonlocal laws presented in the previous sections are adopted to examine the longitudinal and torsional free vibrations of cantilever and doubly-clamped nano-beams of length *L*. Hereafter, the acronyms CC and CF stand for clamp-clamp and clamp-free kinematic boundary conditions. Additionally, the acronyms MEDM, MSDM and SGT denote the mixture Eringen differential model, the mixture stress-driven integral model and the strain gradient theory of elasticity.

5.1. Mixture Eringen integral model

As discussed by Romano et al. [50, 51], in the framework of MEIM it is feasible to accomplish the constitutive boundary conditions Eq. (8), and consequently, the integral formulation Eq. (4) and the differential formulation Eq. (7) are equivalent. While employing the differential condition of kinematic compatibility, the MEDM is given by [50, 51]

$$\begin{aligned}
\frac{T(x,t)}{l_c^2}-\frac{\partial^2 T(x,t)}{\partial x^2}&=\frac{C_t}{l_c^2}\frac{\partial\varphi(x,t)}{\partial x}-\alpha C_t\frac{\partial^3\varphi(x,t)}{\partial x^3}\\
\frac{N(x,t)}{l_c^2}-\frac{\partial^2 N(x,t)}{\partial x^2}&=\frac{C_a}{l_c^2}\frac{\partial u(x,t)}{\partial x}-\alpha C_a\frac{\partial^3 u(x,t)}{\partial x^3}
\end{aligned} \tag{18}$$

By virtue of the dynamic differential conditions of equilibrium Eq. $(3)_1$, the twisting moment and axial force from Eq. (18) can be given by

$$\begin{aligned}
\frac{T(x,t)}{l_c^2}&=\rho J\frac{\partial^3\varphi(x,t)}{\partial x\,\partial t^2}+\frac{C_t}{l_c^2}\frac{\partial\varphi(x,t)}{\partial x}-\alpha C_t\frac{\partial^3\varphi(x,t)}{\partial x^3}\\
\frac{N(x,t)}{l_c^2}&=\rho A\frac{\partial^3 u(x,t)}{\partial x\,\partial t^2}+\frac{C_a}{l_c^2}\frac{\partial u(x,t)}{\partial x}-\alpha C_a\frac{\partial^3 u(x,t)}{\partial x^3}
\end{aligned} \tag{19}$$

As a result, the governing equations of motion for the nano-beam consistent with the MEDM, which is equivalent to MEIM, is determined in terms of the axial displacement *u* and torsional rotation φ as

$$\begin{aligned}
\frac{C_t}{l_c^2}\frac{\partial^2\varphi(x,t)}{\partial x^2}-\alpha C_t\frac{\partial^4\varphi(x,t)}{\partial x^4}&=\frac{\rho J}{l_c^2}\frac{\partial^2\varphi(x,t)}{\partial t^2}-\rho J\frac{\partial^4\varphi(x,t)}{\partial x^2\partial t^2}\\
\frac{C_a}{l_c^2}\frac{\partial^2 u(x,t)}{\partial x^2}-\alpha C_a\frac{\partial^4 u(x,t)}{\partial x^4}&=\frac{\rho A}{l_c^2}\frac{\partial^2 u(x,t)}{\partial t^2}-\rho A\frac{\partial^4 u(x,t)}{\partial x^2\partial t^2}
\end{aligned} \tag{20}$$

Torsional and longitudinal vibration frequencies of nano-beams can now be determined with the aid of classical separation of space and time variables as

$$\varphi(x,t)=\Gamma_t(x)\exp(i\,\omega_t t),\qquad u(x,t)=\Gamma_a(x)\exp(i\,\omega_a t) \tag{21}$$

with $i=\sqrt{-1}$, ω_t and ω_a are the natural frequency of torsional and longitudinal vibrations. Substituting the separation of variables into the governing equations while introducing the dimensionless variable ζ, the dimensionless characteristic parameter λ as well as dimensionless torsional and longitudinal frequencies ϖ_t,ϖ_a

