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Research Paper

Buckling and Free Vibration Analysis of Fiber Metal-laminated Plates Resting on Partial Elastic Foundation

Horae Moraveji Tabasi[1], Jafar Eskandari Jam[2], Keramat Malekzadeh Fard[3]
Mohsen Heydari Beni[4]

[1] *University Complex of Materials and Manufacturing Technology, Malek Ashtar University of Technology, Lavizan, Tehran, Iran.*

[2] *Department of Mechanical Engineering, Malek Ashtar University of Technology, Lavizan, Tehran, Iran.*

[3] *Department of Mechanical Engineering, Malek Ashtar University of Technology, Lavizan, Tehran, Iran.*

[4] *University Complex of Materials and Manufacturing Technology, Malek Ashtar University of Technology, Lavizan, Tehran, Iran.*

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Corresponding author: Jafar Eskandari Jam (jafar.eskandarijam@gmail.com)

Abstract: This research presents, buckling and free vibration analysis of fiber metal-laminated (FML) plates on a total and partial elastic foundation using the generalized differential quadrature method (GDQM). The partial foundation consists of multi-section Winkler and Pasternak type elastic foundation. Taking into consideration the first-order shear deformation theory (FSDT), FML plate is modeled and its equations of motion and boundary conditions are derived using Hamilton's principle. The formulations include Heaviside function effects due to the nonhomogeneous foundation. The novelty of this study is considering the effects of partial foundation and in-plane loading, in addition to considering the various boundary conditions of FML plate. A computer program is written using the present formulation for calculating the natural frequencies and buckling loadings of composite plates without contacting with elastic foundation and composite plates resting on partial foundations. The validation is done by comparison of continuous element model with available results in the literature. The results show that the constant of total or partial spring, elastic foundation parameter, thickness ratio, frequency mode number and boundary conditions play an important role on the critical buckling load and natural frequency of the FML plate resting on partial foundation under in-plane force.

Keywords: Partial elastic foundation, FML composite plate, Free vibration, Buckling, GDQ method.

1. Introduction

The vibration of fiber metal-laminated (FML) plates on elastic foundations is a great concern to the engineering community in recent years because of their many beneficial properties. Applications of these foundations can be found in aerospace structures, aircraft runways, nuclear reactors, building foundation slabs, railway tracks, indoor sports floors, petrochemical and submarine structures, etc. The comprehension of buckling and vibration behaviors of plates resting on the partial elastic foundation is essential because such structures represent real plate-foundation structures in industry. Various investigations are accomplished to design safer and more economic moderately thick laminated composite plate structures supported by non-homogenous elastic foundations. The simplest model for the elastic foundation is Winkler or one-parameter model, which regards the foundation as a series of separated springs without coupling effects between each other. Pasternak improved this model by adding a shear layer to Winkler [1] model. Pasternak [2] or two-parameter model is extensively used to characterize the mechanical behavior of structure-foundation interactions. There have been

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a substantial number of researchers on the plates resting on an elastic foundation, and several methods have been proposed to analyze plates on elastic foundations. Timoshenko [3] was one of the first authors to analyze deflection (not buckling) of rectangular plates subjected to transverse loads, and resting on elastic subgrades. The formulas for buckling loads of elastic plates resting on elastic foundations were also derived in the book by Vlasov and Leontev [4], but again for the two-way foundation. Seide [5] dealt with buckling of rectangular plates, but on two-way elastic foundation and using the energy method. Cheung [6] solved the problems of slabs and tanks (either isotropic or orthotropic) resting either on a semi-infinite elastic continuum or individual springs (of the so-called Winkler's type) by the finite element method. Also, Cheung [7] analyzed plates and beams on elastic continuum by using the finite element method. Akbarov [8] proposed the development of the solution method of the bending problems of the plates fabricated from the composite materials on the "tensionless" elastic foundation and the influence of the plate material properties to displacement distribution. Chilton [9] developed consistent foundation stiffness matrix F for one element of a rectangular plate with simply supported edges on an elastic foundation, based on the Winkler model. Henwood [10] presented a Fourier series solution for a system of first-order partial differential equations which describe the linear elastic behavior of a thick rectangular plate resting on an elastic foundation and carrying an arbitrary transverse load. Also using boundary integral equations by Katsikadelis and Armenkas [11], and boundary element analysis by Puttonen and Varpasuo [12] have been applied to the problem of plates on elastic foundations. Prakash [13] investigated the flexure under a uniform load of clamped rectangular plates resting on elastic foundations via eigenfunction expansions. Xiang [14] used closed-form solutions for the vibration problem of initially stressed thick rectangular plates as described by Mindlin [15] theory. Also, Xing [16] investigated buckling, free vibration, and vibration with initial in-plane loads for moderately thick, simply supported symmetric cross-ply rectangular laminates on Pasternak foundations. The total potential energy functional is derived based on the first-order shear deformation plate theory. Matsunaga [17] analyzed the natural frequencies and buckling stresses of a thick isotropic plate on two-parameter elastic foundations by taking into account the effect of shear deformation, thickness change, and rotatory inertia. Civalek [18] developed the discrete singular convolution (DSC) method is developed for vibration analysis of moderately thick symmetrically laminated composite plates based on the first-order shear deformation theory (FSDT). In another study, Civalek [19] introduced a coupled methodology for the numerical solution of geometrically nonlinear static and dynamic problem of thin rectangular plates resting on elastic foundation. Winkler-Pasternak two-parameter foundation model is considered. In the field of buckling and free vibration analysis of symmetric and antisymmetric laminated composite plates on an elastic foundation. More advanced studies on buckling and free vibration analysis of symmetric and anti-symmetric laminated composite plates on an elastic foundation were conducted by Akavci [20] using Navier technique and a new hyperbolic displacement model. Zenkour [21] using the refined sinusoidal shear deformation plate theory and including plate-foundation interaction, a thermoelastic bending analysis is presented for a simply supported, rectangular, functionally graded material plate subjected to a transverse uniform load and a temperature field, and resting on a two-parameter (Pasternak model) elastic foundation. Malekzadeh [22] works on the three-dimensional (3D) free vibration analyses of functionally graded (FG) plates supported on two-parameter elastic foundation, the formulations are based on the three-dimensional elasticity theory. Malekzadeh [23] studied the effect of non-ideal boundary conditions and initial stresses on the vibration of laminated plates on Pasternak foundation. Dehghan [24] used a combination of the finite element (FE) and differential quadrature (DQ) methods to solve the eigenvalue (buckling and free vibration) equations of rectangular thick plates resting on elastic foundations. Sobhy [25] dealt with the vibration and buckling behavior of exponentially graded materials (EGM) sandwich plate resting on elastic foundations under various boundary conditions. Dehghany [26] investigated the exact solution for free vibration analysis of simply supported rectangular plates on elastic foundation. The solution is on the basis of three-dimensional elasticity theory. Thinh [27] presented new continuous elements for thick laminated plates on a Pasternak or a non-homogeneous foundation using the Dynamic Stiffness Method (DSM). Muntari [28] analyzed a free vibration of functionally graded plates resting on elastic foundation by using a generalized quasi-3D hybrid-type higher order shear deformation theory (HSDT). The flexural and vibration response of a functionally graded (FG) plate resting on Pasternak elastic foundation are studied by Gupta [29] by use of recently developed non-polynomial higher order shear and normal deformation theory. Vibrational behavior of sandwich plates with functionally graded wavy carbon nanotube-reinforced face sheets resting on Pasternak elastic foundation are investigated by Moradi-Dastjerdi [30] based on a mesh-free method and first-order shear deformation (FSDT). Mansouri [31] investigated thermal buckling of general quadrilateral plates fabricated from heterogeneous, orthotropic, and auxetic (with negative Poisson ratio) materials resting on elastic Winkler-Pasternak elastic media. All scopes of the above articles are dynamic and static analysis of plates on the full elastic foundation, but it should be noted that none of them have considered partial foundation.

