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Nonlocal Elasticity Effect on Linear Vibration of Nano-circular Plate Using Adomian Decomposition Method

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Abstract. In this study, the small scale effect on the linear free-field vibration of a nano-circular plate has been investigated using nonlocal elasticity theory. The formulation is based on the classical theory and the linear strain in cylindrical coordinates. To take into account the small scale and the linear geometric effects, the governing differential equation based on the nonlocal elasticity theory was extracted from Hamilton principle while the inertial effect, as well as the shear stresses effect was ignored. Effect of the nonlocal parameter is investigated by solving the governing equation using Adomian decomposition method (ADM) for the clamped and simply supported boundary conditions. By using this method, the first five axisymmetric natural frequencies and displacements of a nano-circular plate are obtained one at a time and some numerical results are given to illustrate the influence of nonlocal parameters on the natural frequencies and displacements of the nano-circular plate. For the purpose of comparison, the linear equations were solved by the analytical method. Excellent agreements were observed between the two methods. This indicates that the latter method can be applied to seek the linear solution of nano-circular plates with high accuracy while simplifying the problem.

Keywords: Linear free vibration, Nano-circular plates, Nonlocal elasticity, Adomian decomposition method.

1. Introduction

Structures with nano-sized elements have attracted the attention of the scientific community due to their superior properties. Conducting experiments with nano-scale size specimens is found to be difficult and expensive. Therefore, the development of appropriate mathematical models for nanostructures is an important issue concerning the application of nanostructures. Generally, three approaches have been developed to model nanostructures: (a) atomistic model (b) hybrid atomistic–continuum mechanics (c) modified continuum mechanics (including small scale effect). Ball [1] and Baughman *et al*. [2] used atomistic model, while Bodily and Sun [3], Li and Chou [4, 5], as well as Pradhan and Phadikar [6], used hybrid atomistic–continuum mechanics to reach a solution in such structures. Atomistic and hybrid atomistic–continuum mechanics used to analyze large scale systems are computationally expensive and are not convenient. However, the modified continuum mechanics approach is less computationally expensive than the former two approaches, and it has been found that its results are in good agreement with atomistic and hybrid approaches [7].

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Since the size effect is not considered in classical continuum theories, then this theory is not applicable to micro and/or nanostructures. Chong [8] and Fleck [9] experimentally showed that size effects play an important role in the mechanical properties of nanostructures. Consequently, ignoring these effects may lead to wrong findings and/or erroneous results. Eringen introduced the nonlocal elasticity theory, namely, Eringen's theory [10, 11] which could be applied to many nano-sized structural problems. This theory is one of the best and most well-known continuum mechanics theories that includes small scale effects with good accuracy. It has been shown that the results of this theory match with those of experimentally observed on nano-structures [12]. In order to apply the continuum mechanics approach to solve micro and nano-structural problems, logical and reasonable modifications that takes into account the scale effect are necessary. For this reason, several theoretical models such as the strain gradient theory, the modified coupled stress theory, and the nonlocal elasticity theory have been proposed which are utilized to analyze such problems.

Chen [13] investigated the small scale effect on the large deflection of the cantilever nano-beam under a vertical end loading. Moreover, Murmu and Adhikari [14] analyzed the vibration of rotating nano-beams using the nonlocal Euler–Bernoulli beam theory. Shishesaz and Hosseini [15] studied the mechanical behavior of functionally graded nano-cylinders under radial pressure based on strain gradient theory. In a related study, Wang *et al.* [16] studied the nonlocal effects on bending vibration of a single-walled carbon nanotube (SWCNT) under pre-stress via nonlocal elasticity and clearly showed that how the effects of material length scale and surface energy on the fundamental frequency of the plate can be changed by decreasing the thickness. Tahani *et al.* [17] employed the modified coupled stress theory to study the size-dependent free vibration characteristics of rectangular micro-plates pre-deformed by an electric field. They showed that if the ratio of plate thickness to the material length scale parameter is larger than 10 and 20, the length scale will not significantly affect the natural frequencies of the flat and electrostatically pre-deformed micro-plates, respectively. Hosseini and Shishesaz [18] studied the thermoelastic behavior of rotating functionally graded micro/nanodisks of variable thickness. They showed that selecting a variable thickness for a nanodisk, reduces its radial stresses. Moreover, due to coupling effect between the radial stress and high-order stresses which appear in the strain gradient theory, total radial stresses at the inner and outer radii were nonzero despite the zero external loads at the two corresponding surfaces. Additionally, owing to a very small radius for the nanodisk, the centrifugal forces were negligible and hence, they barely affect the total radial stresses. Mohammadi *et al.* [19], performed an analysis on the primary and secondary resonance of porous functionally graded nanobeam resting on a nonlinear foundation subjected to mechanical and electrical loads. They used Galerkin method in conjunction with variation iteration method (VIM), Homotopy perturbation method (HPM), as well as Hamiltonian and multiple scale methods to solve the governing equations based on the clamped-clamped, simply-simply and clamped-simply supported boundary conditions. They showed that the length scale parameters have a crucial role in the nonlinear vibrational behavior of such structures.

Ma *et al.* [20] developed the non-classical Mindlin plate model via the modified couple stress theory to investigate the effects of length scale on predicting the natural frequencies of plates. They clearly showed that the predicted values of frequencies can be decreased when the size effect is omitted, especially in a thin plate. Mohammadi *et al.* [21] used the nonlocal continuum plate model to investigate the size effect on free transverse vibration analysis of circular and annular graphene sheets with various boundary conditions. Wang *et al.* [22] used the modified coupled stress theory to investigate the size effect on the axisymmetric nonlinear free vibration of a circular microplate. Kiani [23-25] studied the small-scale effect on the vibration of thin nano-plates subjected to a moving nanoparticle via nonlocal continuum theory. Zarei *et al.* [19] investigated the buckling and free vibration behavior of a circular tapered nano-plate subjected to in-plane forces. They employed the nonlocal elasticity theory and investigated the effects of the nonlocal parameter on the natural frequencies. They showed that increasing the taper parameter causes the buckling load and natural frequencies to be increased while its effect for the clamped boundary condition was more than that of simply supported end condition. Salehipour *et al.* [26] presented closed-form solutions for the in-plane and out-of-plane free vibration of simply supported functionally graded (FG) rectangular micro/nano-plates. Ansari *et al.* [27] used a three-dimensional elastic plate model capturing the small scale effects, developed for the free vibration of functionally graded nanoplates resting on elastic foundations. Shishesaz *et al.* [28] introduced a novel approach to show the small scale effects on the nonlinear free-field vibration of a nano-disk using nonlocal elasticity theory in conjunction with homotopy perturbation method to ease the solution and completely solve the problem. They showed that the effect of dimensionless central amplitude on the nonlinear frequencies increases with an increase in the nonlocal parameter. This effect seemed to be more conspicuous at higher vibrational modes. In addition, Eringen's theory predicted that the thickness to radius ratio of the nano-disk plays an important role in the nonlinear vibrational behavior of the nano-disk. Increasing this ratio will also increase the effect of dimensionless central amplitude on nonlinear frequencies of the nano-disk. This effect is more prominent at higher values of the nonlocal parameter.

