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Dynamic Response Analysis of Fractionally Damped Beams Subjected to External Loads using Homotopy Analysis Method

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Abstract. This paper examines the solution of a damped beam equation whose damping characteristics are well-defined by the fractional derivative (FD). Homotopy Analysis Method (HAM) is applied for calculating the dynamic response (DR). Unit step and unit impulse functions are deliberated for this analysis. Acquired results are illustrated to show the movement of the beam under various sets of parameters with different orders of the FDs. Here FD is defined in the Caputo sense. Obtained results have been compared with the solutions achieved by Adomian decomposition method (ADM) to show the efficiency and effectiveness of the presented method.

Keywords: Fractional damped beam, Fractional derivative, Homotopy Analysis Method, Vibration.

1. Introduction

In current ages, fractional calculus (FC) has played an essential role in various fields, such as solid mechanics, fluid dynamics, ecology and other areas of science and engineering [1-5]. As it is challenging to achieve the analytical solution of the fractional differential equations (FDEs) so one may require consistent and robust numerical methods for the solution of FDEs. Many essential works on FC has been considered in the past couple of years and numerous books had been composed by various authors namely Kiryakov [6], Golmankhaneh [7], Baleanu et al. [8, 9], Miller and Ross [10], Oldham and Spanier [11], Podlubny [12], and Samko et al. [13]. An extensive review of FC is included in these books which may support the student for understanding the elementary ideas of FC. As such, several analytical and numerical techniques have been established for the solution of such types of physical model problems, such as homotopy perturbation method [14-15], Homotopy Perturbation Sumudu Transform Method [16-17], ADM [18-19], sine-cosine method [20] and tanh method [21], etc.

Other related works have been included here for well understanding of the present analysis. Viscoelastically damped structures of half-order FD models have been analyzed by Bagley and Torvik [22, 23]. Also, the fractional model has been used to define relaxation and creep functions of viscoelastic materials by Koeller [24]. The fractional order damping oscillators subjected to impulse load has been examined in [25, 26] by Fourier transformation. The solution of fractional order dynamic systems has been obtained by the implementation of the eigenvector expansion method in [27]. Various numerical and analysis methods have been used in [28-34] to obtain the DR of a fractionally damped model. In current years, many researchers have presented the vibrational analysis of beam made of functionally graded materials and nanocomposites. Mareishi et al. [35] investigated an analytical study on linear and nonlinear free and forced vibrations response of nano-piezo laminated beams enclosed by two piezoelectric layers and resting on nonlinear elastic foundation. A complete analysis of the large amplitude



free and forced vibration response of carbon nanotubes laminated multiscale composite beams subjected to a transverse load using time fractional damping model was presented by He et al. [36]. Lewandowski and Wielentejczyk [37] studied the vibration of viscoelastic beams using FD coupled with Zener rheological model. Freundlich [38] analyzed transient vibrations of a fractional Kelvin-Voigt viscoelastic Bernoulli-Euler cantilever beam with a rigid mass and with base motion.

Lately, the HAM has been introduced which is a useful technique for solving the different problems of fractional order. The HAM was initially proposed in the Ph.D. thesis of Lio [39]. This technique has also been implemented to solve different kinds of equations arising in engineering applications [39-41]. The HAM has a parameter η which gives a simple way to adjust and control the convergence region of the series solution. Moreover, using the so-called η -curve, it is easy to obtain the valid regions of η to attain a convergent series solution. This method gives a rapidly convergent series solution, and usually, a few iterations lead to a very accurate solution. Previously, dynamic analysis of fractional damped beam with loads have been approached by few methods, such as homotopy perturbation method (HPM) and the ADM. It is worth mentioning that the HAM is a versatile method and homotopy perturbation method (HPM), ADM, variational iteration method (VIM) are the particular cases of HAM. As such, our target happens to be used of a generalized computational method viz. HAM. Also, to the best of the authors' knowledge, no DR analysis of fractional damped beam subjected to external loads using the HAM has been done. Moreover, the validity of HAM does not depend on the parameter η .

In this paper, the HAM is applied to obtain the DR of a fractionally damped beam. The same problem has also been analyzed by ref. [32] using the ADM. A damping factor is described with an FD of arbitrary order. Firstly, the basic ideas of the FC are given. Then, application of the HAM for a fractional dynamic system (FDS) subjected to different loads are discussed. Lastly, numerical results, discussions, and conclusions are included.

2. Basics on fractional calculus

Definition 2.1 [10-12]:

The Abel-Riemann (A-R) FD operator D^α of order α is described as

$$D^\alpha u(x) = \begin{cases} \frac{d^m}{dx^m} u(x), & \alpha = m \\ \frac{1}{\Gamma(m-\alpha)} \frac{d}{dx^m} \int_0^x \frac{u(t)}{(x-t)^{\alpha-m+1}} dt, & m-1 < \alpha < m \end{cases} \quad (2.1)$$

where $m \in \mathbb{Z}^+$, $\alpha \in \mathbb{R}^+$ and

$$D^{-\alpha} u(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} u(t) dt, \quad 0 < \alpha \leq 1 \quad (2.2)$$

Definition 2.2 [10-12]:

The A-R fractional order integration operator J^α is described as

$$J^\alpha u(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} u(t) dt, \quad t > 0, \quad \alpha > 0 \quad (2.3)$$