The purpose of this study is to analyze free vibration and buckling of moderately thick fiber metal laminated plates resting on partial elastic foundation by using GDQ method. Governing equations are derived from the principle of minimum total potential energy. Navier technique is used to obtain the closed form solutions for simply supported FML plates resting on full elastic foundation and GDQ method is used to obtain the natural frequencies and critical buckling load for FML plates resting on partial elastic foundation under various boundary conditions. Critical buckling load factors and fundamental frequencies are found by solving an eigenvalue equation. The novelty of this paper is that it solved free vibration and buckling problems of FML plates resting on partial elastic foundation by using GDQ method for all boundary conditions. Besides, applied heavy-side function to obtain governing equations of FML plates efficiently. The results obtained by the present method were compared with solutions derived from the other models and literature, and found to be in good agreement.

2. Theoretical Formulations of FML

2.1. First-Order Shear Deformation Theory

Consider a thick composite laminated rectangular plate of dimensions $a \times b$ resting on a Pasternak elastic foundation, which is composed of *N* elastic orthotropic layers, of length *a*, width *b* and uniform total thickness *h*. As shown in Fig. 1; k_1 is linear stiffness of foundation, k_2 is the shear modulus of the sub-grade. The coordinate system (*x, y, z*), placed at the middle of the plate, is chosen. Two opposite edges *y* = 0 and *y* = *b* are assumed to be supported and boundary conditions of the two remaining edges can be any combination of free, clamped or supported types. The plate has a uniform thickness *h* and in general, is made up of some or many laminate layers [27]; each consists of a unidirectional fiber reinforced composite material.

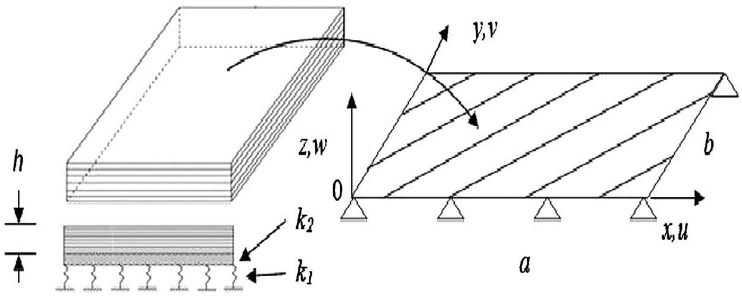

Fig. 1. Schematic of FML plate on elastic foundation [27].

2.2. Displacement field of FML

Based on the FSDT, the displacement field at a point M_o in the middle plane of the plate is express as [32]:

$$u_0(x,y,t) = u(x,y,t) + z\varphi_x(x,y,t),\;\; v_0(x,y,t) = v(x,y,t) + z\varphi_y(x,y,t),\;\; w_0(x,y,t) = w(x,y,t) \tag{1}$$

In above equation, *t* represents the time, u_0, v_0, w_0 are middle plane displacement, φ_x and φ_y are rotations of the normal to the middle plane about *y* and *x*-axes respectively. The strains are related to the displacements by the following expressions:

$$\begin{aligned} &\varepsilon_{xx} = \frac{\partial u_0}{\partial x} + z\frac{\partial \phi_x}{\partial x};\;\; \varepsilon_{yy} = \frac{\partial v_0}{\partial y} + z\frac{\partial \phi_y}{\partial y};\;\; \gamma_{xy} = \left(\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}\right) + z\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right) \\ &\gamma_{xz} = \phi_x + \frac{\partial w_0}{\partial x};\;\; \gamma_{yz} = \phi_y + \frac{\partial w_0}{\partial y} \end{aligned} \tag{2}$$

The constitutive equations for Fiber metal laminated plate are determined by [32]:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{21} & Q_{22} & 0 & 0 & 0 \\ 0 & 0 & Q_{44} & 0 & 0 \\ 0 & 0 & 0 & Q_{55} & 0 \\ 0 & 0 & 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} \tag{3}$$

where $(\sigma_{xx}, \sigma_{yy}, \sigma_{xy}, \sigma_{xz}, \sigma_{yz})$ are the stress and $(\varepsilon_{xx}, \varepsilon_{yy}, \gamma_{xy}, \gamma_{xz}, \gamma_{yz})$ are the strain components of the lamina in the global coordinates. The reduced stiffness coefficients in above equations (see the Appendix) are estimated as:

$$\begin{aligned} &A_{ij} = \sum_{k=1}^{N} (\bar{Q}_{ij})_k [Z_k - Z_{k-1}],\;\; B_{ij} = \frac{1}{2}\sum_{k=1}^{N} (\bar{Q}_{ij})_k [Z_k^2 - Z_{k-1}^2], \\ &D_{ij} = \frac{1}{3}\sum_{k=1}^{N} (\bar{Q}_{ij})_k [Z_k^3 - Z_{k-1}^3],\;\; (i,j = 1,2,6) \\ &A_{ij} = K\sum_{k=1}^{N} (\bar{Q}_{ij})_k [Z_k - Z_{k-1}],\;\; (i,j = 4,5) \end{aligned} \tag{4}$$

in which $\bar{Q}_{ij}$ are transformed elastic coefficients [33], which is related to the coefficients Q_{ij} in the material principal directions. And $K = 5/6$: the shear correction factor, N is the number of layers, Z_{k-1}, Z_k: the coordinates of the top and bottom faces of the k^{th} layer. The mass inertia terms in the governing equations are defined as:

$$I_i = \sum_{k=1}^{N} \int_{Z_k}^{Z_{k+1}} \rho^{(k)} z^i dz, \ (i = 0,1,2) \tag{5}$$

where $\rho^{(k)}$ is the material mass density of the k_{th} layer.

2.3. Governing equations and boundary conditions

For the equations of the motion and boundary conditions, the principle of minimum potential energy states that [34]:

$$\int_{t_2}^{t_1} (\delta T - \delta U + \delta W) dt = 0 \tag{6}$$

By taking the variation of the above equation and integrating by parts the results are obtained in the following forms: Strain energy of FML based on the First Order Shear Deformation theory is expressed as follows:

$$U = \frac{1}{2} \iiint_V \sigma_{ij} \, \varepsilon_{ij} \, dV \tag{7}$$

$$\begin{aligned} \int_0^t \delta U \, dt = \int_0^t \iint_A \int_{-h/2}^{h/2} \Bigg[& \left(N_x \frac{\partial}{\partial x} \delta u_0 + M_x \frac{\partial}{\partial x} \delta \varphi_x \right) + \left(N_y \frac{\partial}{\partial y} \delta v_0 + M_y \frac{\partial}{\partial y} \delta \varphi_y \right) \\ & + N_{xy} \left(\frac{\partial}{\partial y} \delta u_0 + \frac{\partial}{\partial x} \delta v_0 \right) + M_{xy} \left(\frac{\partial}{\partial y} \delta \varphi_x + \frac{\partial}{\partial x} \delta \varphi_y \right) \\ & + Q_{xz} \left(\delta \varphi_x + \frac{\partial}{\partial x} \delta w_0 \right) + Q_{yz} \left(\delta \varphi_y + \frac{\partial}{\partial y} \delta w_0 \right) \Bigg] dV \, dt \end{aligned} \tag{8}$$

where the classical and non-classical force and momentum are defined as follows:

$$\begin{aligned} \{N_x, N_y, N_{xy}\} &= \int_{-h/2}^{h/2} \{\sigma_{xx}, \sigma_{yy}, \sigma_{xy}\} dz \\ \{M_x, M_y, M_{xy}\} &= \int_{-h/2}^{h/2} \{\sigma_{xx}, \sigma_{yy}, \sigma_{xy}\} zdz \\ \{Q_{xz}, Q_{yz}\} &= \int_{-h/2}^{h/2} k_s \{\sigma_{xz}, \sigma_{yz}\} dz, \ \ k_s = \frac{5}{6} \end{aligned} \tag{9}$$