$$\zeta=\frac{x}{L},\qquad \lambda_c=\frac{l_c}{L},\qquad \varpi_t^2=\frac{L^2\rho J}{\pi^2 C_t}\omega_t^2,\qquad \varpi_a^2=\frac{L^2\rho A}{\pi^2 C_a}\omega_a^2 \tag{22}$$

leads to the differential equation governing the base function

$$\alpha\lambda_c^2\frac{d^4\Gamma}{d\zeta^4}(\zeta)+\left(\lambda_c^2\pi^2\varpi^2-1\right)\frac{d^2\Gamma}{d\zeta^2}(\zeta)-\pi^2\varpi^2\Gamma(\zeta)=0 \tag{23}$$

While the base function Γ may be replaced with either torsional or longitudinal base functions Γ_t,Γ_a, the dimensionless frequency ϖ can be also substituted with dimensionless torsional or longitudinal frequencies ϖ_t,ϖ_a. The analytical solution of the preceding equation governing the spatial base function can be expressed by

$$\Gamma(\zeta)=\Lambda_1\sin\gamma_1\zeta+\Lambda_2\cos\gamma_1\zeta+\Lambda_3\sinh\gamma_2\zeta+\Lambda_4\cosh\gamma_2\zeta \tag{24}$$

where $\Lambda_k\ (k=1..4)$ are unknown constants needed to be determined by suitable boundary conditions, along with

$$\begin{aligned}\gamma_1^2\Big|_{\text{MEDM}}&=\frac{\left(\lambda_c^2\pi^2\varpi^2-1\right)+\sqrt{\left(\lambda_c^2\pi^2\varpi^2-1\right)^2+4\pi^2\varpi^2\alpha\lambda_c^2}}{2\alpha\lambda_c^2}\\ \gamma_2^2\Big|_{\text{MEDM}}&=\frac{-\left(\lambda_c^2\pi^2\varpi^2-1\right)+\sqrt{\left(\lambda_c^2\pi^2\varpi^2-1\right)^2+4\pi^2\varpi^2\alpha\lambda_c^2}}{2\alpha\lambda_c^2}\end{aligned} \tag{25}$$

Also, employing the differential condition of kinematic compatibility as well as kinetic fields as Eq. (19), the constitutive boundary conditions at $\zeta=0$ and $\zeta=1$, in terms of the spatial base function, write as

$$\begin{aligned}&-\lambda_c\pi^2\varpi^2\Gamma(0)-\left(1-\lambda_c^2\pi^2\varpi^2\right)\frac{d\Gamma}{dx}(0)+\alpha\lambda_c^2\frac{d^3\Gamma}{dx^3}(0)=\alpha\lambda_c\frac{d^2\Gamma}{dx^2}(0)-\alpha\frac{d\Gamma}{dx}(0)\\ &-\lambda_c\pi^2\varpi^2\Gamma(1)+\left(1-\lambda_c^2\pi^2\varpi^2\right)\frac{d\Gamma}{dx}(1)-\alpha\lambda_c^2\frac{d^3\Gamma}{dx^3}(1)=\alpha\lambda_c\frac{d^2\Gamma}{dx^2}(1)+\alpha\frac{d\Gamma}{dx}(1)\end{aligned} \tag{26}$$

In the framework of MEDM, the classical boundary conditions for a doubly clamped nano-beam can be expressed in terms of the spatial base function as $\Gamma(0)=\Gamma(1)=0$. Also, the classical boundary conditions for a cantilever nano-beam in terms of the spatial base function are given by

$$\Gamma(0)=0,\qquad \left(1-\lambda_c^2\pi^2\varpi^2\right)\frac{d\Gamma}{dx}(1)-\alpha\lambda_c^2\frac{d^3\Gamma}{dx^3}(1)=0 \tag{27}$$

A set of algebraic equations in terms of the unknown constants $\Lambda_k\ (k=1..4)$ can be obtained by employing the aforementioned boundary conditions in the solution form of the spatial base function as Eq. (24). It is known that to obtain a non-trivial solution, the system of algebraic equations should be singular. For any given positive dimensionless characteristic parameter λ and the mixture parameter α, the smallest nontrivial solution of the system of algebraic equations corresponds to the fundamental natural frequency of the nano-beam.