Presence of the nonlocal scale effect reduces the rigidity of nano-plates. In fact, this is the stiffness softening model but not the whole of nonlocal continuum theory. Li [29] used the Eringen's gradient type of the nonlocal elasticity theory to model the torsion of cylindrical nanostructures such as nanorods, nanoshafts, and/or nanotubes. For the first time, he proposed the effective nonlocal twisting moment for the torsional deformation. It was demonstrated that the static maximum angular deformation decreases with an increase in the nonlocal nanoscale parameter. In other words, the torsional rigidity of the nonlocal nanostructures tends to harden when the effective nonlocal effects are considered. He also showed that this effect was consistent with some other existing approaches, including the strain gradient elasticity and the modified couple stress theories. Li *et al.* [30] derived the governing equation of a nanobeam by introducing certain simplifying assumptions based on the two-dimensional differential constitutive relations of nonlocal elasticity and plane-stress condition. They applied their model to the nano-cantilever beams subjected to several typical external forces and studied the nonlocal effect on the bending behavior of their model. They showed that although the nonlocal effect does appear in the nano-cantilever beam subjected to a concentrated force, the bending behavior of the beam subjected to general transverse distributed loads is significantly influenced by the nonlocal scale effect factor. It was also demonstrated that depending on the category of applied loads, the equivalent stiffness of a nanostructure predicted by the nonlocal theory may be larger or smaller than that predicted by the classical theory. Shen and Li [31] developed a modified semi-continuum Euler beam model with relaxation phenomenon and the bending deformation for the extreme-thin beam with micro/nano-scale thickness. They considered the external loads in the form of concentrated force and uniformly distributed loads for different boundary constraints. They showed that the semi-continuum, classical continuum and the nonlocal continuum models are in good agreements with each other unless the thickness is too small. They proved that the equivalent Young's modulus may increase or decrease with increasing the thickness.

Adomian decomposition method (ADM) introduced by Adomian [32, 33] has been used to solve the differential, integral and integro-differential equations, linear or nonlinear, by many researchers. Here, the main controversy is the determination of Adomian's polynomials and the convergence of the introduced series. The theoretical treatment of the convergence of Adomian decomposition method has been considered in [34-39]. Hsu *et al.* [40, 41] used modified ADM to analyze the free vibration of non-uniform Euler- Bernoulli beams with general elastically end constraints, as well as the uniform Timoshenko beams under various supporting boundary conditions. Mao and Pietrzko [42] studied the design of shaped piezoelectric modal sensor for beams with arbitrary boundary conditions using ADM. In another study, they applied ADM to investigate the free vibrations of a stepped Euler-Bernoulli beam consisting of two uniform sections [43]. Later on, they investigated the free vibration of rotating Euler-Bernoulli beams with the linearly varying thickness and/or width of the cross-section along the length by using the modified ADM [44]. Jamali *et al.* [45] used Adomain decomposition method for buckling analysis of nano-composite cut out plates. Other studies have also shown that ADM is an efficient tool for solving boundary value problems and many other differential equations that arise in structural vibration, fluid mechanics, viscoelasticity, control theory, acoustics, and so on. The Adomian technique is very simple in its principles. The accuracy and its simplicity in calculating the natural frequencies and mode shapes make this method outstanding among many other methods.

In spite of the vast literature on free transverse vibration of nano-beams, nano-plates, and nano-shells, few works have been dedicated to the nonlocal elasticity of free vibration of a nano-circular plate, using semi-analytical methods such as ADM. Vibrational analysis of such plates is undoubtedly a prerequisite for designing an efficient nano-machines. This has motivated the authors to investigate this problem and discover the applicability of ADM for the solution of such problems. The main objective of this work is to study the axisymmetric free vibration of a nano-circular plate using nonlocal elasticity. Here, the effects of the nonlocal parameter and different boundary conditions on the nano-circular plate frequencies are investigated. For this purpose, Erigen's nonlocal elasticity theory along with Hamilton's principle is used. Using ADM, the linear differential governing equation will be solved and the corresponding results will be compared with those of analytical solution, using separation of variables. The effect of the nonlocal parameter on the axisymmetric frequencies will be investigated for the clamped and simply supported boundary conditions. These results will be useful for the design of nanomotors, nanorotors, and other rotary nano-structures.

2. Nonlocal Constitutive Relations

According to the Eringen theory, the stress field at point x in an elastic medium not only depends on the strain field at that point but also on strains at all other points in the body. Eringen also attributed this fact to the atomic theory of lattice dynamics and experimental observations on phonon dispersion [10]. On this basis, the nonlocal stress tensor σ at point x is expressed as:

$$\sigma = \int_V K\left(\left|x'-x\right|,\frac{e_0 a}{l}\right)\,\sigma^*\left(x'\right)\,\mathrm{d}x' \tag{1}$$

where $\sigma^*(x')$ is the classical, macroscopic stress tensor at point x', the kernel function $K\left(\left|x'-x\right|,e_0 a/l\right)$ represents the nonlocal modulus, $\left|x'-x\right|$ is considered as the relative distance (in Euclidean norm) and e_0 is a material constant, a and l are the internal and external characteristic lengths (the lattice spacing and wavelength, respectively). The macroscopic stress $\sigma^*(x')$ at point x' in a Hookean solid is related to the strain ε at that point by the generalized Hooke's law:

$$\sigma^* = C : \varepsilon \tag{2}$$

where C is the fourth-order elasticity tensor and : indicates the 'double-dot product'.
Equations (1) and (2) define the nonlocal constitutive behavior of a solid based on Hook's law. Equation (1) represents the weighted average of the strain field contributions of all points in the body to the stress field at a point. However, the integral constitutive relation in Eq. (1) makes the elasticity problems difficult to solve. To simplify the governing nonlocal relation, we have taken the nonlocal modulus $K\,(|x'-x|,\tau)$ as the Green's function of linear differential operator [46]:

$$L_0 K\left(\left|x'-x\right|,\tau\right) = \delta\left(\left|x'-x\right|\right) \tag{3}$$

where L_0 is the linear differential operator and δ is the Dirac-delta function. It is worth to mention that the Green's function is selected in conjunction with the properties of nonlocal modulus. Now, applying L_0 to Eq. (1), we obtain:

$$L_0\sigma^* = \sigma \tag{4}$$

The differential operator L_0 has different forms for different expressions of nonlocal modulus. Considering Eq. (4), based on the nonlocal modulus depicted in Eq. (3) we have:

$$L_0 = 1-(e_0 a)^2\nabla^2 \tag{5}$$

where ∇^2 is the Laplacian operator. Therefore, according to Eqs. (4) and (5), the nonlocal constitutive relation, namely Eq. (1), can be expressed in the differential form as:

$$\sigma^* = \sigma-(e_0 a)^2\nabla^2\sigma \tag{6}$$

The material parameter e_0 is a constant with a specific value for each material. The parameter e_0 is estimated such that the relations based on the nonlocal elasticity model will provide an acceptable approximation of the atomic dispersion curves of the plane waves and those of atomic lattice dynamics. Furthermore, the values of key parameter e_0 can be found by comparing the nonlocal results with the molecular dynamic simulation results and atomistic-based techniques. The terms e_0 and e_0a are referred to as the nonlocal parameters or the scale coefficient. The common term of the nonlocal parameter is e_0a. Table 1 shows some nonlocal parameter values proposed and used by different researchers.

Table 1. Nonlocal parameter proposed by various researchers

	Magnitudes	Reference
	0.7 nm	[47]
e_0	0-2.0 nm	[48]
	0-2.0 nm	[49]
	0-2.2 nm	[50]

Additional values of non-local parameters for different materials can be found in Ref. [51].

3. Problem Formulation

The derivation in this section is for an isotropic axisymmetric plate. The Kirchhoff plate theory assumptions are [52]:
1. The deflection of the mid-plane is small compared with the thickness of the plate. The slope of the deflected surface is therefore very small and its squared value is negligible compared with unity.
2. The mid-plane remains unstrained subsequent to bending.
3. A straight line initially normal to the mid-surface remains straight and normal to that surface after bending.
4. The stress σ_z normal to the mid-plane is small compared with the other stress components and may be neglected.

Since based on linear classical plate theory, the deflection is the order of the plate thickness (though it is still smaller than the other plate dimensions) then the first two assumptions do not hold anymore, while the third and fourth assumptions are still applicable.