Also, the kinetic energy of the FML moderately thick plate can be expressed as:

$$\begin{aligned} \delta T = \iint_A \int_{-h/2}^{h/2} \rho \Bigg[& \left(\frac{\partial u_0}{\partial t} + z \frac{\partial \varphi_x}{\partial t} \right) + \left(\frac{\partial}{\partial t} \delta u_0 + z \frac{\partial}{\partial t} \delta \varphi_x \right) \\ & + \left(\frac{\partial v_0}{\partial t} + z \frac{\partial \varphi_y}{\partial t} \right) + \left(\frac{\partial}{\partial t} \delta v_0 + z \frac{\partial}{\partial t} \delta \varphi_y \right) + \left(\frac{\partial w_0}{\partial t} + \frac{\partial}{\partial t} \delta w_0 \right) \Bigg] dV \end{aligned} \tag{10}$$

The work done by external forces (spring and in-plane loads) is obtained as follows:

$$\begin{aligned} \delta W &= \delta W_1 + \delta W_2 = \delta \left[\frac{1}{2} \int_0^a N_1 \left(\frac{\partial w}{\partial x} \right)^2 dx + \frac{1}{2} \int_0^b N_2 \left(\frac{\partial w}{\partial y} \right)^2 dy \right] \\ \delta W_1 &= \delta w_0 N_1 \frac{\partial w_0}{\partial x} - \int_0^a \delta w_0 \frac{\partial}{\partial x} \left(N_1 \frac{\partial w_0}{\partial x} \right) dx \\ \delta W_2 &= \delta w_0 N_2 \frac{\partial w_0}{\partial y} - \int_0^b \delta w_0 \frac{\partial}{\partial y} \left(N_2 \frac{\partial w_0}{\partial y} \right) dy \end{aligned} \tag{11}$$

Using Heaviside step Function [35] H to drive governing equations of FML rectangular plates on partial elastic foundation as follows:

$$H(x, y, x_0, y_0, c, d) = \left[u\left(x - (x_0 - c)\right) - u\left((x_0 + c) - x\right) \right] \left[u\left(y - (y_0 - d)\right) - u\left((y_0 + c) - y\right) \right] \tag{12}$$

As illustrated in Fig. 2, x_0, y_0 are the center of partial elastic foundation area, c is the length and d is the width of partial elastic foundation area.

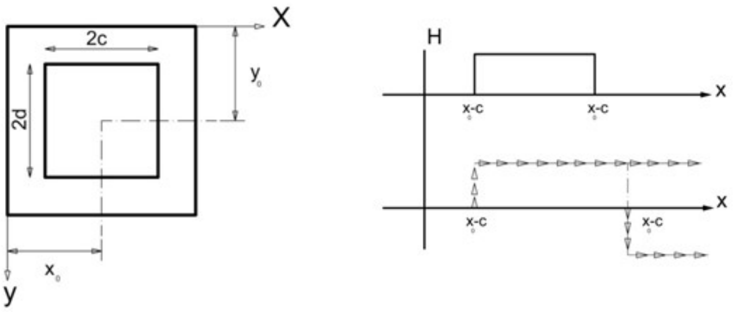

Fig. 2. Applying Heaviside step function for FML on partial elastic foundation.

The equation of motion for FML on full elastic foundations using analytical method can be expressed as follows:

$$\begin{aligned}
&\delta u_0 : \frac{\partial}{\partial x}\left(A_{11}\frac{\partial u_0}{\partial x}+B_{11}\frac{\partial f_x}{\partial x}+A_{12}\frac{\partial v_0}{\partial y}+B_{12}\frac{\partial \varphi_y}{\partial y}\right)+\frac{\partial}{\partial y}\left(A_{44}\left(\frac{\partial u_0}{\partial y}+\frac{\partial v_0}{\partial x}\right)+B_{44}\left(\frac{\partial \varphi_x}{\partial y}+\frac{\partial \varphi_y}{\partial x}\right)\right)=I_0\frac{\partial^2 u_0}{\partial t^2}+I_1\frac{\partial^2 \varphi_x}{\partial t^2}\\
&\delta v_0 : \frac{\partial}{\partial x}\left(A_{44}\left(\frac{\partial u_0}{\partial y}+\frac{\partial v_0}{\partial x}\right)+B_{44}\left(\frac{\partial \varphi_x}{\partial y}+\frac{\partial \varphi_y}{\partial x}\right)\right)+\\
&\frac{\partial}{\partial y}\left(A_{21}\frac{\partial u_0}{\partial x}+B_{21}\frac{\partial \varphi_x}{\partial x}+A_{22}\frac{\partial v_0}{\partial y}+B_{22}\frac{\partial \varphi_y}{\partial y}\right)=I_0\frac{\partial^2 v_0}{\partial t^2}+I_1\frac{\partial^2 \varphi_y}{\partial t^2}\\
&\delta w_0 : \frac{\partial}{\partial x}\left(\frac{5}{6}A_{55}\left(\frac{\partial w_0}{\partial x}+\varphi_x\right)\right)+\frac{\partial}{\partial y}\left(\frac{5}{6}A_{66}\left(\frac{\partial w_0}{\partial y}+\varphi_y\right)\right)+N^*\left(\frac{\partial^2 w_0}{\partial x^2}+\frac{\partial^2 w_0}{\partial y^2}\right)-k_w\times w_0\times H_{heavi}=I_0\frac{\partial^2 w_0}{\partial t^2}\\
&\delta \varphi_x : \frac{\partial}{\partial x}\left(B_{11}\frac{\partial u_0}{\partial x}+D_{11}\frac{\partial \varphi_x}{\partial x}+B_{12}\frac{\partial v_0}{\partial y}+D_{12}\frac{\partial \varphi_y}{\partial y}\right)+\frac{\partial}{\partial y}\left(B_{44}\left(\frac{\partial u_0}{\partial y}+\frac{\partial v_0}{\partial x}\right)+D_{44}\left(\frac{\partial \varphi_x}{\partial y}+\frac{\partial \varphi_y}{\partial x}\right)\right)-\\
&\frac{5}{6}A_{55}\left(\frac{\partial w_0}{\partial x}+\varphi_x\right)=I_1\frac{\partial^2 u_0}{\partial t^2}+I_2\frac{\partial^2 \varphi_x}{\partial t^2}\\
&\delta \varphi_y : \frac{\partial}{\partial x}\left(B_{44}\left(\frac{\partial u_0}{\partial y}+\frac{\partial v_0}{\partial x}\right)+D_{44}\left(\frac{\partial \varphi_x}{\partial y}+\frac{\partial \varphi_y}{\partial x}\right)\right)+\frac{\partial}{\partial y}\left(B_{21}\frac{\partial u_0}{\partial x}+D_{21}\frac{\partial \varphi_x}{\partial x}+B_{22}\frac{\partial v_0}{\partial y}+D_{22}\frac{\partial \varphi_y}{\partial y}\right)-\\
&\frac{5}{6}A_{66}\left(\frac{\partial w_0}{\partial y}+\varphi_y\right)=I_1\frac{\partial^2 v_0}{\partial t^2}+I_2\frac{\partial^2 \varphi_y}{\partial t^2}
\end{aligned}\tag{13}$$

where N^* is in-plane compressive loading on the sides of FML plate. For the Pasternak foundation model:

$$F_{foundation}=K_w w+K_p\nabla^2 w \tag{14}$$

If the foundation is modeled as linear Winkler foundation, the coefficient K_w in Eq. (14) is zero. The boundary conditions which should be prescribed at the edges of FML plate are obtained as follow:

$$\begin{aligned}
\delta u_0 &= 0 \quad or \quad N_{xx}dy+N_{xy}dx=0\\
\delta v_0 &= 0 \quad or \quad N_{yy}dx+N_{xy}dy=0\\
\delta w_0 &= 0 \quad or \quad Q_{xz}dy+Q_{yz}dx=0\\
\delta \varphi_x &= 0 \quad or \quad M_{xx}dy+M_{xy}dx=0\\
\delta \varphi_y &= 0 \quad or \quad M_{yy}dx+M_{xy}dy=0
\end{aligned}\tag{15}$$

It should be noted that K_w and K_p are the coefficient of Winkler and Pasternak elastic foundation.