5.2. Mixture stress-driven nonlocal integral model

Now, let us consider the longitudinal and torsional free vibration for the MSDM. As demonstrated by Barretta et al. [63] in the framework of MSDM, it is possible to apply the constitutive boundary conditions Eq. (11) and, accordingly, the integral formulation Eq. (9) and the differential formulation Eq. (10) are equivalent. While applying the differential condition of kinematic compatibility, the MSDM is given by

$$\begin{aligned}\frac{1}{l_c^2}\frac{\partial\varphi(x,t)}{\partial x}-\frac{\partial^3\varphi(x,t)}{\partial x^3}&=\frac{T(x,t)}{l_c^2C_t}-\alpha\frac{\partial^2}{\partial x^2}\frac{T(x,t)}{C_t}\\ \frac{1}{l_c^2}\frac{\partial u(x,t)}{\partial x}-\frac{\partial^3u(x,t)}{\partial x^3}&=\frac{N(x,t)}{l_c^2C_a}-\alpha\frac{\partial^2}{\partial x^2}\frac{N(x,t)}{C_a}\end{aligned} \tag{28}$$

By virtue of the dynamic differential conditions of equilibrium Eq. (3)1, the twisting moment and axial force can be given by

$$\begin{aligned} \frac{T(x,t)}{l_c^2} &= \alpha\rho J \frac{\partial^3\varphi(x,t)}{\partial x\,\partial t^2} + \frac{C_t}{l_c^2}\frac{\partial\varphi(x,t)}{\partial x} - C_t\frac{\partial^3\varphi(x,t)}{\partial x^3} \\ \frac{N(x,t)}{l_c^2} &= \alpha\rho A \frac{\partial^3 u(x,t)}{\partial x\,\partial t^2} + \frac{C_a}{l_c^2}\frac{\partial u(x,t)}{\partial x} - C_a\frac{\partial^3 u(x,t)}{\partial x^3} \end{aligned} \tag{29}$$

Therefore, the governing equation of motion for a nano-beam consistent with MSDM can be expressed in terms of the axial displacement u and torsional rotation φ as

$$\begin{aligned} \frac{C_t}{l_c^2}\frac{\partial^2\varphi(x,t)}{\partial x^2} - C_t\frac{\partial^4\varphi(x,t)}{\partial x^4} &= \frac{\rho J}{l_c^2}\frac{\partial^2\varphi(x,t)}{\partial t^2} - \alpha\rho J\frac{\partial^4\varphi(x,t)}{\partial x^2\partial t^2} \\ \frac{C_a}{l_c^2}\frac{\partial^2 u(x,t)}{\partial x^2} - C_a\frac{\partial^4 u(x,t)}{\partial x^4} &= \frac{\rho A}{l_c^2}\frac{\partial^2 u(x,t)}{\partial t^2} - \alpha\rho A\frac{\partial^4 u(x,t)}{\partial x^2\partial t^2} \end{aligned} \tag{30}$$

Applying the classical separation of spatial and time variables as Eq. (21) to the governing equation while employing the definition of dimensionless variables and parameters as Eq. (22), we get

$$\frac{d^4\Gamma}{d\zeta^4}(\zeta) + \left(\alpha\pi^2\varpi^2 - \frac{1}{\lambda_c^2}\right)\frac{d^2\Gamma}{d\zeta^2}(\zeta) - \frac{\pi^2\varpi^2}{\lambda_c^2}\Gamma(\zeta) = 0 \tag{31}$$