Following the methods proposed by Podlubny [12], we may have

$$J^\alpha t^n = \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)} t^{n+\alpha}, \quad (2.4)$$

$$D^\alpha t^n = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} t^{n-\alpha}, \quad (2.5)$$

Definition 2.3 [10-12]:

The Caputo FD operator D^α of order α is defined as

$${}^c D^\alpha u(x) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^x \frac{u^m(t)}{(x-t)^{\alpha-m+1}} dt, & m-1 < \alpha < m, \\ \frac{d^m}{dt^m} u(x), & \alpha = m. \end{cases} \quad (2.6)$$

Definition 2.4 [10-12]:

$$(a) \quad D_t^\alpha J_t^\alpha f(t) = f(t). \quad (2.7)$$

$$(b) \quad J_t^\alpha D_t^\alpha f(t) = f(t) - \sum_{k=0}^m f^{(k)}(0^+) \frac{t^k}{k!}, \quad \text{for } t > 0$$

Definition 2.5:

The Mittag-Leffler function (MLF) of two parameters is defined by [12] as follow

$$E_{a,b}(x) = \sum_{i=0}^{\infty} \frac{x^i}{\Gamma(ai + b)}, \quad (a > 0, b > 0) \quad (2.8)$$

3. Homotopy Analysis Method (HAM)

Let us consider the fractional PDE as

$$FD(u(x, t)) = 0, \quad (3.1)$$

Where FD is a partial fractional operator, x and t are the variables and $u(x, t)$ is the unknown function. The zero-order deformation equation (DE) is constructed and can be written as [41]

$$(1-q)L[v(x, t; q) - u_0(x, t)] = q\hbar FD(v(x, t; q)) \quad (3.2)$$

where $q \in [0, 1]$ is the bounded parameter, $\hbar$ is a non-zero auxiliary parameter, L is a linear operator, $u_0(x, t)$ is the initial condition (IC) of $u(x, t)$ and $v(x, t; q)$ is the unknown function. If $q = 0$ and $q = 1$, then we obtain

$$v(x, t; 0) = u_0(x, t), \quad v(x, t; 1) = u(x, t). \quad (3.3)$$

As q increases from 0 to 1, then $v(x, t; q)$ varies from $u_0(x, t)$ to the solution of Eq. (3.1). Expanding $v(x, t; q)$ and using Taylor's series concerning q , one has

$$v(x, t; q) = u_0(x, t) + \sum_{n=1}^{\infty} u_n(x, t) q^n \quad (3.4)$$

where

$$u_n(x, t) = \frac{1}{n!} \left. \frac{\partial^n v(x, t; q)}{\partial q^n} \right|_{q=0} \quad (3.5)$$

If L , $u_0(x, t)$, and $\hbar$ are properly chosen then Eq. (3.4) converges at $q = 1$. So we get

$$u(x, t) = u_0(x, t) + \sum_{n=1}^{\infty} u_n(x, t) \quad (3.6)$$

Eq. (3.6) is the solution of Eq. (3.1). Substituting $\hbar = -1$, Eq. (3.2) reduces to

$$(1-q)L[v(x, t; q) - u_0(x, t)] + qFD(v(x, t; q)) = 0 \quad (3.7)$$

The above equation is generally implemented in HPM. So the HPM is the particular instance of the HAM. Now we can define the vector as

$$\vec{u}_n = \{u_0, u_1, \dots, u_n\} \quad (3.8)$$

Differentiating Eq. (3.2) n times with respect to q , putting $q = 0$ and then dividing by $n!$, we have the n -th order DE as follows

$$L[u_n(x, t) - \chi_n u_{n-1}(x, t)] = \hbar FR(u_{n-1}(\vec{x}, t)) \quad (3.9)$$

where

$$FR(u_{n-1}(\vec{x}, t)) = \frac{1}{(n-1)!} \left. \frac{\partial^{n-1} FR(v(x, t; q))}{\partial q^{n-1}} \right|_{q=0} \quad (3.10)$$



and

$$\chi_n = \begin{cases} 0, & n \leq 1, \\ 1, & n > 1, \end{cases} \quad (3.11)$$

Operating Reimann-Liouville integral oprator J^α on both sides of Eq. (3.9), we obtain

$$u_n(x, t) = u_n^*(x, t) + \chi_n u_{n-1}(x, t) + \chi_n \sum_{i=0}^{m-1} u_{n-1}^i(x, t) \frac{t^i}{i!} \quad (3.12)$$

where

$$u_n^*(x, t) = \hbar J^\alpha \left[FR \left(u_{n-1}(\vec{x}, t) \right) \right] \quad (3.13)$$

Eq.(3.13) is the particular solution. In this way, we can obtain $u_1(x, t), u_2(x, t), \dots, u_n(x, t)$. The n th-order approximate solution is written as

$$u(x, t) = \sum_{i=0}^n u_i(x, t) \quad (3.14)$$

if $n \rightarrow \infty$ the solution converges, but few terms of the series give the required solution, in general.