3. Solution Procedure

3.1. Analytical solution

For the analytical solution of Eq. (13), the Navier method is used under specified boundary conditions and for full elastic foundation and without elastic foundation.

The displacement functions that satisfy the equations of the simply supported are selected as the following Fourier series:

$$\begin{Bmatrix} u(x,y,z) \\ v(x,y,z) \\ w(x,y,z) \\ \varphi_x(x,y,z) \\ \varphi_y(x,y,z) \end{Bmatrix} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \begin{Bmatrix} U_{mn}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}e^{i\omega t} \\ V_{mn}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}e^{i\omega t} \\ W_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}e^{i\omega t} \\ X_{mn}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}e^{i\omega t} \\ Y_{mn}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}e^{i\omega t} \end{Bmatrix} \tag{16}$$

where $U_{mn}, V_{mn}, W_{mn}, X_{mn}, Y_{mn}$ are arbitrary parameters to be determined, and ω is the eigen-frequency or buckling load factor associated with (m, n) $_{th}$ eigen mode.

3.2. Numerical solution (GDQ)

In the past decade, Bellman et al. introduced differential quadrature (DQM) as a reliable and effective method [36, 37]. In DQM preliminary formulations, weight coefficients were calculated using an algebraic equation system which limits the use of large grid numbers in DQM. So, for this defect, general quadrature method appeared. Shu [38] devised an explicit formula for the weighting coefficients with an infinite number of grid points leading to GDQM. Early applications of GDQ were applied mostly to regular domain problems; in addition, Shu and Richards [39] developed a domain decomposition technique to be used in the multi-domain problems. By this method, the main domain is divided into a number of sub-domains or elements, before discretizing each sub-domain for using GDQ. The mth-order derivative of the function f(x_i) can be expressed as follows [38]:

$$f^{(m)}(x_i) = \sum_{j=1}^{N} C_{ij}^{(m)} f(x_j);\ i = 1 \ldots N \tag{17}$$

where "N" is the number of grid points along "x"- direction and superscript "m" is the order of the derivative; and also, $C_{ij}^{(m)}$ is the weighting coefficient of x-direction which is written as follows:

$$C_{ij}^{(m)} = \begin{cases} m\left(C_{ij}^{(1)}C_{ii}^{(m-1)} - \dfrac{C_{ij}^{(m-1)}}{x_i - x_j}\right) & j \neq i \\ -\displaystyle\sum_{j=1,\ j\neq i}^{N} C_{ij}^{(m)} & j = i \end{cases} ;\quad i,j = 1,2,\ldots,N \tag{18}$$

Also, the base polynomial "g_j" is obtained as follows:

$$g_j(x) = \prod_{k=1,k\neq j}^{N} \frac{x - x_k}{x_j - x_k};\ j = 1\ldots \tag{19}$$

$$g_j(x) = \frac{M(x)}{(x - x_j)M^{(1)}(x_j)} \qquad i, j = 1, 2, \ldots, n \ \text{ and } i \neq j \tag{20}$$

where M(x_i) and $M^{(1)}$(x_i) are developed as:

$$M(x_i) = \prod_{k=1}^{N}(x_i - x_k) \tag{21a}$$

$$M^{(1)}(x_i) = (x_i - x_1)(x_i - x_2)\ldots(x_i - x_{i-1})(x_i - x_{i+1})\ldots(x_i - x_{N-1})(x_i - x_N) \tag{21b}$$

Substituting Eq. (2). into Eqs. (13) and (14), we get the below eigenvalue equations for any fixed value of m and n, for free vibration problem:

$$[K_{ij}] + \omega^2[M_{ij}] = \{0\},\ i,j = 1:5 \tag{22}$$

and for buckling problem:

$$[K_{ij}] + \omega[N_{ij}] = \{0\};\quad i,j = 1:5 \tag{23}$$

Stiffness matrix [K], mass matrix [M] and load matrix [N] are obtained by applying GDQ into the equations of motion and the boundary conditions.

4. Results and Discussion

4.1 Results validation with other articles

Initially, the analysis is carried out considering symmetric laminates resting on elastic foundations: results are then compared with those available in the literature and reported in Tables 1 and 2 for, free vibrations and buckling loads. The first step to validate the present method with articles without elastic foundation has been investigated, the second step the present method has been validated with articles that FML plates are on full elastic foundations, eventually the graphs are obtained for both free vibration and buckling analysis that came to result which its trend is acceptable for partial elastic foundations.

4.1.1. Free Vibration Analysis

The fundamental frequencies of the FML moderately thick plates resting on partial elastic foundations are calculated by the free vibration Eq. (21) as an eigenvalue problem. In table 1, non-dimensional fundamental frequencies of a symmetrically laminated cross-ply plate ($0^0/90^0/90^0/0^0$) are shown as compared with different shear deformation theories for different Orthotropy ratios.

Table 1. Non-dimensional fundamental frequency of SSSS cross-ply laminated $[0^0/90^0/90^0/0^0]$ square plate

Non-dimensional frequency $\Omega=(\omega\times a^2/h)\sqrt{\rho/E_2}$					
$G_{12}/E_2=0.6$	$G_{12}/E_2=0.6$	$G_{13}/E_2=0.6$	$G_{23}/E_2=0.5$	$a=b=1\text{m}$	$\upsilon_{12}=0.25$
E_1/E_2	Khdeir [40]	Thinh [27]	Present		Discrepancy %
			Analytical	GDQM	
10	8.2982	8.2981	8.8485	8.8482	6%
20	9.5671	9.5671	10.0328	10.0326	4%
30	10.326	10.326	10.6318	10.6311	2%
40	10.854	10.854	11.0045	11.0045	1%

4.1.2. Buckling Analysis

The buckling load factors of the system are calculated by the stability Eq. (21) as an eigenvalue problem. It can be seen from the Tables 1 and 2 that the present theory yields result very close to those of the other shear deformation theories, even with the presence of the elastic foundation.

Table 2. Comparison of uniaxial buckling loads for square symmetric laminates resting on Pasternak foundation

a/b	$b/h=10$				
	Xiang [16]	Aiello [41]	Present		
			Analytical	GDQM	Discrepancy %
1	50.7515	50.7511	52.1648	52.1668	2%
2	49.2666	49.2666	51.2038	51.2058	3%

$G_{12}=G_{13}=0.6E_{22};G_{23}=0.5E_{22};\nu_{12}=0.25,\beta=N_x/N_y$ $\left(N_{Cr(non-dim.)}=N_{cr}b^2/E_{22}h^3,\delta_p=10,\delta_w=100\right)$, Lay-up $[0^0/90^0/0^0]$, $E_{11}/E_{22}=40$;

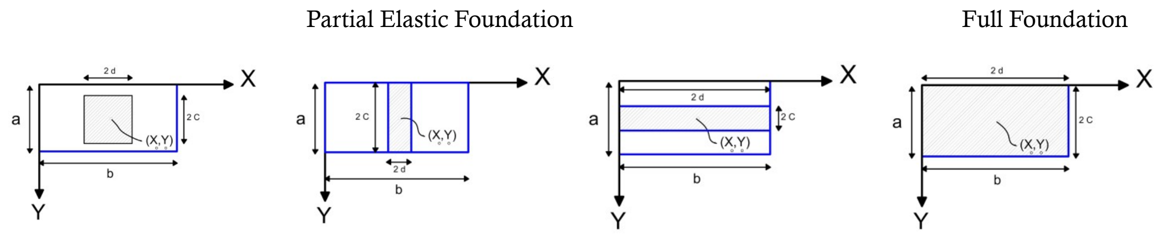

Fig. 3. Illustrates the FML plate resting on the partial elastic foundation for different foundation area.