Based on the classical plate theory (CPT) The plate displacement field in r, θ , and z directions can be written as [19, 21, 22, 28]:

$$\begin{aligned} u_r(r,\theta,z,t) &= u(r,\theta,t)-z\frac{\partial u_z}{\partial r}(r,\theta,t), \\ u_\theta(r,\theta,z,t) &= v(r,\theta,t)-z\frac{1}{r}\frac{\partial u_z}{\partial r}(r,\theta,t), \\ u_z(r,\theta,z,t) &= w(r,\theta,t). \end{aligned} \tag{7}$$

where u_r, u_θ, and u_z, are the radial, tangential, and transverse displacements of the plate, respectively. Similarly, u, v, and w correspond to the radial, tangential, and transverse displacements of the mid-surface of the plate. The nonzero linear strains are [21, 22, 28, 53]:

$$\begin{aligned} \varepsilon_{rr} &= \frac{\partial u_r}{\partial r}, \\ \varepsilon_{\theta\theta} &= \frac{u_r}{r}+\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}, \\ \gamma_{r\theta} &= \frac{1}{r}\left(\frac{\partial u_r}{\partial \theta}-u_\theta\right)+\frac{\partial u_\theta}{\partial r}. \end{aligned} \tag{8}$$

Substituting the displacement field Eq. (7) into Eq. (8), and realizing that for the axisymmetric loading, $\partial/\partial\theta = 0$ and $u_\theta = 0$, then we have [53]:

$$\varepsilon_{rr} = \frac{\partial u}{\partial r} - z\frac{\partial^2 w}{\partial r^2},$$
$$\varepsilon_{\theta\theta} = \frac{u}{r} - \frac{z}{r}\frac{\partial w}{\partial r}, \qquad (9)$$
$$\gamma_{r\theta} = 0.$$

Based on Eq. (6), the plane stress constitutive relations for the nano-plate are obtained as [21, 22, 28]:

$$\left(1-(e_0a)^2\nabla^2\right)\sigma_{rr} = \frac{E}{1-\upsilon^2}(\varepsilon_{rr}+\upsilon\varepsilon_{\theta\theta}),$$
$$\left(1-(e_0a)^2\nabla^2\right)\sigma_{\theta\theta} = \frac{E}{1-\upsilon^2}(\upsilon\varepsilon_{rr}+\varepsilon_{\theta\theta}). \qquad (10)$$

where E and υ are the Young's modulus and the Poisson's ratio of the plate, respectively. In polar coordinates, the Laplace operator ∇^2 is defined as:

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} \qquad (11)$$

Additionally, the stress resultants are defined as [52, 53]:

$$N_{rr} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{rr}\mathrm{d}z \quad , \quad N_{\theta\theta} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{\theta\theta}\mathrm{d}z,$$
$$M_{rr} = \int_{-\frac{h}{2}}^{\frac{h}{2}} z\sigma_{rr}\mathrm{d}z \quad , \quad M_{\theta\theta} = \int_{-\frac{h}{2}}^{\frac{h}{2}} z\sigma_{\theta\theta}\mathrm{d}z\ . \qquad (12)$$

Using strain displacement relations Eq. (9) in conjunction with stress–strain relations Eq. (10) and stress resultants Eq. (12), the non-zero stress resultants in terms of displacements are:

$$\left(1-(e_0a)^2\nabla^2\right)M_{rr} = -D\left(\frac{\partial^2 w}{\partial r^2} + \frac{\upsilon}{r}\frac{\partial w_0}{\partial r}\right),$$
$$\left(1-(e_0a)^2\nabla^2\right)M_{\theta\theta} = -D\left(\upsilon\frac{\partial^2 w_0}{\partial r^2} + \frac{1}{r}\frac{\partial w_0}{\partial r}\right). \qquad (13)$$

In Eq. (13), D is flexural rigidity and is defined in Eq.(14) [52, 53]:.

$$D = \frac{Eh^3}{12\left(1-\upsilon^2\right)} \qquad (14)$$

Now, Hamilton's principle is used to derive the governing equation [53]. This principle can be written in the following analytical form:

$$\int_0^t \delta\{T-U+W\}dt = 0 \qquad (15)$$

Here T, U and W correspond to the kinetic energy, strain energy of the plate and work done by the body force, respectively. On successful substitution of the terms for each case, one can recast Eq. (15) as [53]:

$$\int_0^t\int_A\left(\rho h\frac{\partial w}{\partial t}\delta\frac{\partial w}{\partial t} - \sigma_{rr}\delta\varepsilon_{rr} - \sigma_{\theta\theta}\delta\varepsilon_{\theta\theta} - f\delta w_0\right)dAdt = 0 \qquad (16)$$

where ρ, h and f are the density, thickness and the body force acting on the nanoplate. Using Eqs. (9)-(12) into Eq. (16) and integrating the resulting expression by parts and collecting the coefficients of δw_0, the equilibrium equation is obtained as [21]:

$$D\left(\frac{\partial^4 w}{\partial r^4} + \frac{2}{r}\frac{\partial^3 w}{\partial r^3} - \frac{1}{r^2}\frac{\partial^2 w}{\partial r^2} + \frac{1}{r^3}\frac{\partial w}{\partial r}\right) + \rho h\ddot{w} - (e_0a)^2\rho h\nabla^2\ddot{w} = 0 \qquad (17)$$

where the boundary conditions for the simply supported and clamped edges at $r = R$ are:

$$\text{simply supprted:} \quad w = 0 \quad , \quad M_{rr} - (e_0a)^2\nabla^2 M_{rr} = 0,$$
$$\text{clamped edge:} \quad w = 0 \quad , \quad \frac{\partial w}{\partial r} = 0. \qquad (18)$$

We now introduce the non-dimensional variables denoted by asterisks as [28]:

$$r = Rr^* \quad , \quad \mathrm{t} = R^2 \sqrt{\frac{\rho h}{D}}\, t^* \quad , \quad \mathrm{w} = \frac{w^*}{R} \quad , \quad \frac{e_0 a}{R} = \mu. \tag{19}$$

Substituting Eq. (19) into Eqs. (17)-(18), the non-dimensional governing equation and boundary conditions are obtained as follow:

$$\frac{\partial^4 w^*}{\partial r^{*4}} + \frac{2}{r}\frac{\partial^3 w^*}{\partial r^{*3}} - \frac{1}{r^{*2}}\frac{\partial^2 w^*}{\partial r^{*2}} + \frac{1}{r^{*3}}\frac{\partial w^*}{\partial r^*} + \frac{\partial^2 w^*}{\partial t^{*2}} - \mu^2 \frac{\partial^2}{\partial r^{*2}}\frac{\partial^2 w^*}{\partial t^{*2}} - \mu^2 \frac{1}{r^*}\frac{\partial}{\partial r^*}\frac{\partial^2 w^*}{\partial t^{*2}} = 0 \tag{20}$$

$$w^*\left(r^*,0\right) = f\left(r^*\right) \qquad , \qquad w_{t^*}\left(r^*,0\right) = g\left(r^*\right). \tag{21}$$

4. Solution Procedure

4.1. Adomian decomposition method

In recent years, the ADM has received much attention in applied mathematics in general, and in the area of series solutions in particular. The method is proved to be powerful, effective, and can easily handle a wide class of linear or nonlinear, ordinary or partial differential equations, as well as linear and nonlinear integral equations. The ADM demonstrates fast convergence of the solution and therefore provides several significant advantages. This method attacks the problem in a direct way in a straightforward fashion without using linearization, perturbation or any other restrictive assumption that may change the physical behavior of the model under discussion. Adomian [32, 33], introduced and developed the ADM and is well addressed in the literature. Hsu [40, 41], Cherruault [37], as well as Wazwaz [54, 55], applied this method to a wide class of linear and nonlinear ordinary differential equations, partial differential equations and integral equations. In this study, this method will be successfully used to handle Eq. (20), where its applicability and versatility will be examined.