where the base function Γ and dimensionless frequency ϖ can be substituted with either torsional or longitudinal base functions Γ_t, Γ_a and dimensionless frequencies ϖ_t, ϖ_a. The analytical solution of the preceding equation governing the spatial base function can be also expressed by Eq. (24). However, the constants γ_1^2, γ_2^2 should be modified for MSDM as

$$\begin{aligned} \gamma_1^2\big|_{\text{MSDM}} &= \frac{1}{2}\left(\alpha\pi^2\varpi^2 - \frac{1}{\lambda_c^2}\right) + \frac{1}{2}\sqrt{\left(\alpha\pi^2\varpi^2 - \frac{1}{\lambda_c^2}\right)^2 + 4\frac{\pi^2\varpi^2}{\lambda_c^2}} \\ \gamma_2^2\big|_{\text{MSDM}} &= -\frac{1}{2}\left(\alpha\pi^2\varpi^2 - \frac{1}{\lambda_c^2}\right) + \frac{1}{2}\sqrt{\left(\alpha\pi^2\varpi^2 - \frac{1}{\lambda_c^2}\right)^2 + 4\frac{\pi^2\varpi^2}{\lambda_c^2}} \end{aligned} \tag{32}$$

Imposing the differential condition of kinematic compatibility along with the kinetic fields as Eq. (29), the constitutive boundary conditions at $\zeta = 0$ and $\zeta = 1$, in terms of the spatial base function, are given by

$$\begin{aligned} \frac{d^2\Gamma}{dx^2}(0) - \frac{1}{\lambda_c}\frac{d\Gamma}{dx}(0) &= -\alpha\pi^2\varpi^2\Gamma(0) + \left(\alpha^2\pi^2\varpi^2\lambda_c - \frac{\alpha}{\lambda_c}\right)\frac{d\Gamma}{dx}(0) + \alpha\lambda_c\frac{d^3\Gamma}{dx^3}(0) \\ \frac{d^2\Gamma}{dx^2}(1) + \frac{1}{\lambda_c}\frac{d\Gamma}{dx}(1) &= -\alpha\pi^2\varpi^2\Gamma(1) + \left(-\alpha^2\pi^2\varpi^2\lambda_c + \frac{\alpha}{\lambda_c}\right)\frac{d\Gamma}{dx}(1) - \alpha\lambda_c\frac{d^3\Gamma}{dx^3}(1) \end{aligned} \tag{33}$$

In the framework of MSDM, the well-known classical boundary conditions for a doubly clamped nano-beam remain unchanged. However, the classical boundary conditions for a cantilever beam in terms of the spatial base function write as

$$\Gamma(0) = 0, \qquad \left(\frac{1}{\lambda_c^2} - \alpha\pi^2\varpi^2\right)\frac{d\Gamma}{dx}(1) - \frac{d^3\Gamma}{dx^3}(1) = 0 \tag{34}$$

Again, enforcing the classical and constitutive boundary conditions, a homogeneous algebraic system in terms of the unknown constants $\Lambda_k\ (k = 1..4)$ can be found. For a non-trivial solution, the determinant of the coefficients of the homogeneous algebraic system has to vanish. Consequently, for any specified positive dimensionless characteristic parameter λ and the mixture parameter α, the smallest nontrivial solution of the system of algebraic equations corresponds to the fundamental natural frequency of the nano-beam.

5.3. Strain gradient theory of elasticity

Finally, substituting the twisting moment and axial force associated with SGT as Eq. (16-17)3 in the differential conditions of equilibrium Eq. (15), the governing equation of motion can be determined in terms of the axial displacement u and torsional rotation φ as

$$C_t\left(\frac{\partial^2\varphi(x,t)}{\partial x^2}-l_s^2\frac{\partial^4\varphi(x,t)}{\partial x^4}\right)=\rho J\frac{\partial^2\varphi(x,t)}{\partial t^2}$$
$$C_a\left(\frac{\partial^2 u(x,t)}{\partial x^2}-l_s^2\frac{\partial^4 u(x,t)}{\partial x^4}\right)=\rho A\frac{\partial^2 u(x,t)}{\partial t^2} \tag{35}$$