Because the present work is the first research on analysis of free vibration and buckling of FML plates resting on partial elastic foundations so the validation of the present method is done as is shown in Fig. 4 and Fig. 5. It can be seen from Fig. 4 that as the foundation area increases the non-dimensional frequency also increases. In another word the trend of non-dimensional frequency when the partial foundation parameter is zero (no foundation) and when the partial foundation area is equal to plate area (full foundation) is in the exact agreement with the results are obtained from the other articles which have been mentioned. The non-dimensional frequency increases as the partial elastic foundation area increase from zero to full elastic foundation. The free vibration and buckling results are in exact agreement for both analytical and GDQ method.

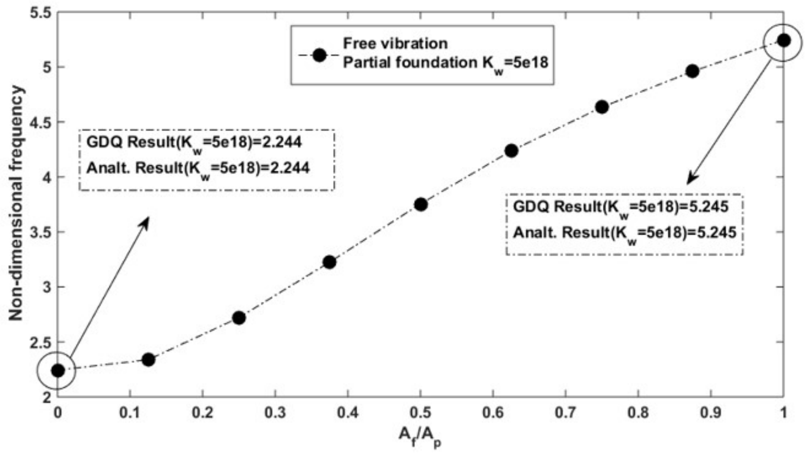

Fig. 4. The effect of increasing partial elastic foundation area on non-dimensional frequency for FML plates under SSSS BCs. ($n=1$,$b/h=10$,, $c=a/2, d=b/2, x_0=a/2, y_0=b/2$)

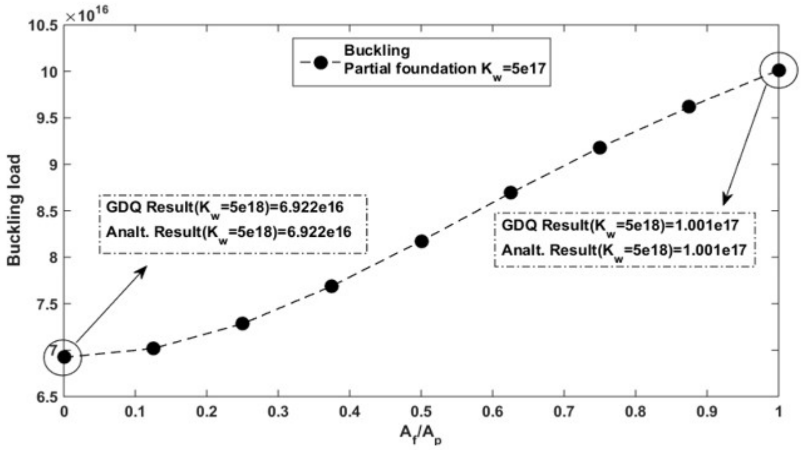

Fig. 5. The effect of increasing partial elastic foundation area on buckling load for FML plates under SSSS BCs. ($n=1$,$b/h=10$,, $c=a/2, d=b/2, x_0=a/2, y_0=b/2$)

In Fig. 6, the variations of the non-dimensional fundamental frequencies of FML plate on the elastic foundation with changing the elastic modulus ratio, are given. It is seen from the figures that the increase of the degree of elastic modulus produces a decrease of non-dimensional frequency values.

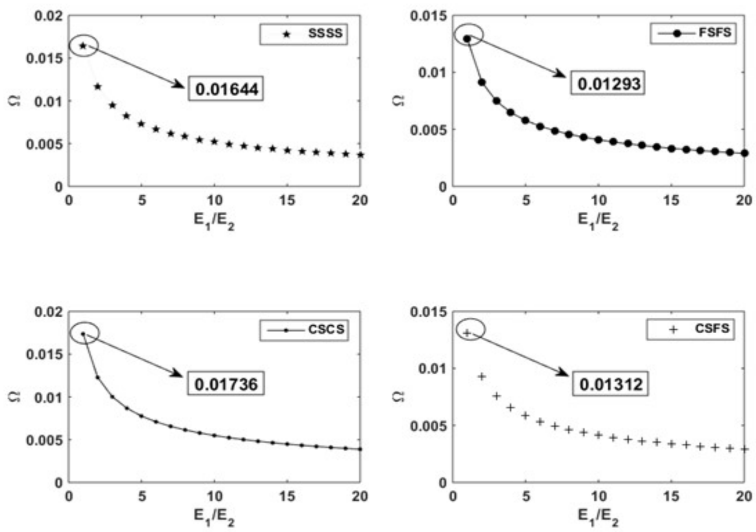

Fig. 6. The effect of elastic modulus on the non-dimensional fundamental frequency of FML plates on the full elastic foundation for various BCs. $(K_w=4e10, a=2.5, b=1, h=0.1, n=1)$, Lay-up $[0^0/90^0/0^0/AL/0^0/90^0/0^0]$

The effect of partial elastic foundation stiffness on the vibration of thick laminated plates is shown in Figures 8. The figures show that frequencies of laminates increase when foundation parameters increase. Table 3 shows that for getting convergent results of free vibration, fifteen grid points are enough. Table 4 shows that to get convergent results of Buckling load, 365 grid points are enough.

Table 3. Effect of the number of grid points on evaluating convergence of the non-dimensional frequency of the FML moderately thick plate resting on partial elastic foundation ($a=2,,b=1,h=0.1b,K_w=5e18$)

Boundary conditions	N=5	N=7	N=9	N=11	N=15
SSSS	3.2346	3.2352	3.2352	3.2352	3.2352
FSFS	3.153	3.148	3.147	3.146	3.146
CSCS	3.2595	3.2569	3.2597	3.2597	3.2597

Table 4. Effect of the number of grid points on evaluating convergence of the Buckling load of the FML moderately thick plate resting on partial elastic foundation $\left(k_w=10,k_p=100,n=1,a=2.5,b=1,h=0.1\right)$

Boundary conditions	N=280	N=290	N=340	N=350	N=356	N=360	N=362	N=365
SSSS	4.507	4.507	4.507	4.507	4.507	4.507	4.507	4.507
FSFS	2.548	3.236	2.632	2.646	2.154	2.154	2.154	2.154
CSCS	1.511	5.728	5.730	7.555	6.044	5.714	5.714	5.714
CSFS	4.314	4.322	4.361	4.369	4.0918	4.0918	4.0918	4.0918

In Figures 7, the variations of non-dimensional fundamental frequencies of FML plate on the elastic foundation with the change in foundation area are given. It is seen from the figures that the increase of the area of elastic foundation causes an increase of non-dimensional frequency values.