The ADM consists of decomposing the unknown function $w^*(r, t)$ of an equation into a sum of an infinite number of components defined by the decomposition series:

$$w^*\left(r,t\right) = \sum_{n=0}^{\infty} w_n\left(r,t\right) \tag{22}$$

where the components $w_n(r,t)$ $(n \gg 0)$ are to be determined in a recursive manner. The ADM concerns itself with finding the components w_n (n =1, 2, 3, …..∞), individually. Hereon, as will be observed throughout the text, the determination of these components can be achieved in an easy way through a recursive relation that usually involves simple integrals. To give a clear overview of ADM, the linear differential equation is written in an operator form by:

$$L\left(w^*\right) + R\left(w^*\right) = g \tag{23}$$

where L is, mostly, the lower order derivative which is assumed to be invertible, R is another linear differential operator, and g is a source term. Applying the inverse operator L^{-1} to both sides of Eq. (23), and then using the given conditions we have:

$$w^* = f - L^{-1}\left(R\left(w^*\right)\right) \tag{24}$$

where the function f represents the terms arising from integrating the source term g and from using the given conditions that are assumed to be prescribed. As indicated before, the ADM defines the solution w by an infinite series of components given by:

$$w^* = \sum_{n=0}^{\infty} w_n \tag{25}$$

where the components w_n (n=1, 2, 3, …..∞) are usually recurrently determined. Substituting Eq. (25) into both sides of Eq. (24) gives:

$$\sum_{n=0}^{\infty} w_n = f - L^{-1}\left(R\sum_{n=0}^{\infty} w_n\right) \tag{26}$$

for simplicity, Eq. (26) can be rewritten as:

$$w_0 + w_1 + w_2 + \ldots = f - L^{-1}\left(R\left(w_0 + w_1 + w_2 + \ldots\right)\right) \tag{27}$$

To construct the recursive relation needed for the determination of components w_0, w_1, w_2, …, using the ADM, the formal recursive relation is defined by:

$$w_0 = f \quad , \quad w_{n+1} = -L^{-1}\left(Rw_n\right) \quad \{n \geq 1\} \tag{28}$$

It is clearly observed that Eq. (27) reduces Eq. (23) into an elegant determination of computable components. Having determined these components, they are substituted into Eq. (22) to obtain the solution in a series form. Based on the results in [39], Eq. (25) converges to the exact solution, when

$$0 \ll \alpha_i < 1 \qquad i = 0,1,2,\ldots \tag{29}$$

where

$$\alpha_i = \begin{cases} \dfrac{\|w_{i+1}\|}{\|w_i\|} & \|w_i\| \neq 0 \\ 0 & \|w_i\| \neq 0 \end{cases} \tag{30}$$

4.2. Formulation with ADM

Using the ADM, Eq. (20) can be rewritten in the operator form as:

$$L_{t^*}\left(w^*\right) = \mu^2 L_{t^*}\left(\frac{\partial^2 w^*}{\partial r^{*2}} + \frac{1}{r^*}\frac{\partial w^*}{\partial r^*}\right) - \left(\frac{\partial^4 w^*}{\partial r^{*4}} + \frac{2}{r^*}\frac{\partial^3 w^*}{\partial r^3} - \frac{1}{r^{*2}}\frac{\partial^2 w^*}{\partial r^2} + \frac{1}{r^{*3}}\frac{\partial w^*}{\partial r^*}\right) \tag{31}$$

where the differential operators L_t^* is defined as

$$L_{t^*} = \frac{\partial^2}{\partial t^{*2}} \tag{32}$$

We assume that the integral operator L_t^{-1} exists and may be regarded as two-fold indefinite integrals defined by:

$$L_{t^*}^{-1}(.) = \int_0^{t^*}\int_0^{s}(.)\,dsdt \tag{33}$$

such that;

$$\begin{aligned} &L_{t^*}^{-1}L_{t^*}w^*\left(r^*,t^*\right) = w^*\left(r^*,t^*\right) - w^*\left(r^*,0\right) - t^*\frac{\partial w^*}{\partial t^*}\left(r^*,0\right) \\ &L_{t^*}^{-1}\left(\mu^2 L_{t^*}\left(\frac{\partial^2 w^*}{\partial r^{*2}} + \frac{1}{r^*}\frac{\partial w^*}{\partial r^*}\right)\right) = \mu^2\left(\frac{\partial^2}{\partial r^{*2}} + \frac{1}{r^*}\frac{\partial}{\partial r^*}\right)\left(w^*\left(r^*,t^*\right) - w^*\left(r^*,0\right) - t^*\frac{\partial w^*}{\partial t^*}\left(r^*,0\right)\right) \end{aligned} \tag{34}$$

Applying L_t^{-1} to both sides of Eq. (31) while using Eq. (34) and the initial conditions (Eq. (21)), the result can be obtained as:

$$\begin{aligned} w^*\left(r^*,t^*\right) = f\left(r^*\right) + t^* g\left(r^*\right) + \mu^2\left(\frac{\partial^2}{\partial r^{*2}} + \frac{1}{r^*}\frac{\partial}{\partial r^*}\right)\left(w^*\left(r^*,t^*\right) - f\left(r^*\right) - t^* g\left(r^*\right)\right) \\ -\int_0^{t^*}\int_0^{s}\left(\frac{\partial^4 w^*}{\partial r^{*4}} + \frac{2}{r^*}\frac{\partial^3 w^*}{\partial r^{*3}} - \frac{1}{r^{*2}}\frac{\partial^2 w^*}{\partial r^{*2}} + \frac{1}{r^{*3}}\frac{\partial w^*}{\partial r^*}\right)dsdt \end{aligned} \tag{35}$$

The ADM defines the unknown function $w^*(r, t)$ by the following series:

$$w^*\left(r^*,t^*\right) = \sum_{n=0}^{\infty} w_n\left(r^*,t^*\right) \tag{36}$$

such that Eq. (35) can be recast as:

$$\begin{aligned} \sum_{n=0}^{\infty} w_n\left(r^*,t^*\right) = f\left(r^*\right) + t^* g\left(r^*\right) + \mu^2\left(\frac{\partial^2}{\partial r^{*2}} + \frac{1}{r^*}\frac{\partial}{\partial r^*}\right)\left(\sum_{n=0}^{\infty} w_n\left(r^*,t^*\right) - f\left(r^*\right) - t^* g\left(r^*\right)\right) \\ -\int_0^{t^*}\int_0^{s}\left(\frac{\partial^4}{\partial r^{*4}} + \frac{2}{r^*}\frac{\partial^3}{\partial r^{*3}} - \frac{1}{r^{*2}}\frac{\partial^2}{\partial r^{*2}} + \frac{1}{r^{*3}}\frac{\partial}{\partial r^*}\right)\sum_{n=0}^{\infty} w_n\left(r^*,t^*\right)dsdt \end{aligned} \tag{37}$$

Following the discussions presented above, the recursive relation is written as:

$$w^*{}_0\left(r^*,t^*\right) = \left(1 - \mu^2\left(\frac{\partial^2}{\partial r^{*2}} + \frac{1}{r^*}\frac{\partial}{\partial r^*}\right)\right)\left(f\left(r^*\right) + tg\left(r^*\right)\right) \tag{38}$$

$$w^*_{n+1}\left(r^*,t^*\right)=\int_0^{t^*}\int_0^{s}\left\{\mu^2\left(\frac{\partial^2}{\partial r^{*2}}+\frac{1}{r^*}\frac{\partial}{\partial r^*}\right)-\left(\frac{\partial^4}{\partial r^{*4}}+\frac{2}{r^*}\frac{\partial^3}{\partial r^{*3}}-\frac{1}{r^{*2}}\frac{\partial^2}{\partial r^{*2}}+\frac{1}{r^{*3}}\frac{\partial}{\partial r^*}\right)\right\}w_n\left(r^*,t^*\right)dsdt \qquad (n\geq 1) \tag{39}$$

Approximating the above solution by the *K*-term truncated series, then Eq. (39) is recast in the following form:

$$w^*_K\left(r^*,t^*\right)=\sum_{n=0}^{K}w_n\left(r^*,t^*\right) \qquad (n\geq 0) \tag{40}$$

The dimensionless center displacements of the nano-circular plate are obtained as:

$$w^*_c\left(t^*\right)=w^*_K\left(0,t^*\right) \tag{41}$$

The convergence of Eq. (41) can be checked using Eqs. (29) and (30).