4. Implementation of HAM on fractional damped beam equation

In this study, the equation of motion of the beam is derived by taking the assumption of the Bernoulli-Euler theory. In order to develop the solution of fractional damped viscoelastic beam [32], the fractional damped beam equation of order α is considered as follows:

$$\rho A \frac{\partial^2 u}{\partial t^2} + c \frac{\partial^\alpha u}{\partial t^\alpha} + EI \frac{\partial^4 u}{\partial x^4} = F(x, t) \quad (4.1)$$

where ρ, A, c, E and I denote the density, area, damping coefficient, Young's modulus and moment of inertia of the beam, respectively. Eq. (4.1) defines the dynamic of the viscoelastic beam. $F(x, t)$ and $u(x, t)$ are the external force and transverse displacement respectively. $\partial^\alpha u / \partial t^\alpha$ is the FD of $u(x, t)$ of order $\alpha \in (0, 1)$. Here the IC is taken as

$$u(x, 0) = 0, u_t(x, 0) = 0 \quad (4.2)$$

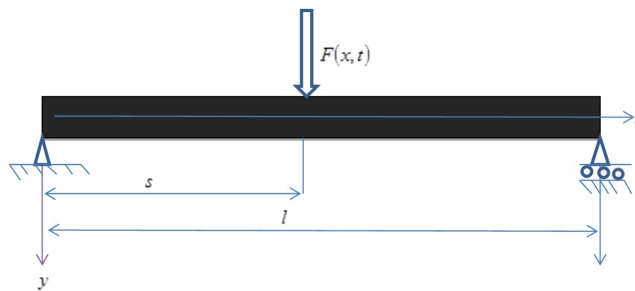


Fig. 1. Schematic diagram of a beam subjected to external load

Now Eq. (4.1) can be revised as

$$\frac{\partial^2 u}{\partial t^2} + \frac{c}{\rho A} \frac{\partial^\alpha u}{\partial t^\alpha} + \frac{EI}{\rho A} \frac{\partial^4 u}{\partial x^4} = \frac{F(x, t)}{\rho A} \quad (4.3)$$

Applying the HAM and taking the auxiliary linear operator as given below

$$L[\varphi(x, t; q)] = \frac{\partial^2 u}{\partial t^2} \quad (4.4)$$

Let us define the operator as

$$NFD[v(x, t; q)] = D_t^2 u + \frac{c}{\rho A} D_t^\alpha u + \frac{EI}{\rho A} D_x^4 u - \frac{F(x, t)}{\rho A} \quad (4.5)$$

According to Eq. (3.7), zeroth-order DE is constructed as

$$(1-q)L[v(x,t;q)-u_0(x,t)]=q\hbar NFD(v(x,t;q)). \quad (4.6)$$

From Eqs. (3.9)-(3.11), m th-order DE is written as

$$L[u_n(x,t)-\chi_n u_{n-1}(x,t)]=\hbar FR\left(u_{n-1}(\vec{x},t)\right) \quad (4.7)$$

Solving Eq. (4.7) for $n \geq 1$, we have

$$u_n(x,t)=\chi_n u_{n-1}(x,t)+\hbar L^{-1}FR\left(u_{n-1}(\vec{x},t)\right) \quad (4.8)$$

where

$$FR\left(u_{n-1}(\vec{x},t)\right)=D_t^2 u_{n-1}+\frac{c}{\rho A} D_t^\alpha u_{n-1}+\frac{EI}{\rho A} D_x^4 u_{n-1}-(1-\chi_n)\frac{F(x,t)}{\rho A} \quad (4.9)$$

Using Eqs. (4.2) and (4.8), we obtain

$$u_0(x,t)=u(x,0)=0 \quad (4.10)$$

$$\begin{aligned} u_1(x,t) &= \chi_1 u_0(x,t) + \hbar L^{-1}FR\left(u_0(\vec{x},t)\right) = \hbar L^{-1}\left[D_t^2 u_0 + \frac{c}{\rho A} D_t^\alpha u_0 + \frac{EI}{\rho A} D_x^4 u_0 - (1-\chi_1)\frac{F(x,t)}{\rho A}\right] \\ &= \hbar L^{-1}\left[\frac{-F(x,t)}{\rho A}\right] \end{aligned} \quad (4.11)$$

$$\begin{aligned} u_2(x,t) &= \chi_2 u_1(x,t) + \hbar L^{-1}FR\left(u_1(\vec{x},t)\right) \\ &= u_1(x,t) + \hbar L^{-1}\left[D_t^2 u_1 + \frac{c}{\rho A} D_t^\alpha u_1 + \frac{EI}{\rho A} D_x^4 u_1 - (1-\chi_2)\frac{F(x,t)}{\rho A}\right] \\ &= u_1(x,t) + \hbar u_1(x,t) + \hbar L^{-1}\left[\frac{c}{\rho A} D_t^\alpha u_1 + \frac{EI}{\rho A} D_x^4 u_1\right] \\ &= \hbar L^{-1}\left[\frac{F(x,t)}{\rho A}\right] + \hbar^2 L^{-1}\left[\frac{F(x,t)}{\rho A}\right] + \hbar L^{-1}\left[\frac{c}{\rho A} D_t^\alpha u_1 + \frac{EI}{\rho A} D_x^4 u_1\right] \end{aligned} \quad (4.12)$$

$$\begin{aligned} u_3(x,t) &= \chi_3 u_2(x,t) + \hbar L^{-1}FR\left(u_2(\vec{x},t)\right) \\ &= u_2(x,t) + \hbar L^{-1}\left[D_t^2 u_2 + \frac{c}{\rho A} D_t^\alpha u_2 + \frac{EI}{\rho A} D_x^4 u_2 - (1-\chi_3)\frac{F(x,t)}{\rho A}\right] \\ &= u_2(x,t) + \hbar u_2(x,t) + \hbar L^{-1}\left[\frac{c}{\rho A} D_t^\alpha u_2 + \frac{EI}{\rho A} D_x^4 u_2\right] \end{aligned} \quad (4.13)$$

So, the solution of Eq. (4.1) can be written as

$$u(x,t)=\sum_{n=0}^{\infty} u_n(x,t) \quad (4.14)$$

From the above simulations, it is clear that the convergence of the presented method depends on $\hbar$, initial condition and auxiliary linear operator L . One may see the reference [40] for convergence analysis of the HAM.