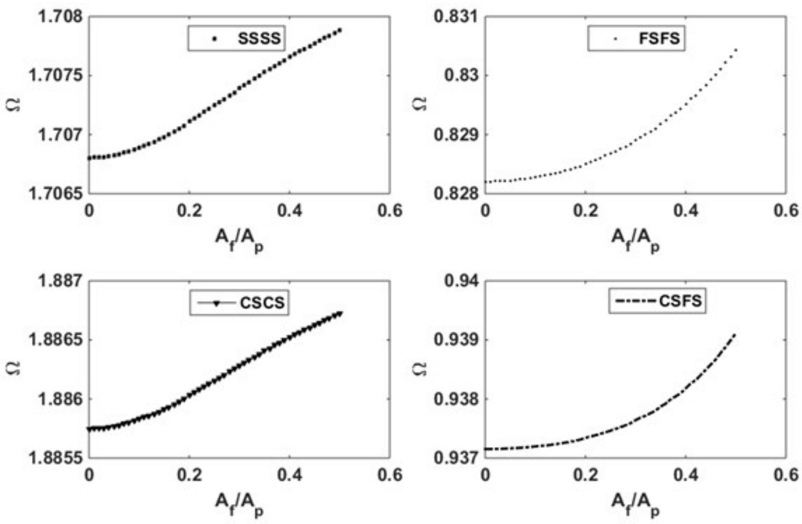

Fig. 7. The effect of the partial elastic foundation area (A_f) changes on the non-dimensional fundamental frequency of FML plates for various BCs. $(n=1,a/b=2.5\ ,b/h=10,c=a/2,x_0=a/2,y_0=b/2,K_w=2e15)$

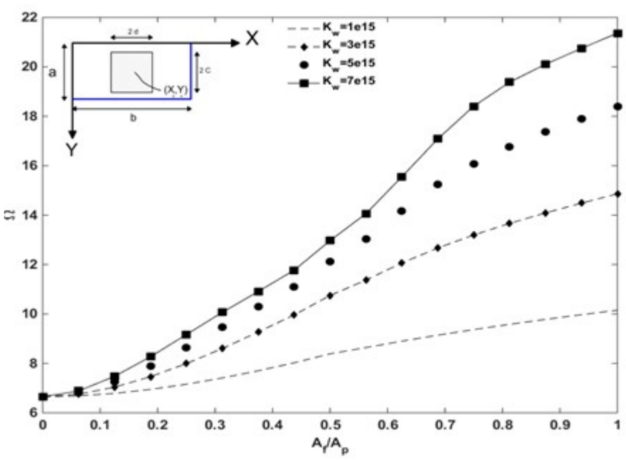

Fig. 8. The effect of the partial elastic foundation area (A_f) changes on the non-dimensional fundamental frequency of FML plates for various elastic foundation coefficient. $(n=1,a/b=2\ ,b/h=10,c=a/2,x_0=a/2,y_0=b/2)$

Figure 8 shows the effect of the partial elastic foundation area changes on the non-dimensional fundamental frequency of FML plates for various elastic foundation coefficient. According to Fig. 8, it can be seen that a steep gently goes up with an increase in foundation coefficient from 1e18 to 7e18, in fact increasing the stiffness of the elastic foundation is directly related to the stability of the FML plates and thus finally increases the natural frequency. Also, it shows when A_f to A_p ratio is lower than 15%, the increase in the non-dimensional frequency is lower than when A_f to A_p ratio is between 15% and 85%. Also, it can be mentioned that when the elastic foundation area to the plate area reaches more than 85%, free vibration and buckling behavior of the plate exactly similar with lower than 15% A_f to A_p ratio.

In figures 9 to 12, the mode-shape of non-dimensional frequencies of FML plates resting on a partial elastic foundation, for specific foundation area (A_f) are given. Spring location in Fig. 9 is in the center of rectangular FML plate causes the plate behaves like plate without any deflection for 3 modes. The perpendicular axes referred to non-dimensional deflection of the plate which is gained from eigenvalue problem and has no unit.

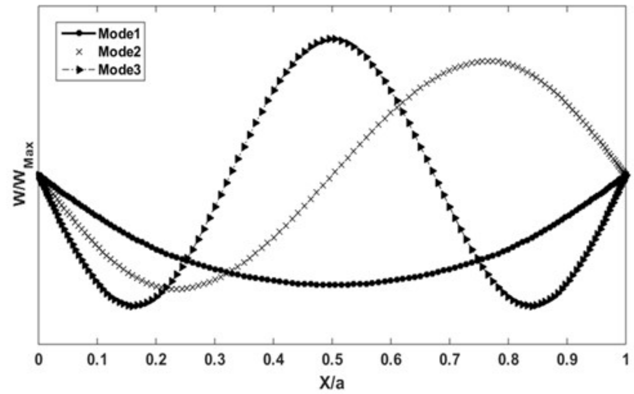

Fig. 9. Mode-shape results of FML plates resting on a partial elastic foundation under SSSS B.Cs.
$(c = a/4, d = b/4, x_0 = a/2, y_0 = b/2, K_w = 5e18, a = 4, b = 1)$

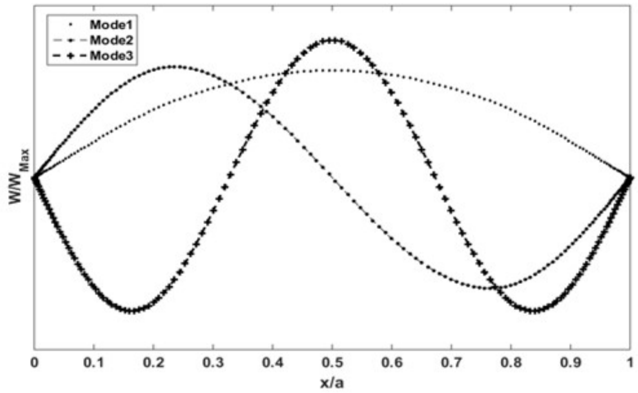

Fig. 10. Mode-shape results of FML plates resting on a partial elastic foundation under CSCS B.Cs.
$(c = a/4, d = b/4, x_0 = a/2, y_0 = b/2, K_w = 5e18, a = 4, b = 1)$

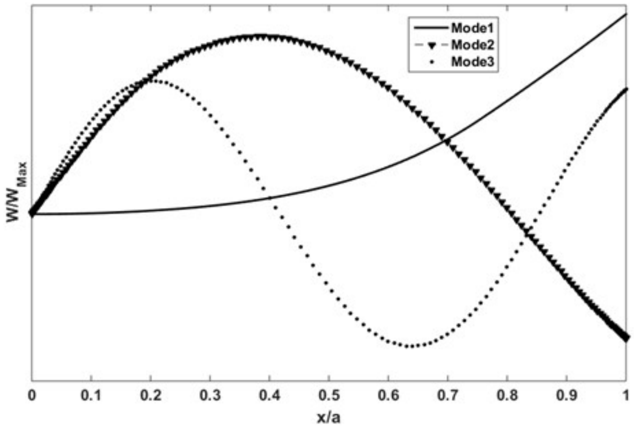

Fig. 11. Mode-shape results of FML plates resting on a partial elastic foundation under CSFS B.Cs.
$(c = a/4, d = b/4, x_0 = a/2, y_0 = b/2, K_w = 5e18, a = 4, b = 1)$

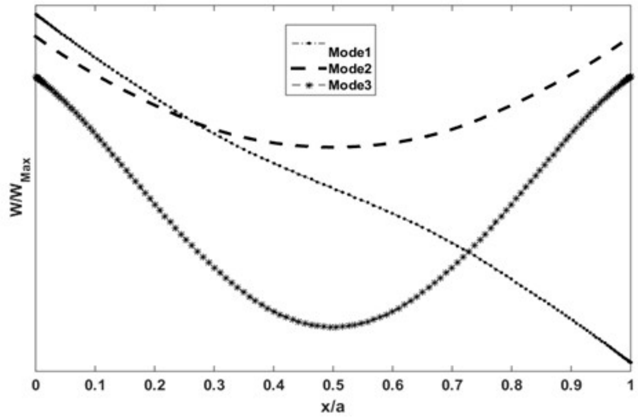

Fig. 12. Mode-shape results of FML plates resting on a partial elastic foundation under SFSF B.Cs. $(c = a/4, d = b/4, x_0 = a/2, y_0 = b/2, K_w = 5e18, a = 4, b = 1)$