4.3 Modal Analysis with ADM

To solve Eq. (20) for the natural frequencies of the nano-circular plate, the assumption of a harmonic motion for the n^{th} mode is used while the transverse displacements of the nano-circular plate are expressed in the following form:

$$w^*\left(r^*,t^*\right)=\varphi_n\left(r^*\right)e^{i\omega_n t^*} \tag{42}$$

Here, $\varphi_n(r)$ is the n^{th} axisymmetric mode shape and ω_n is the n^{th} non-dimensional natural frequency. Substituting Eq. (42) into Eq. (20) yields:

$$\left(\frac{d^4\varphi_n}{dr^{*4}}+\frac{2}{r^*}\frac{d^3\varphi_n}{dr^3}-\frac{1}{r^{*2}}\frac{d^2\varphi_n}{dr^{*2}}+\frac{1}{r^{*3}}\frac{d\varphi_n}{dr^*}\right)-\omega_n^2\left(\varphi_n\right)+\mu^2\omega_n^2\left(\frac{d^2\varphi_n}{dr^{*2}}+\frac{1}{r^*}\frac{d\varphi_n}{\partial r^*}\right)=0 \tag{43}$$

According to ADM, Eq. (43) can be considered in the operator form of as:

$$L_{r^*}\left(\varphi_n\right)+R_{r^*}\left(\varphi_n\right)=0 \tag{44}$$

where the differential operators L_{r^*} and R_{r^*} are defined in Eq. (45) to overcome the singularity behavior at $r^*=0$.

$$\begin{aligned}L_{r^*}&=\frac{d^4}{dr^{*4}}+\frac{2}{r^*}\frac{d^3}{dr^{*3}}-\frac{1}{r^{*2}}\frac{d^2}{dr^{*2}}+\frac{1}{r^{*3}}\frac{d}{dr^*}=\left(\frac{1}{r^*}\frac{d}{dr}\left(r^*\frac{d}{dr^*}\right)\right)\left(\frac{1}{r^*}\frac{d}{dr^*}\left(r^*\frac{d}{dr^*}\right)\right)\\ R_{r^*}&=-\omega_n^2\left(1-\mu^2\left(\frac{d^2}{dr^{*2}}+\frac{1}{r^*}\frac{d}{\partial r}\right)\right)=-\omega_n^2\left(1-\mu^2\left(\frac{1}{r^*}\frac{d}{dr^*}\left(r^*\frac{d}{dr^*}\right)\right)\right)\end{aligned} \tag{45}$$

In the first relation in Eq. (45), the inverse operator $L_{r^*}^{-1}$ is considered as a four-fold integral operator defined by:

$$L_{r^*}^{-1}(.)=\int_0^{r^*}\frac{1}{r}\int_0^{r}r\int_0^{r}\frac{1}{r}\int_0^{r}r(.)drdrdrdr \tag{46}$$

Applying Eq. (44) to Eq. (43) gives:

$$\varphi_n\left(r^*\right)-\varphi_n(0)-\frac{r^{*2}}{2!}\varphi_n''(0)=\omega_n^2\left(\int_0^{r^*}\frac{1}{r}\int_0^{r}r\int_0^{r}\frac{1}{r}\int_0^{r}r\varphi_n(r)drdrdrdr\right)-\mu^2\omega_n^2\left(\int_0^{r^*}r^{-1}\int_0^{r}r\left\{\varphi_n(r)-\varphi_n(0)\right\}drdr\right) \tag{47}$$

Based on the geometrical aspect of the circular nanodisk and the axisymmetric loading condition, Eq. (47) can be simplified to:

$$\varphi_n\left(r^*\right)=\varphi_n(0)+\frac{r^{*2}}{2!}\varphi_n''(0)+\omega_n^2\left(\int_0^{r^*}r^{-1}\int_0^{r}r\int_0^{r}r^{-1}\int_0^{r}r\varphi_n(r)drdrdrdr\right)-\mu^2\omega_n^2\left(\int_0^{r^*}r^{-1}\int_0^{r}r\left\{\varphi_n(r)-\varphi_n(0)\right\}drdr\right) \tag{48}$$

Based on ADM, the unknown functions $\varphi_n(r^*)$ are expressed by the following series:

$$\varphi_n\left(r^*\right)=\sum_{i=0}^{\infty}\psi_{n,i}\left(r^*\right) \tag{49}$$

Moreover, the recursive relation is:

$$\begin{aligned}
&\psi_{n,0}\left(r^*\right)=\left(\varphi_n\left(0\right)+\frac{r^{*2}}{2!}\varphi_n''(0)\right)+\mu^2\omega_n^2\left(\int_0^{r^*} r^{-1}\int_0^{r} r\left\{\varphi_n\left(0\right)\right\}drdr\right)\\
&\psi_{n,i+1}\left(r^*\right)=\omega_n^2\left(\int_0^{r^*} r^{-1}\int_0^{r} r\int_0^{r} r^{-1}\int_0^{r} r\varphi_{n,i}\left(r\right)drdrdrdr\right)-\mu^2\omega_n^2\left(\int_0^{r^*} r^{-1}\int_0^{r} r\varphi_{n,i}\left(r\right)drdr\right)\qquad\left(i\geq 1\right)
\end{aligned}\tag{50}$$

Approximating the above solution by the K-term truncated series, Eq. (49) can be recast in the following form:

$$\varphi_{n,K}\left(r^*\right)=\sum_{i=0}^{K}\psi_{n,i}\left(r^*\right)\tag{51}$$

where Eq. (51) implies that $\sum_{i=K}^{\infty}\varphi_{n,i}\left(r^*\right)$ is negligibly small. In practice, the number of series summation limit "K" is determined by the convergence requirement. Based on the boundary conditions for the nano-circular plate (Eq. (18)), one can obtain the following matrix expression to seek out the solution.

$$\begin{bmatrix}A_{11,K}\left(\omega_{n,K}\right) & A_{12,K}\left(\omega_{n,K}\right)\\ A_{21,K}\left(\omega_{n,K}\right) & A_{22,K}\left(\omega_{n,K}\right)\end{bmatrix}\begin{Bmatrix}\varphi_{n,K}\left(0\right)\\ \varphi_{n,K}''\left(0\right)\end{Bmatrix}=\begin{Bmatrix}0\\0\end{Bmatrix}\tag{52}$$

Now, for a non-trivial solution, equating the determinant of the coefficient matrix in Eq. (52) equal to zero will determine the natural frequencies of the vibrating system, as expressed in Eq. (53).

$$\det\begin{bmatrix}A_{11,K}\left(\omega_{n,K}\right) & A_{12,K}\left(\omega_{n,K}\right)\\ A_{21,K}\left(\omega_{n,K}\right) & A_{22,K}\left(\omega_{n,K}\right)\end{bmatrix}=0\tag{53}$$

where $\omega_{n,K}$ is the n^{th} estimated dimensionless natural frequency of the nano-circular plate corresponding to the approximate term K which is obtained by solving the frequency Eq. (53). Here, K is determined by the following equation:

$$\left|\omega_{n,K}-\omega_{n,K-1}\right|=\varepsilon\tag{54}$$

In the above equation, $\omega_{n,K-1}$ is the n^{th} estimated dimensionless natural frequency corresponding to the approximate term K-1, and ε is a preset small value. If Eq. (54) is satisfied, then $\omega_{n,K}$ is the n^{th} dimensionless natural frequency of the free vibration problem; that is $\omega_n=\omega_{n,K}$.