5. Response analysis

Here the forcing function $F(x,t)$ is considered as

$$F(x,t)=f(x)g(t) \quad (5.1)$$

where $f(x)$ is a function and $g(t)$ is a time depending process. In next sections, response analysis of the beam under different loads will be discussed.

5.1. Step function

Let us take $g(t)$ as

$$g(t) = Bu(t) \quad (5.1.1)$$

where B is a constant term and $u(t)$ is Heaviside function. Substituting Eqs. (4.2), (5.1) and (5.1.1) into Eqs. (4.11), (4.12) and (4.13) by taking $\hbar = -1$, we have

$$u_0(x, t) = 0 \quad (5.1.2)$$

$$u_1(x, t) = \frac{fBt^2}{2\rho A} \quad (5.1.3)$$

$$u_2(x, t) = \frac{-cfB}{\rho^2 A^2} \frac{t^{4-\alpha}}{\Gamma(5-\alpha)} - \frac{EIBf^{(4)}}{\rho^2 A^2} \frac{t^4}{\Gamma(5)} \quad (5.1.4)$$

$$u_3(x, t) = \frac{c^2 fB}{\rho^3 A^3} \frac{t^{6-2\alpha}}{\Gamma(7-2\alpha)} + \frac{2cEIBf^{(4)}}{\rho^3 A^3} \frac{t^{6-\alpha}}{\Gamma(7-\alpha)} + \frac{E^2 I^2 Bf^{(8)}}{\rho^3 A^3} \frac{t^6}{\Gamma(7)} \quad (5.1.5)$$

where $f^{(i)}$ is the i -th derivative of f with respect to x . So, the solution in general form of Eq. (4.1) subjected to a unit step load is given by

$$u(x, t) = \frac{B}{\rho A} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \left(\frac{EI}{\rho A} \right)^i f^{(4i)} t^{2(i+1)} \sum_{j=0}^{\infty} \left(\frac{-c}{\rho A} \right)^j \frac{(i+j)! t^{(2-\alpha)j}}{j! \Gamma((2-\alpha)j + 2i + 2)} \quad (5.1.6)$$

Eq. (5.1.6) may be modified as

$$u(x, t) = \frac{B}{\rho A} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \left(\frac{EI}{\rho A} \right)^i f^{(4i)} t^{2(i+1)} \sum_{j=0}^{\infty} E_{2-\alpha, 2+\alpha i}^i \left(\frac{-c}{\rho A} t^{(2-\alpha)} \right) \quad (5.1.7)$$

This result is exactly same as Zu-Feng and Xiao-yan [32], where

$$E_{a,b}^i(x) = \sum_{j=0}^{\infty} \frac{(i+j)! x^j}{j! \Gamma(qj + ai + b)}, \quad i = 0, 1, 2, \dots \quad (5.1.8)$$

This is the MLF of two parameters a and b .

5.2. Impulse function

Consider the function $g(t)$ as

$$g(t) = \delta(t) \quad (5.2.1)$$

where $\delta(t)$ is a unit impulse function. From Eqs. (4.11)-(4.13) and substituting $\hbar = -1$, we obtain

$$u_0(x, t) = 0 \quad (5.2.2)$$

$$u_1(x, t) = \frac{ft}{\rho A} \quad (5.2.3)$$

$$u_2(x, t) = \frac{-cf}{\rho^2 A^2} \frac{t^{3-\alpha}}{\Gamma(4-\alpha)} - \frac{EI f^{(4)}}{\rho^2 A^2} \frac{t^3}{\Gamma(4)} \quad (5.2.4)$$

$$u_3(x, t) = \frac{c^2 f}{\rho^3 A^3} \frac{t^{5-2\alpha}}{\Gamma(7-2\alpha)} + \frac{2cEI f^{(4)}}{\rho^3 A^3} \frac{t^{5-\alpha}}{\Gamma(6-\alpha)} + \frac{E^2 I^2 f^{(8)}}{\rho^3 A^3} \frac{t^5}{\Gamma(6)} \quad (5.2.5)$$

Finally, we may write the solution of Eq. (4.1) subjected to impulse load function as

$$u(x, t) = \frac{1}{\rho A} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \left(\frac{EI}{\rho A} \right)^i f^{(4i)} t^{2i+1} \sum_{j=0}^{\infty} \left(\frac{-c}{\rho A} \right)^j \frac{(i+j)! t^{(2-\alpha)j}}{j! \Gamma((2-\alpha)j + 2i + 2)} \quad (5.2.6)$$

Eq. (5.2.6) can be revised as

$$u(x, t) = \frac{1}{\rho A} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \left(\frac{EI}{\rho A} \right)^i f^{(4i)} t^{2i+1} \sum_{j=0}^{\infty} E_{2-\alpha, 2+2i}^i \left(\frac{-c}{\rho A} t^{(2-\alpha)} \right) \quad (5.2.7)$$

The above Eq. (5.2.7) is the same as the solution of Zu-Feng and Xiao-yan [32].