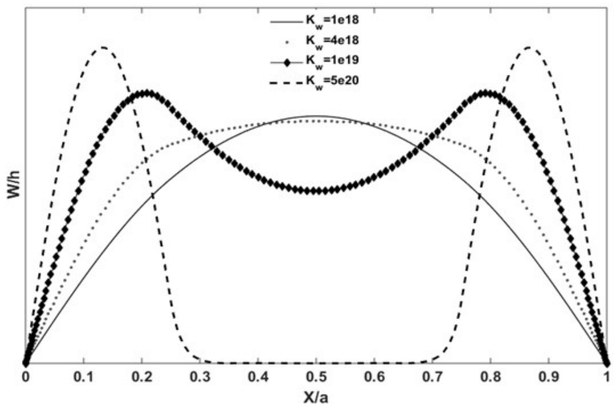

Fig. 13. Mode-shape results of FML plates resting on partial elastic foundation under SSSS B.Cs. and for various foundation coefficient. $(c = a/4, d = b/4, x_0 = a/2, y_0 = b/2, a = 4, b = 1)$

In Fig. 13 with increasing in foundation stiffness parameter, deflection of FML plate in foundation area goes to zero, in the other word, the FML plates strength has been raised.

Table 5 shows that with increasing the partial elastic foundation area along the length of FML plate, critical buckling load has increased, also with increasing FML plate length the critical buckling load has decreased. It shows that with increase in plate length and by considering elastic foundation parameter constant value, the FML plate stability is reduced and gets buckling sooner.

Table 5. The effect of the partial elastic foundation location changes on critical Buckling load of FML plates for various length to width (a/b). $m = 1, n = 1, K_w = 4e16$

a/b	Partial Elastic Foundation Location $c = b/2$					
	a/16	a/14	a/10	a/8	a/4	a/2
1	8.023005	8.024621	8.032160	8.040474	8.099477	8.219245
2	5.568144	5.571025	5.582970	5.595991	5.690287	5.882078
3	5.098088	5.100986	5.114408	5.129330	5.235130	5.451341
4	4.950136	4.953152	4.967077	4.982530	5.092595	5.319583
5	4.886416	4.889465	4.935185	4.919618	5.030515	5.263916

In Fig.14 FML plate resting on partial elastic foundation under CSCS boundary condition gets its critical buckling load right at the point where the frequency becomes zero.

Finally, we get to an important result from Table 6 and 7, if the rising spring stiffness and costs do not matter to us with fewer springs instead of keeping foundation stiffness constant, the result is the same as the fundamental frequency. In table 7, with considering $K_{wP} = K_{wT}$ in Table 7, the optimum condition of A_f / A_p is 65% so that if K_w increasing optimal then this ratio can be reduced by up to 50%. This is a very important result for other boundary conditions with less than 4% discrepancy. Where K_w, K_{wP} and K_{wT} are elastic foundation stiffness, partial elastic foundation, and total elastic foundation respectively.

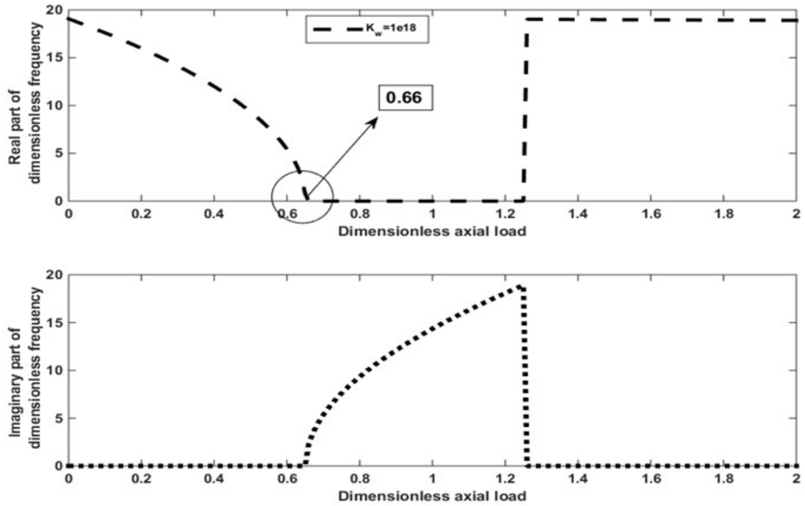

Fig. 14. Critical Buckling Load with increasing axial load for FML plates resting on a Partial Elastic foundation $n=1, a/b=4, b/h=10, c=a/4, d=b/8, x_0=a/2, y_0=b/2, K_w=1e18, BCs=CSCS$

Table 6. Foundation Area for FML plate resting on a partial elastic foundation for different foundation coefficient ($c=a/4, d=b/4, x0=a/2, y0=b/2, n,m=1, BCs.\ SSSS$)

a/b	Foundation Parameter K	Non-dimensional Freq. Ω	Optimum Foundation area A_f / A_p
1	K_{wT} = 2e18 K_{wP} = 6e18	2.568 2.507	0. 5-0.52
2	K_{wT} = 2e18 K_{wP} = 4e18	2.223 2.221	0.52-0.55
3	K_{wT} = 2e18 K_{wP} = 3e18	2.170 2.157	0.53-0.56
4	K_{wT} = 2e18 K_{wP} = 2e18	2.153 2.151	0.55-0.57

Table 7. Foundation Area for FML plate resting on partial elastic foundation for equal foundation coefficient ($c=a/4, d=b/4, x0=a/2, y0=b/2, n,m=1, BCs.SSSS$)

Boundary Condition	Foundation Parameter K	Non-dimensional Freq. Ω	Optimum Foundation area A_f / A_p
SSSS	K_{wT} = 2e18 K_{wP} = 2e18	2.223 2.220	0.65-0.7
FSFS	K_{wT} = 2e18 K_{wP} = 2e18	2.093 2.076	0.85-0.9
CSCS	K_{wT} = 2e18 K_{wP} = 2e18	2.228 2.225	0.7-0.75
CSFS	K_{wT} = 2e18 K_{wP} = 2e18	2.098 2.089	0.85-0.9

9. Conclusions

The results show that the constant of total or partial spring, elastic foundation parameter, thickness ratio, frequency mode number and boundary conditions play an important role on the critical buckling load and natural frequency of the FML plate resting on partial foundation under in-plane force. The following important results can be obtained from this study: It can be seen from the figures that increasing the area of partial elastic foundation along the x-direction to whole plate area, enhance the natural frequencies and critical buckling load. An interesting and vigorous GDQ method has been presented which has then strongly been applied to solve the free vibration and buckling problem of FML moderately thick plates resting on partial elastic foundations. Obtained results using the present method have been evaluated against those available in literature and excellent agreements have been found. It is concluded, through the results, that the present method can provide accurate results for natural frequencies and buckling loads of FML plates on a partial elastic foundation. Also, another important result of the current study is that in SSSS boundary condition if the proportion of the partial elastic foundation area to whole plate area is approximately 60% then the natural frequency and buckling results approach to the equivalent stability results opposed to full foundation with the same K_w for FML plates.

Conflict of Interest

The authors declared no potential conflicts of interest with respect to the research, authorship and publication of this article.