Again, utilizing the classical separation of spatial and time variables as Eq. (21) in the governing equations while applying the dimensionless variables and parameters as Eq. (22), leads to

$$\lambda_s^2\frac{d^4\Gamma}{d\zeta^4}(\zeta)-\frac{d^2\Gamma}{d\zeta^2}(\zeta)-\pi^2\varpi^2\Gamma(\zeta)=0 \tag{36}$$

where $\lambda_s=l_s/L$ is the strain gradient characteristic parameter. The analytical solution of the preceding equation governing the spatial base function can be also expressed by Eq. (24). However, the constants γ_1^2,γ_2^2 should be adapted for the strain gradient theory of elasticity as

$$\gamma_1^2\Big|_{\mathrm{SGT}}=\frac{-1+\sqrt{1+4\pi^2\varpi^2\lambda_s^2}}{2\lambda_s^2}$$
$$\gamma_2^2\Big|_{\mathrm{SGT}}=\frac{1+\sqrt{1+4\pi^2\varpi^2\lambda_s^2}}{2\lambda_s^2} \tag{37}$$

The natural higher-order boundary conditions consistent with SGT in terms of the spatial base function at $\zeta=0$ and $\zeta=1$ are expressed as

$$\frac{d^2\Gamma}{d\zeta^2}(0)=\frac{d^2\Gamma}{d\zeta^2}(1)=0 \tag{38}$$

In the framework of SGT, while the classical boundary conditions for a doubly clamped nano-beam remain unaffected, the classical boundary conditions for a cantilever nano-beam in terms of the spatial base function can be expressed as

$$\Gamma(0)=0,\qquad \frac{d\Gamma}{d\zeta}(1)-\lambda_s^2\frac{d^3\Gamma}{d\zeta^3}(1)=0 \tag{39}$$

5.4. Case studies, natural frequencies and comparisons

Figs. 2 and 3 depict the plots of the dimensionless fundamental natural frequency ϖ associated with MEDM, MSDM and SGT for CC and CF boundary conditions, respectively. While the dimensionless characteristic parameter λ is ranging in the interval $]0,0.1[$, the mixture parameter α is assumed to range in the set $\{0.01,0.1,0.5,1\}$. It is obviously deduced from Figs. 2 and 3 that, for a given value of the mixture parameter α, while both MSDM and SGT exhibit a stiffening behavior in terms of the dimensionless characteristic parameter λ, the MEDM reveals a softening behavior.

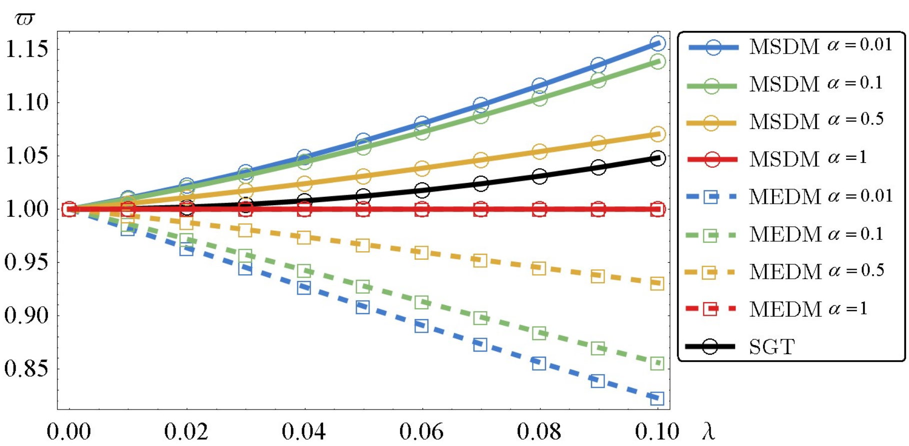

Fig. 2. Doubly clamped nano-beam: dimensionless fundamental natural frequency ϖ versus the characteristic parameter λ