6. Results and discussions

As discussed above, Eqs. (5.1.7) and (5.2.7) are the solutions of the fractional damped equation with unit step and impulse function load. In order to illustrate the response analysis of titled equation more precisely, some numerical results are presented. Here, we have assumed simply supported beam, and for that, the expression of $f(x)$ can be written as

$$f(x) = \sin\left(\frac{\pi x}{L}\right) \quad (6.1)$$

Here all the calculations have been computed by taking a finite number of terms solutions of Eqs. (5.1.7) and (5.2.7). For simplification, let us presume $c/m = 2\eta\omega^{3/2}$ and $\omega^2 = EI/\rho A$, where ω and η are the frequency and damping ratio. The values of these parameters are considered as $B = 1$, $\rho A = 1$, $L = \pi$ and $m = 1$ which is the same as [31]. The behavior of movement versus time for various values of $\alpha = 0.2, 0.4, 0.6, 0.8, 1$ with unit step load is given in Figure 2. For this, we have considered $x = \eta = 0.5$. The plots for $\omega = 5 \text{ rad/s}$ and $\omega = 10 \text{ rad/s}$ are depicted in Figures 2a and 2b respectively.

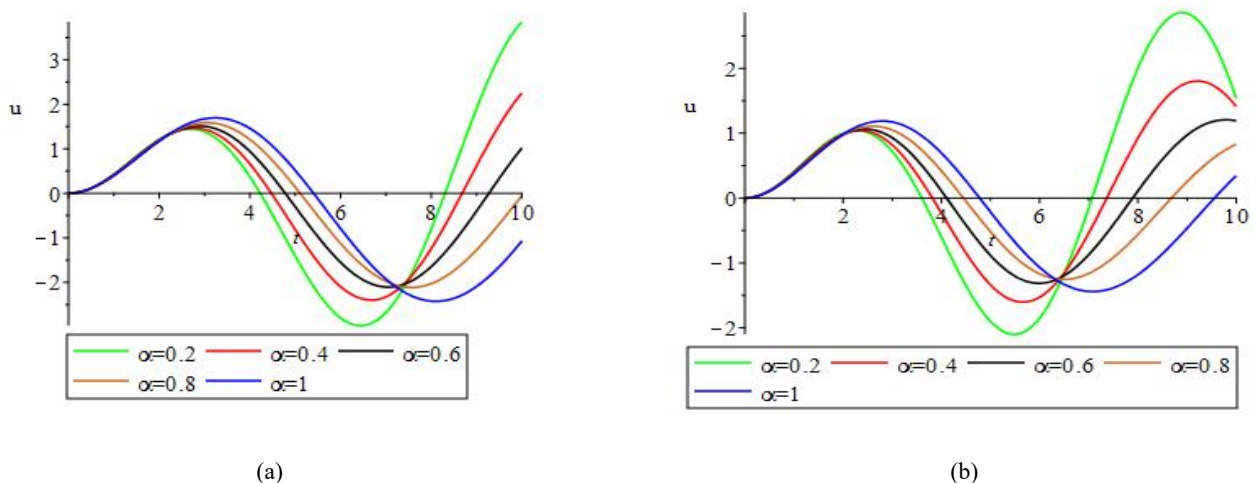


Fig. 2. Unit step response of Eq. (5.1.7) truncated at $i = 10$ and $j = 10$ along $x = 0.5$, $\eta = 0.5$, (a) $\omega = 5 \text{ rad/s}$ (b) $\omega = 10 \text{ rad/s}$

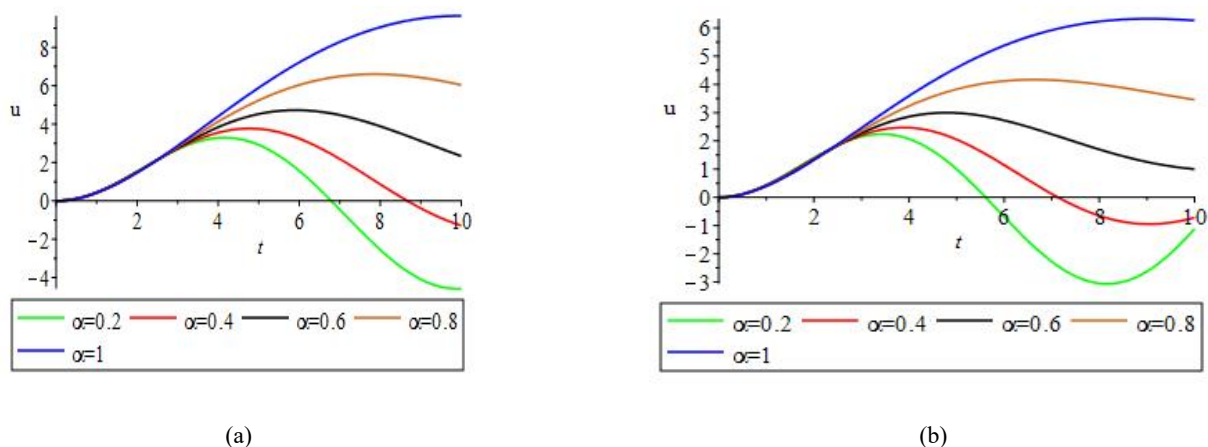


Fig. 3. Unit step response of Eq. (5.1.7) truncated at $i = 10$ and $j = 10$ along $x = 0.5$, $\eta = 0.05$, (a) $\omega = 5 \text{ rad/s}$ (b) $\omega = 10 \text{ rad/s}$