5. Verification

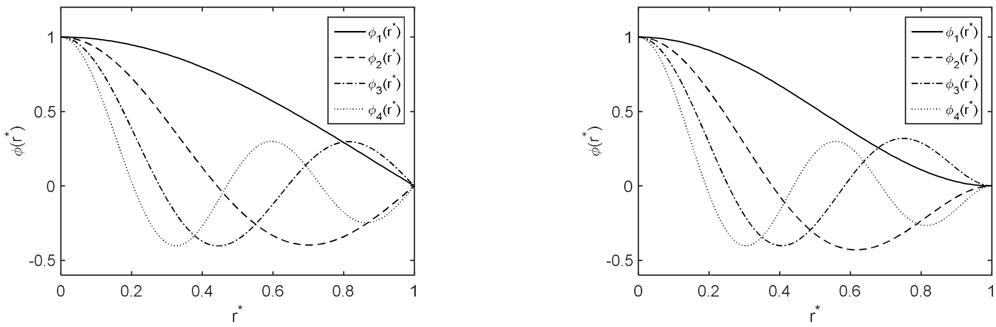
To verify the results of the presented model, the solution based on the ADM for the nano-circular plate with simply supported and clamped edge conditions were compared with those of the analytical method. For this purpose, It was observed that Eqs. (20) and (21) are reduced into Eqs. (38-39). Having determined the two expressions given in Eq. (36), they are substituted back into Eq. (36) to obtain the series solution. It is worth to indicate that the functions $f(r^*)$ and $g(r^*)$ satisfy the boundary conditions of the problem while the general solution of the equation is the sum of $w_i\,(r^*, t^*)$ terms. Moreover, if all $w_i\,(r^*, t^*)$ functions satisfy the boundary conditions, then their sum must also satisfy the same boundary conditions. Usually $f(r^*)$ and $g(r^*)$ are expanded in terms of the known orthogonal functions $\Phi_1(r^*)$, $\Phi_2(r^*)$, … (as the generalization of Fourier series expansion) and are selected such that to satisfy the boundary conditions. Since the linear problem of free axisymmetric flexural vibration of a circular plate with simply supported and clamped edge boundary condition has an exact analytical solution, then the basic functions $\Phi_i(r^*)$ were selected as the linear free oscillation mode shapes of a circular plate with similar boundary conditions [56]. Consequently,

$$\Phi_i\left(r^*\right)=I_0\left(\lambda_i r^*\right)-\left(\frac{I_0\left(\lambda_i\right)}{J_0\left(\lambda_i\right)}J_0\left(\lambda_i r^*\right)\right)\tag{55}$$

where J_0 and I_0 are the Bessel and modified Bessel functions of the first kind with zero order, respectively. The first four positive real values of λ_i for simply supported and clamped edge boundary conditions are given in Table 2. Additionally, the associated normalized functions $\Phi_i(r^*)$ for these two boundary conditions are shown in Fig. 1.

Table 2. The positive real value of λ_i for simply supported and clamped edge boundary conditions.

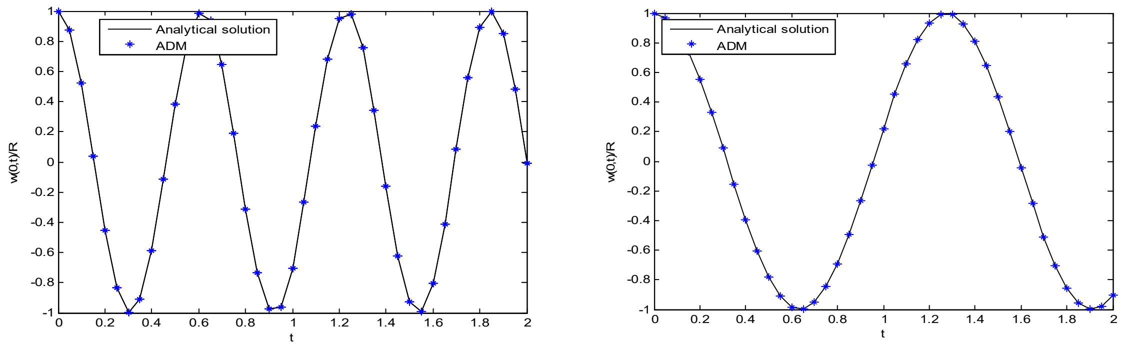
Boundary condition	λ_1	λ_2	λ_3	λ_4
Clamped	3.1962	6.3064	9.4395	12.5771
Simply supported	2.2473	5.4599	8.6164	11.7645



(a) (b)

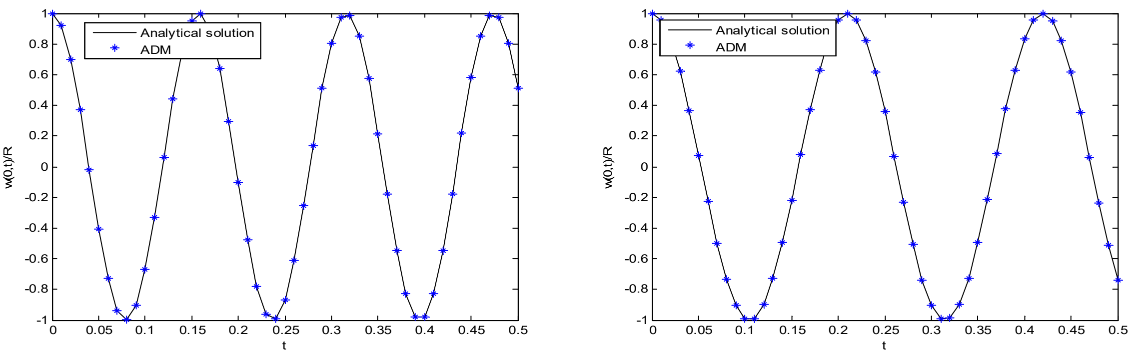
Fig. 1. Variations in functions $\Phi_i(r^*)$ vs. r^* for (a) clamped edge and (b) simply supported boundary conditions.

Figures 2 and 3 show a comparison between dimensionless central vibrational displacements of a nano-circular plate with simply supported and clamped boundary conditions. These results are obtained based on Adomian decomposition and analytical methods. For this purpose, it was assumed that $w(r^*,0)/R=\Phi_1(r^*)$. As clearly shown, excellent agreements are observed between the results of both methods, for the two types of used boundary conditions.



(a) (b)

Fig. 2. Comparison between dimensionless central vibrational displacements of a nano-circular plate obtained by Adomian decomposition and analytical methods for the initial condition of $w(r^*,0)/R=\Phi_1(r^*)$, (a) clamped nano-circular plate (b) simply supported nano-circular plate.



(a) (b)

Fig. 3. Comparison between dimensionless central vibrational displacements of a nano-circular plate obtained by Adomian decomposition and analytical methods for the initial condition of $w(r^*,0)/R=\Phi_1(r^*)$, (a) clamped nano-circular plate (b) simply supported nano-circular plate

The first axisymmetric dimensionless natural frequency parameters with two different boundary conditions are presented in Tables 3 and 4, for various values of the nonlocal parameter e_oa. It can be observed that the axisymmetric dimensionless natural frequencies determined by the proposed method converge very rapidly. The first dimensionless natural frequencies can be obtained for the approximate terms of K = 34 and K=28 for the simply supported and Clamped edge boundary conditions, respectively. As observed, the converged solutions based on the approximate terms and those obtained in Ref. [16] are very consistent for both types of boundary conditions. In addition, from Tables 3 and 4, it can be realized that for a given value of the nonlocal parameter, the dimensionless frequency parameters are monotonically increasing with decreasing the nonlocal parameter. This comparative study, clearly states that the present approach can yield reliable answers to the vibrational study of a nano-plate based on Eringen's model.

Table 3. Comparison of first axisymmetric dimensionless natural frequencies for Simply supported boundary conditions and nonlocal parameter for small value ε = 0.0001.

Boundary condition	$e_0a(nm)$	Mohammadi *et al.* [21]	PEM				
			K=5	K=10	K=15	K=33	K=34
Simply supported	0	4.9345	5.2213	5.1964	4.9843	4.9348	4.9348
	0.5	4.8997	5.1847	5.1599	4.9493	4.9001	4.9000
	1	4.7979	5.0778	5.0530	4.8470	4.7983	4.7982
	1.5	4.6409	4.9130	4.8881	4.6892	4.6412	4.6412
	2	4.4455	4.7078	4.6830	4.4929	4.4457	4.4458

Table 4. Comparison of first axisymmetric dimensionless natural frequencies for two different boundary conditions and nonlocal parameter for small value ε = 0.0001.