The stiffening effect of MSDM is more noticeable in comparison to SGT. Furthermore, as the mixture parameter α has the effect of stiffening the nano-beam in MEDM, the MSDM reveals a softening behavior in terms of the mixture parameter α. Both mixture nonlocal integral models and strain gradient theory provide the results of the local beam for vanishing dimensionless characteristic parameter $\lambda \to 0$, no matter what the value of the mixture parameter α is. The discrepancy between the fundamental frequencies of the size-dependent models is enhanced for increasing values of the dimensionless characteristic parameter λ. As the mixture parameter $\alpha \to 1$, both mixture nonlocal integral models provide the local beam model independent of the value of dimensionless characteristic parameter λ. MSDM tends to the corresponding SDM for a vanishing mixture parameter $\alpha \to 0$ [52, 53]. On the contrary, the mixture parameter α cannot vanish in MEDM since the corresponding EDM is ill-posed [39, 50-51]. The numerical values of dimensionless fundamental natural frequency ϖ evaluated by MEDM, MSDM and SGT for CC and CF boundary conditions are collected in Tables 1 and 2 for doubly clamped and cantilever nano-beams, respectively.

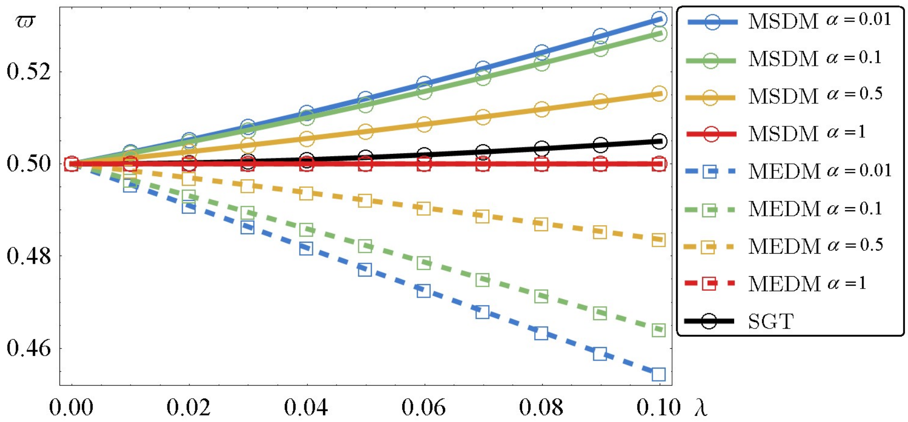

Fig. 3. Cantilever nano-beam: dimensionless fundamental natural frequency ϖ versus the characteristic parameter λ

6. Conclusions

The main outcomes of the present study can be enumerated as follows.

- Natural axial and torsional frequencies determined by MSDM and SGT reveal a stiffening behavior in terms of the small-scale parameter. On the contrary, MEIM leads to natural frequencies decreasing with the scale parameter.
- The stiffening effect of MSDM is more noteworthy compared with the results of SGT.
- The mixture parameter has the effect of increasing the fundamental frequency determined by MEIM. Conversely, MSDM reveals a softening behavior with respect to the mixture parameter.
- Both two-phase nonlocal models and SGT provide the local response as the nonlocal parameter goes to zero, independently of the mixture parameter.
- As the mixture parameter tends to 1, both two-phase nonlocal models provide local frequency responses independently of the nonlocal parameter.
- MSDM tends to the corresponding well-posed SDM for a vanishing mixture parameter. Conversely, the mixture parameter cannot vanish in the MEIM since the corresponding EIM is ill-posed for bounded beams.

The proposed stress-driven local/nonlocal mixture model provides an effective approach for the dynamical analysis of modern beam-like nano-devices utilized in Nanotechnology.

Conflict of Interest

The author(s) declared no potential conflicts of interest with respect to the research, authorship and publication of this article.

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