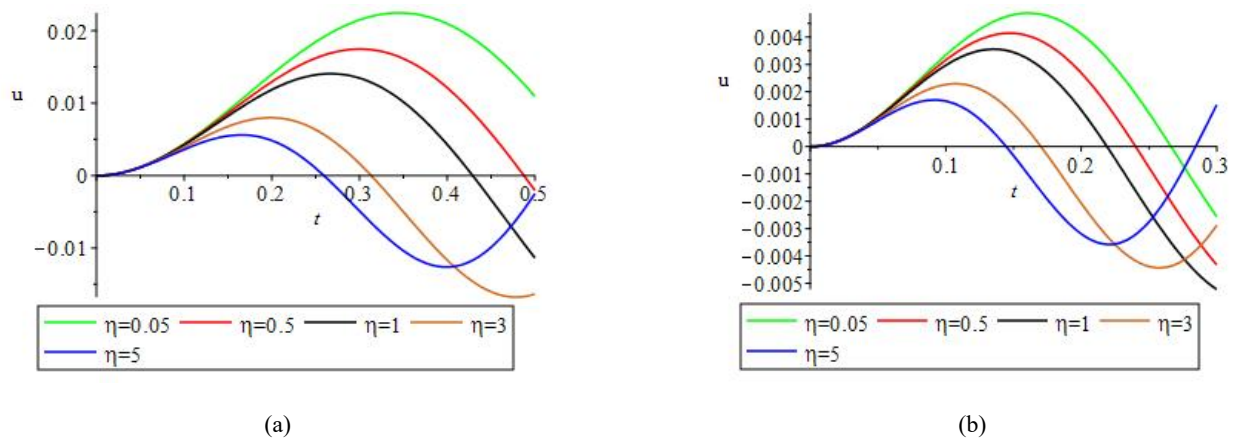


Fig. 4. Unit step response of Eq. (5.1.7) truncated at $i = 10$ and $j = 10$ along $x = 0.5$, (a) $\omega = 5 \text{ rad/s}$ (b) $\omega = 10 \text{ rad/s}$ and $\eta = 0.05, 0.5, 1, 3, 5$ at $\alpha = 0.2$

The same calculations are computed with $\eta = 0.05$, and the obtained results are presented in Figure 3. The plots for $\omega = 5 \text{ rad/s}$ and $\omega = 10 \text{ rad/s}$ are given in Figures 3a and 3b respectively. DR versus time graphs for $\eta = 0.05, 0.5, 1, 3, 5$ are illustrated in Figure 4. In this simulation $\alpha = 0.2$ and $x = 0.5$ are taken. The same calculation has been done with $\alpha = 0.5$, and the acquired results are shown in Figure 5. A similar calculation has been done for unit impulse load using the same parameters with the same values similar to unit step load, and obtained results are depicted in Figures 6 - 9 respectively.

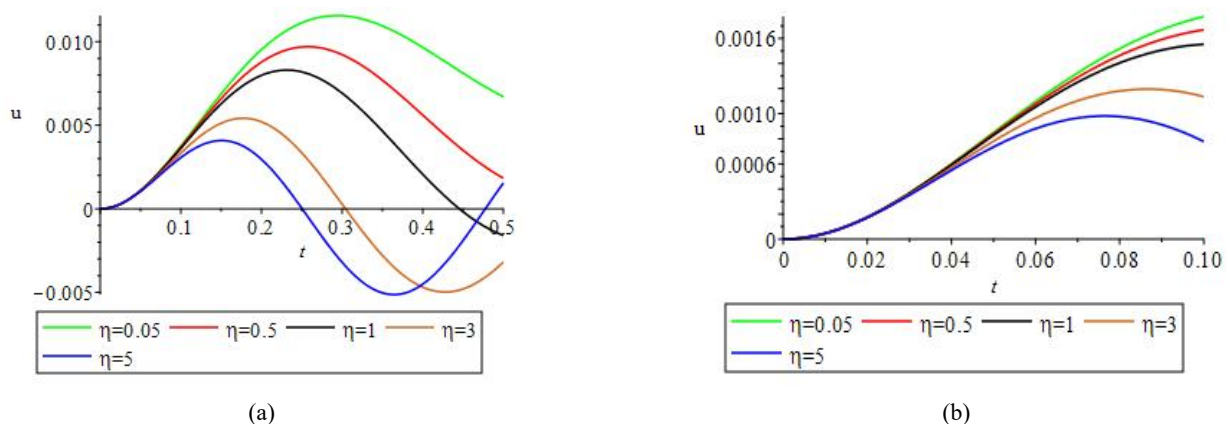


Fig. 5. Unit step response of Eq. (5.1.7) truncated at $i = 10$ and $j = 10$ along $x = 0.5$, (a) $\omega = 5 \text{ rad/s}$ (b) $\omega = 10 \text{ rad/s}$ and $\eta = 0.05, 0.5, 1, 3, 5$ at $\alpha = 0.5$

It is noted from Figures 2 that the beam undergoes more fluctuations for a less value of α . Similar behaviors have been noticed by keeping α constant and changing the values of η as displayed in Figures 4 and 5. It can be observed that increasing the value of η decreases the fluctuations. However, from Figures 3, we observe that the beam undergoes more fluctuations for a higher value of α at $\eta = 0.05$. Same observations have been observed for unit impulse load which are depicted in Figures 6-9. Also, it is observed that all the mode of vibrations of the titled problem is matching with Zu-Feng and Xiao-yan [32].