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Nomenclature

σ_{ij}	Stress tensor	M_{ij}	Bending moment tensor
ε_{ij}	Strain tensor	ρ	Density
u	Displacement along x-direction	K_w	Winkler foundation coefficient
v	Displacement along y-direction	K_P	Pasternak foundation coefficient
w	Displacement along z-direction	n	Semi-wavelength mode number
φ_y	Rotation about y	U	Potential energy
φ_x	Rotation about x	W	External work
E	Young modulus	n	Transverse wave mode number
I	Moment of Inertia	T	Kinetic energy
A_{ij}	Extensional stiffness tensor	δ	Variation
B_{ij}	Coupling stiffness tensor	N_{ij}	Axial forces components
D_{ij}	Bending stiffness tensor	m	Semi-wave width mode number
M_{ij}	Mass tensor	H	Heaviside step Function
N_{cr}	Axial critical buckling load	a	Length
K_{wT}	Coefficient of partial elastic foundation	b	Width
K_{wP}	Coefficient of partial elastic foundation	x_0	Longitudinal distance of the center of foundation from the origin of the coordinate
N_i	Node number along x-direction	y_0	Transverse distance of the center of foundation from the origin of the coordinate
N_i	Node number along y-direction	ω	Frequency

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Appendix

$$\sigma_{xx} = \bar{Q}_{11}\varepsilon_{xx} + \bar{Q}_{12}\varepsilon_{yy} \tag{A.1}$$

$$N_{xx} = \int \sigma_{xx} dz = \int \left(\bar{Q}_{11}\varepsilon_{xx} + \bar{Q}_{12}\varepsilon_{yy}\right) dz = \int \bar{Q}_{11} \frac{\partial u_0}{\partial x} dz + \int \bar{Q}_{11} z \frac{\partial \varphi_x}{\partial x} dz + \int \bar{Q}_{12} \frac{\partial v_0}{\partial y} dz + \int \bar{Q}_{12} z \frac{\partial \varphi_y}{\partial y} dz \tag{A.2}$$

$$N_{xx} = A_{11} \frac{\partial u_0}{\partial x} + B_{11} \frac{\partial \varphi_x}{\partial x} + A_{12} \frac{\partial v_0}{\partial y} + B_{12} \frac{\partial \varphi_y}{\partial y} \tag{A.3}$$

$$\sigma_{yy} = \bar{Q}_{21}\varepsilon_{xx} + \bar{Q}_{22}\varepsilon_{yy} \tag{A.4}$$

$$N_{yy}=\int\sigma_{yy}dz=\int\overline{Q}_{21}\varepsilon_{xx}dz+\int\overline{Q}_{22}\varepsilon_{yy}dz=\int\overline{Q}_{21}\varepsilon_{xx}dz+\int\overline{Q}_{22}\varepsilon_{yy}dz$$
$$=\int\overline{Q}_{21}\frac{\partial u_0}{\partial x}dz+\int\overline{Q}_{21}z\frac{\partial\varphi_x}{\partial x}dz+\int\overline{Q}_{22}\frac{\partial v_0}{\partial y}dz+\int\overline{Q}_{22}\frac{\partial\varphi_y}{\partial y}dz \tag{A.5}$$

$$N_{yy}=A_{21}\frac{\partial u_0}{\partial x}+B_{21}\frac{\partial\varphi_x}{\partial x}+A_{22}\frac{\partial v_0}{\partial y}+B_{22}\frac{\partial\varphi_y}{\partial y} \tag{A.6}$$

$$\sigma_{xy}=\overline{Q}_{44}\gamma_{xy}\quad,\quad N_{xy}=\int\sigma_{xy}dz \tag{A.7}$$

$$N_{xy}=\int\overline{Q}_{44}\gamma_{xy}dz=\int\overline{Q}_{44}\frac{\partial u_0}{\partial y}dz+\int\overline{Q}_{44}\frac{\partial v_0}{\partial x}dz+\int\overline{Q}_{44}z\frac{\partial\varphi_x}{\partial y}dz+\int\overline{Q}_{44}z\frac{\partial\varphi_y}{\partial x}dz \tag{A.8}$$

$$N_{xy}=A_{44}\left(\frac{\partial u_0}{\partial y}+\frac{\partial v_0}{\partial x}\right)+B_{44}\left(\frac{\partial\varphi_x}{\partial y}+\frac{\partial\varphi_y}{\partial x}\right)\quad,\quad\sigma_{xz}=\overline{Q}_{55}\gamma_{xz} \tag{A.9}$$

$$Q_{xz}=K_s\int\overline{Q}_{55}\gamma_{xz}dz=\frac{5}{6}\int\overline{Q}_{55}\left(\varphi_x+\frac{\partial w_0}{\partial x}\right)dz \tag{A.10}$$

$$Q_{xz}=\frac{5}{6}A_{55}\left(\varphi_x+\frac{\partial w_0}{\partial x}\right)\quad,\quad\sigma_{yz}=\overline{Q}_{66}\gamma_{yz} \tag{A.11}$$

$$Q_{yz}=K_s\int\overline{Q}_{66}\gamma_{yz}dz=\frac{5}{6}\int\overline{Q}_{66}\left(\varphi_y+\frac{\partial w_0}{\partial y}\right)dz\quad,\quad Q_{yz}=\frac{5}{6}A_{66}\left(\varphi_y+\frac{\partial w_0}{\partial y}\right) \tag{A.12}$$

$$\sigma_{xx}=\overline{Q}_{11}\varepsilon_{xx}+\overline{Q}_{12}\varepsilon_{yy}\quad,\quad M_{xx}=\int\sigma_{xx}zdz \tag{A.13}$$

$$M_{xx}=\int\overline{Q}_{11}\varepsilon_{xx}zdz+\int\overline{Q}_{12}\varepsilon_{yy}zdz=\int\overline{Q}_{11}\frac{\partial u_0}{\partial x}zdz+\int\overline{Q}_{11}\frac{\partial\varphi_x}{\partial x}z^2dz+\int\overline{Q}_{12}\frac{\partial v_0}{\partial y}zdz+\int\overline{Q}_{12}\frac{\partial\varphi_y}{\partial y}z^2dz \tag{A.14}$$

$$M_{xx}=B_{11}\frac{\partial u_0}{\partial x}+D_{11}\frac{\partial\varphi_x}{\partial x}+B_{12}\frac{\partial v_0}{\partial y}+D_{12}\frac{\partial\varphi_y}{\partial y} \tag{A.15}$$

$$\sigma_{yy}=\overline{Q}_{21}\varepsilon_{xx}+\overline{Q}_{22}\varepsilon_{yy},\quad M_{yy}=\int\sigma_{yy}zdz \tag{A.16}$$

$$M_{yy}=\int\overline{Q}_{21}\frac{\partial u_0}{\partial x}zdz+\int\overline{Q}_{21}\frac{\partial\varphi_x}{\partial x}z^2dz+\int\overline{Q}_{22}\frac{\partial v_0}{\partial y}zdz+\int\overline{Q}_{22}\frac{\partial\varphi_y}{\partial y}z^2dz \tag{A.17}$$

$$M_{yy}=B_{21}\frac{\partial u_0}{\partial x}+D_{21}\frac{\partial\varphi_x}{\partial x}+B_{22}\frac{\partial v_0}{\partial y}+D_{22}\frac{\partial\varphi_y}{\partial y} \tag{A.18}$$

$$\sigma_{xy}=\overline{Q}_{44}\gamma_{xy},\quad M_{xy}=\int\sigma_{xy}zdz \tag{A.19}$$

$$M_{xy}=\int\overline{Q}_{44}\frac{\partial u_0}{\partial y}zdz+\int\overline{Q}_{44}\frac{\partial v_0}{\partial x}zdz+\int\overline{Q}_{44}\frac{\partial\varphi_x}{\partial y}z^2dz+\int\overline{Q}_{44}\frac{\partial\varphi_y}{\partial x}z^2dz \tag{A.20}$$

$$M_{xy}=B_{44}\left(\frac{\partial u_0}{\partial y}+\frac{\partial v_0}{\partial x}\right)+D_{44}\left(\frac{\partial\varphi_x}{\partial y}+\frac{\partial\varphi_y}{\partial x}\right) \tag{A.21}$$