Boundary condition	$e_0a(nm)$	Mohammadi *et al.* [21]	PEM				
			K=5	K=10	K=15	K=27	K=28
Clamped	0	10.2158	10.7665	10.7416	10.2918	10.2163	10.2163
	0.5	10.1283	10.6746	10.6497	10.2039	10.1289	10.1288
	1	9.8784	10.4122	10.3873	9.9528	9.8789	9.8788
	1.5	9.4999	10.0148	9.9899	9.5724	9.5003	9.5003
	2	9.0348	9.5265	9.5015	9.1049	9.0353	9.0352

6. Numerical Results

Now, we obtain the numerical results on the first five axisymmetric dimensionless frequencies for a nano-circular plate with two boundary conditions, namely, simply supported and clamped edge constraints. Five different parameter conditions for e_0a or μ, are selected, values of which are given in Table 5 [48-50]. The dependency of dimensionless linear frequencies on nonlocal parameter μ for the first five axisymmetric modes, ω_i (i =1, 2, 3, 4, 5), are summarized in Tables 5 and 6. As can be observed, the convergence in result occurs very rapidly. Additionally, results indicate that the effect of nonlocal parameter μ on the plate is to make it behave as a stiffer plate. According to these results, the non-classical plate theory predicts frequencies closer to those of classical plate theory, provided this parameter becomes smaller. Also, examining the results in Table 4, one can conclude that the scale effect is less prominent in lower vibration modes compared with higher modes. It is worth to mention that for all values of nonlocal parameter μ, the vibration frequencies predicted by the classical theory are greater than those predicted by Eringen's theory. This means that a plate modeled based Eringen's theory behaves stiffer than that modeled by classical plate theory.

Figures. 4 and 5 show the dimensionless displacements for the center of nano-circular plate based on the clamped and simply supported conditions, for the two initial conditions of $w(r^*,0)/R=\Phi_1(r^*)$ and $w(r^*,0)/R=\Phi_2(r^*)$. Here, it is assumed that μ=0.05 and 0.2. The results from the classical plate theory are also superimposed for the purpose of comparison (μ=0). The difference between the transverse displacements predicted by the classical and non-classical models can be clearly observed for the two given boundary conditions. This difference is less prominent at lower values of nonlocal parameter μ. This indicates that the nonlocal parameter plays an important role in the determination of dimensionless displacements at the center of the nano-disk, and hence, to attain a correct solution, application of the nonlocal elasticity theory is mandatory.

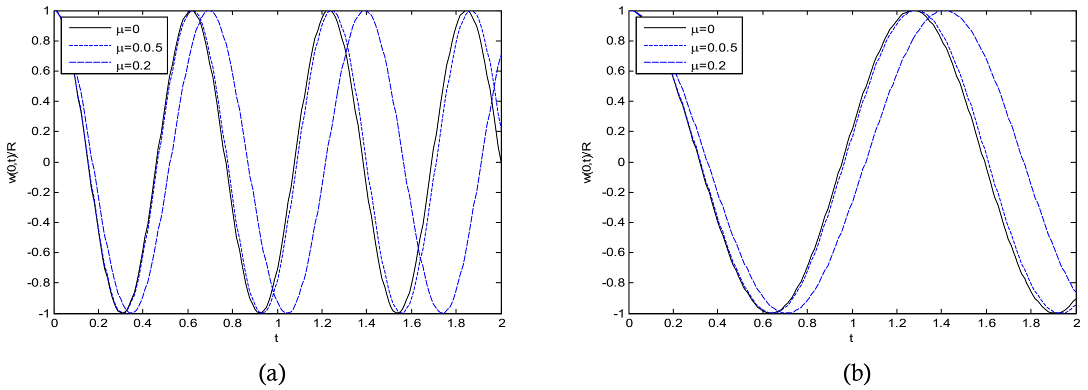


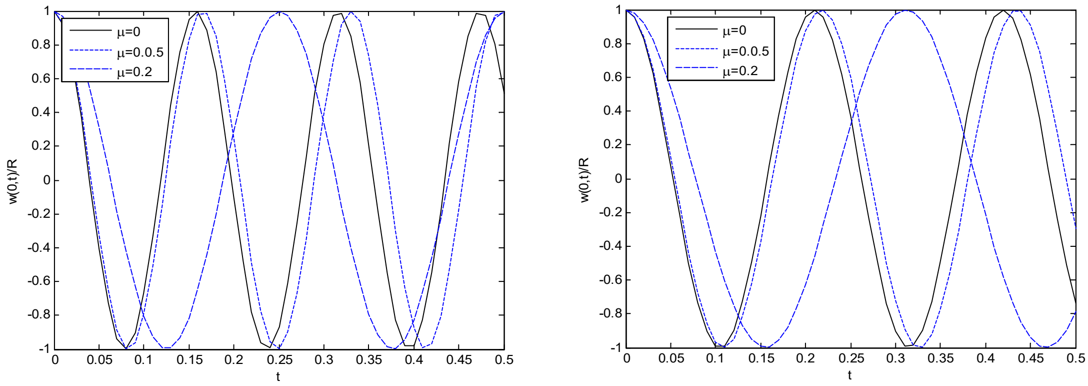
Fig. 4. The effect of nonlocal parameter μ on the dimensionless displacements of the center of nano-circular of the nano-circular plate with initial condition of $w(r^*,0)=\Phi_1(r^*)$, (a) clamped and (b) simply supported end.

Table 5. The Effect of nonlocal parameter μ on the first five axisymmetric dimensionless natural frequencies of the nano-circular plate based on the clamped boundary conditions.

Boundary condition	μ	PEM, $K = 29$	PEM, $K = 31$	PEM, $K = 26$	PEM, $K = 24$	PEM, $K = 22$
		ω_1	ω_2	ω_3	ω_4	ω_5
Clamped	0	10.2138	39.7711	89.1039	158.1836	247.0053
	0.05	10.1012	38.0047	80.7701	134.2385	194.6988
	0.1	9.8794	33.8497	65.1939	98.9751	133.2165
	0.15	9.5566	29.1927	51.8517	74.5920	96.9929
	0.2	9.0338	25.0443	42.1350	58.8814	75.3299

Table 6. The effect of nonlocal parameter μ on the first five axisymmetric dimensionless natural frequencies of the nano-circular plate based on the Simply supported boundary conditions.

Boundary condition	μ	PEM, $K = 29$	PEM, $K = 28$	PEM, $K = 32$	PEM, $K = 31$	PEM, $K = 29$
		ω_1	ω_2	ω_3	ω_4	ω_5
Simply supported	0	4.9344	29.9455	74.3712	138.5286	222.4226
	0.05	4.8821	28.8795	68.2841	119.3771	178.2914
	0.1	4.7799	26.2564	56.3010	89.6747	123.8503
	0.15	4.6133	23.1263	45.4626	68.2535	90.7521
	0.2	4.4450	20.1755	37.2803	54.1461	70.6869



(a) (b)

Fig. 5. The effect of nonlocal parameter μ on the dimensionless center displacements of the nano-circular plate with the initial condition of $w(r^*,0)=\Phi_2(r^*)$, (a) clamped and (b) simply supported end.

It is shown that the nonlocal parameter has a major effect on the natural frequencies and displacements of the nano-circular plate and its effect must not be excluded in the study of nano-circular plate vibrational behavior. In other words, in nano-size structures, one aspect of the material properties is the material's dependency on the other factors such as molecule bonds and so on. In such events, the application of classical formulation which lacks these effects can yield erroneous results. Obviously, in these cases, to solve the problem, one of the remedies is the application of one of the non-classical continuum theories, such as nonlocal elasticity theory, when applicable. It is worth to mention that based on the nonlocal constitutive equations, stress at any point is a function of the strain at all other points of in the object.