Table 1. Comparison of the present method with ref. [32] subject to unit step load at $\alpha = 1$, $\eta = 0.5$ and $\omega = 5$

x	t	Present method	Zu-Feng and Xiao Yan [32]
0.5	0.2	0.653412041E-2	0.653412041E-2
	0.4	0.109235227E-1	0.109235227E-1
	0.6	0.161516434	0.161516434
	0.8	3.23345965	3.23345965
	1	21.29457263	21.29457263
1	0.2	0.1146846027E-1	0.1146846027E-1
	0.4	0.1917258577E-1	0.1917258577E-1
	0.6	0.2834880011	0.2834880011
	0.8	5.675255550	5.675255550
	1	37.37549130	37.37549130

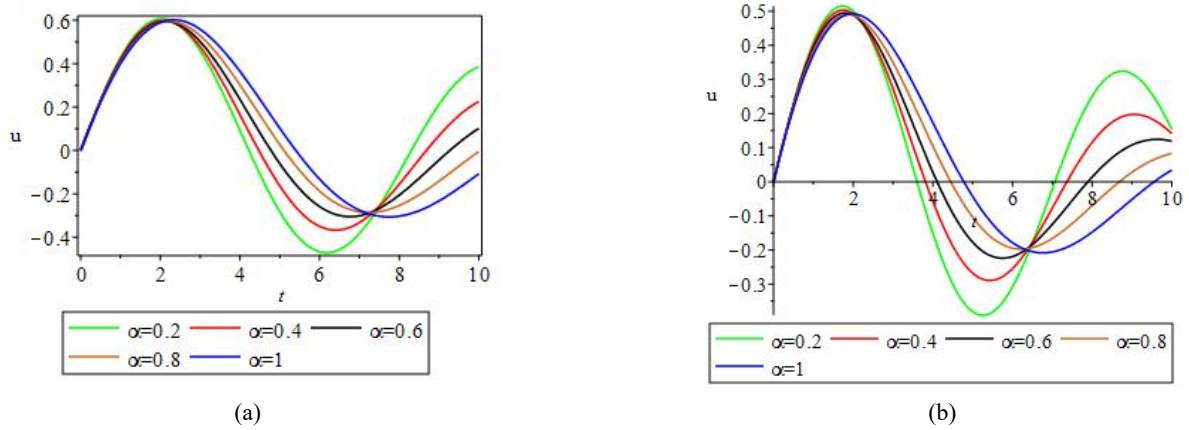


Fig. 6. Unit step response of Eq. (5.2.7) truncated at $i = 10$ and $j = 10$ along $x = 0.5$, $\eta = 0.5$, (a) $\omega = 5 \text{ rad/s}$ (b) $\omega = 10 \text{ rad/s}$

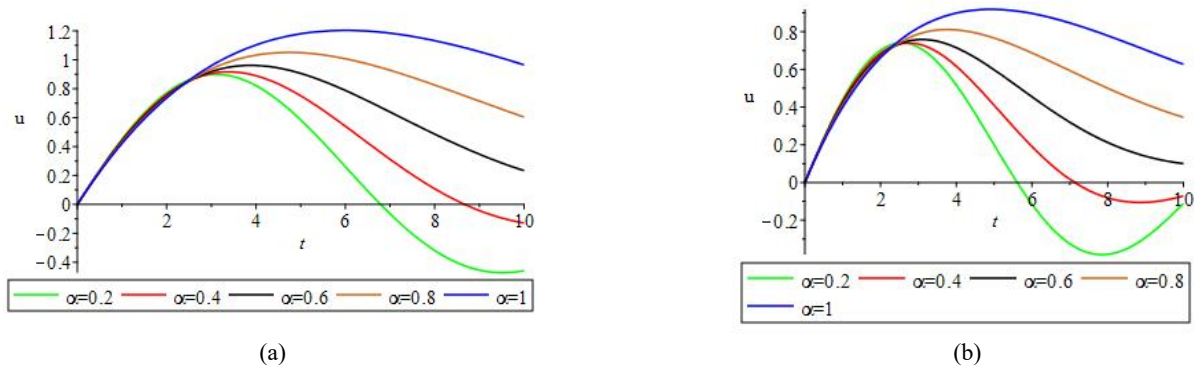


Fig. 7. Unit step response of Eq. (5.2.7) truncated at $i = 10$ and $j = 10$ along $x = 0.5$, $\eta = 0.05$, (a) $\omega = 5 \text{ rad/s}$ (b) $\omega = 10 \text{ rad/s}$

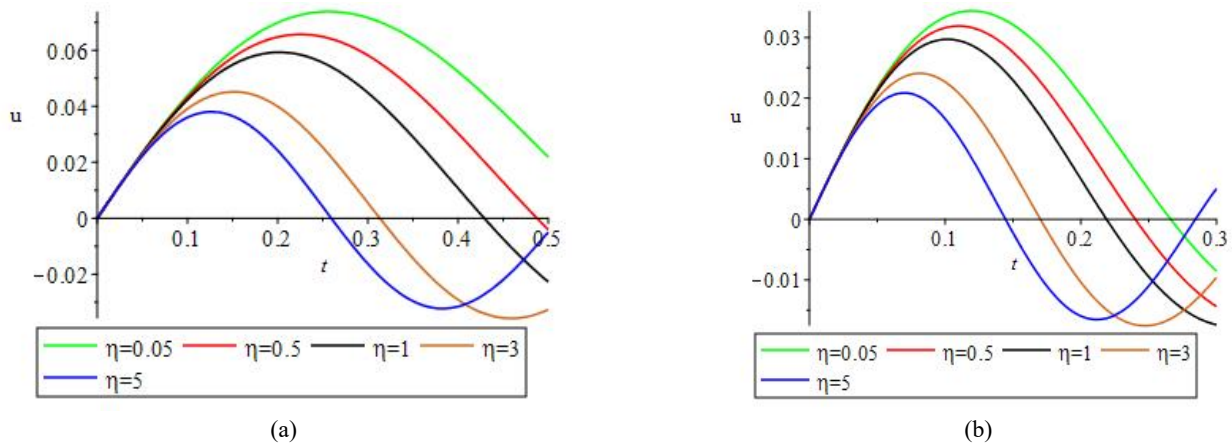


Fig. 8. Unit step response of Eq. (5.2.7) truncated at $i = 10$ and $j = 10$ along $x = 0.5$, (a) $\omega = 5 \text{ rad/s}$ (b) $\omega = 10 \text{ rad/s}$ and $\eta = 0.05, 0.5, 1, 3, 5$ at $\alpha = 0.2$

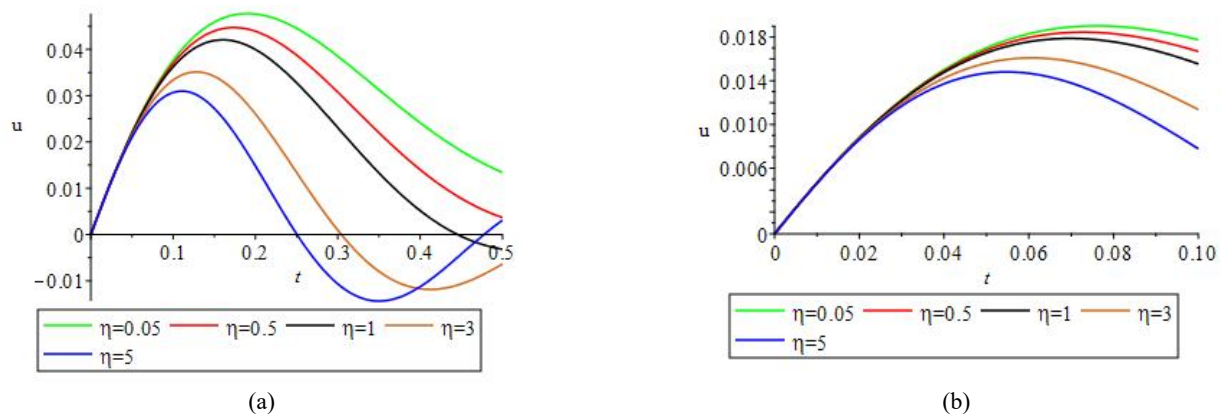


Fig. 9. Unit step response of Eq. (5.2.7) truncated at $i = 10$ and $j = 10$ along $x = 0.5$, (a) $\omega = 5 \text{ rad/s}$ (b) $\omega = 10 \text{ rad/s}$ and $\eta = 0.05, 0.5, 1, 3, 5$ at $\alpha = 0.5$

Table 2. Comparison of the presented method with ref. [32] subjected to unit impulse load at $\alpha = 1$, $\eta = 0.5$ and $\omega = 5$

x	t	Present method	Zu-Feng and Xiao Yan [32]
0.5	0.2	0.7231122656E-1	0.7231122656E-1
	0.4	0.7055511796E-1	0.7055511796E-1
	0.6	0.1101946847E-1	0.1101946847E-1
	0.8	0.4566609744E-1	0.4566609744E-1
	1	-0.5336854210E-1	-0.5336854210E-1
1	0.2	0.1269181429	0.1269181429
	0.4	0.1238358824	0.1238358824
	0.6	0.1934098634E-1	0.1934098634E-1
	0.8	-0.8015154084E-1	-0.8015154084E-1
	1	-0.9367060396E-1	-0.9367060396E-1

7. Conclusions

In this study, Homotopy Analysis Method (HAM) has been applied to find the solution of the titled equations. Unit step and unit impulse functions are taken to demonstrate the present method. The results obtained by the HAM are compared with ref. [32]. It is seen that the obtained results are exactly same as the solution of [32]. HAM offers us the proper way to control the convergence region of the solution by presenting a parameter η . Moreover, any discretization is not required by HAM which helps us to overcome the difficulties of the round-off errors, high computer memory, and more times. The results show that the HAM is the powerful and efficient tool for solving fractional linear PDEs.

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Conflict of Interest

The author(s) declared no potential conflicts of interest with respect to the research, authorship and publication of this article.

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