7. Conclusion

A new solution method based on the application of ADM for the linearly free vibration analysis of nano-circular plate was formulated based on Hamilton's principle while using the Eringen's model. The model was size-dependent with an additional material length scale parameter to capture the size effect. The governing equilibrium equation was reduced to that of linearly free vibration of a classical circular plate provided μ=0. The ADM was introduced to solve the governing equations based on the simply supported and clamped edge boundary conditions. The method was proved to be useful and easily applicable to the solution of linear vibrational behavior of nano-circular plate, to seek out the frequency characteristics and dimensionless displacements of the center of the nano-circular when size effect is included. Using this method, the effect of nonlocal parameter μ on the linear natural frequencies and the transverse deflection of the nano-plate were obtained. Results indicate that the present analysis is concise, accurate, and can provide a unified and systematic procedure which is simple and more straightforward than the other analyses. On the other hand, results indicate that for all values of the nonlocal parameter μ, the linear natural frequencies predicted by the classical theory are greater than those predicted by Eringen's theory. This means that a plate modeled based on Eringen's theory behaves stiffer than that modeled by the classical plate theory. Moreover, the effect of nonlocal parameter μ on the transverse deflection of the nano-plate was more prominent at higher vibrational modes. Similar behavior was observed for the differences in frequencies predicted by the classical and non-classical models, for both types of clamped and simply supported boundary conditions. Results of this study indicate that ADM is a powerful technique for studying vibrational behavior of nano-circular as well as micro-circular plates, and many other similar problems arising in mathematical physics.

Conflict of Interest

The authors declared no potential conflicts of interest with respect to the research, authorship, and publication of this article.

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References

[1] Ball P. Roll up for the revolution. *Nature*. 414(6860), 2001, 142-4.
[2] Baughman RH, Zakhidov AA, De Heer WA. Carbon nanotubes-the route toward applications. *Science*. 297(5582), 2002, 787-92.
[3] Bodily B, Sun C. Structural and equivalent continuum properties of single-walled carbon nanotubes. *International Journal of Materials and Product Technology*. 18(4-6), 2003, 381-97.
[4] Li C, Chou T-W. A structural mechanics approach for the analysis of carbon nanotubes. *International Journal of Solids and Structures*. 40(10), 2003, 2487-99.
[5] Li C, Chou T-W. Single-walled carbon nanotubes as ultrahigh frequency nanomechanical resonators. *Physical review B*. 68(7), 2003, 073405.
[6] Pradhan S, Phadikar J. Nonlinear analysis of carbon nanotubes, Proceedings of Fifth International Conference on Smart Materials. Structures and Systems, Indian Institute of Science, Bangalore. 2008, 24-6.
[7] Wang C, Tan V, Zhang Y. Timoshenko beam model for vibration analysis of multi-walled carbon nanotubes. *Journal of Sound and Vibration*. 294(4-5), 2006, 1060-72.
[8] Chong A, Yang F, Lam DC, Tong P. Torsion and bending of micron-scaled structures. *Journal of Materials Research*. 16(4), 2001, 1052-8.
[9] Fleck N, Muller G, Ashby M, Hutchinson J. Strain gradient plasticity: theory and experiment. *Acta Metallurgica et Materialia*. 42(2), 1994, 475-87.
[10] Eringen AC. On differential equations of nonlocal elasticity and solutions of screw dislocation and surface waves. *Journal of Applied Physics*. 54(9), 1983, 4703-10.
[11] Eringen AC. Linear theory of nonlocal elasticity and dispersion of plane waves. *International Journal of Engineering Science*. 10(5), 1972, 425-35.
[12] Moory-Shirbani M, Sedighi HM, Ouakad HM, Najar F. Experimental and mathematical analysis of a piezoelectrically actuated multilayered imperfect microbeam subjected to applied electric potential. *Composite Structures*. 184, 2018, 950-60.
[13] Chen M. Large Deflection of a Cantilever Nanobeam under a Vertical End Load. *Applied Mechanics and Materials*. 353-356, 2013, 3387-90.
[14] Murmu T, Adhikari S. Scale-dependent vibration analysis of prestressed carbon nanotubes undergoing rotation. *Journal of Applied Physics*. 108(12), 2010, 123507.
[15] Shishesaz M, Hosseini M, Mechanical behavior of functionally graded nano-cylinders under radial pressure based on strain gradient theory. *Journal of Mechanics*, 2018, doi: https://doi.org/10.1017/jmech.2018.10.
[16] Wang K, Kitamura T, Wang B. Nonlinear pull-in instability and free vibration of micro/nanoscale plates with surface energy–a modified couple stress theory model. *International Journal of Mechanical Sciences*. 99, 2015, 288-96.
[17] Tahani M, Askari AR, Mohandes Y, Hassani B. Size-dependent free vibration analysis of electrostatically pre-deformed rectangular micro-plates based on the modified couple stress theory. *International Journal of Mechanical Sciences*. 94, 2015, 185-98.
[18] Hosseini M, Shishesaz M, Thermoelastic analysis of rotating functionally graded micro/nanodisks of variable thickness. *Thin-Walled Structures*, 134, 2019, 508–523.
[19] Mohammadi M, Hosseini M, Shishesaz M, Hadi A, Rastgoo A, Primary and secondary resonance analysis of porous functionally graded nanobeam resting on a nonlinear foundation subjected to mechanical and electrical loads. *European Journal of Mechanics/A Solids*, 77, 2019, 103793.
[20] Ma H, Gao X-L, Reddy J. A non-classical Mindlin plate model based on a modified couple stress theory. *Acta Mechanica*. 220(1-4), 2011, 217-35.
[21] Mohammadi M, Ghayour M, Farajpour A. Free transverse vibration analysis of circular and annular graphene sheets with various boundary conditions using the nonlocal continuum plate model. *Composites Part B: Engineering*. 45(1), 2013, 32-42.
[22] Wang Y-G, Lin W-H, Liu N. Large amplitude free vibration of size-dependent circular microplates based on the modified couple stress theory. *International Journal of Mechanical Sciences*. 71, 2013, 51-7.
[23] Kiani K. Nonlocal continuum-based modeling of a nanoplate subjected to a moving nanoparticle. Part I: Theoretical formulations. *Physica E: Low-dimensional Systems and Nanostructures*. 44(1), 2011, 229-48.
[24] Kiani K. Nonlocal continuum-based modeling of a nanoplate subjected to a moving nanoparticle. Part II: Parametric studies. *Physica E: Low-dimensional Systems and Nanostructures*. 44(1), 2011, 249-69.
[25] Kiani K. Small-scale effect on the vibration of thin nanoplates subjected to a moving nanoparticle via nonlocal continuum theory. *Journal of Sound and Vibration*. 330(20), 2011, 4896-914.
[26] Salehipour H, Nahvi H, Shahidi A. Exact analytical solution for free vibration of functionally graded micro/nanoplates via three-dimensional nonlocal elasticity. *Physica E: Low-dimensional Systems and Nanostructures*. 66, 2015, 350-8.

[27] Ansari R, Shahabodini A, Shojaei MF. Nonlocal three-dimensional theory of elasticity with application to free vibration of functionally graded nanoplates on elastic foundations. *Physica E: Low-dimensional Systems and Nanostructures*. 76, 2016, 70-81.
[28] Shishesaz M, Shariati M, Yaghootian A, Alizadeh A. Nonlinear Vibration Analysis of Nano-Disks Based on Nonlocal Elasticity Theory Using Homotopy Perturbation Method. *International Journal of Applied Mechanics*. 2019, doi: 10.1142/S175882511950011X.
[29] Li C. A nonlocal analytical approach for torsion of cylindrical nanostructures and the existence of higher-order stress and geometric boundaries. *Composite Structures*. 118, 2014, 607-21.
[30] Li C, Yao L, Chen W, Li S. Comments on nonlocal effects in nano-cantilever beams. *International Journal of Engineering Science*. 87, 2015, 47-57.
[31] Shen J, Li C. A semi-continuum-based bending analysis for extreme-thin micro/nano-beams and new proposal for nonlocal differential constitution. *Composite Structures*. 172, 2017, 210-20.
[32] Adomian G. Nonlinear Stochastic Operator Equations: Acad. press. Can Diego, CA. 1986.
[33] Adomian G. Solving Frontier Problems of Physics: The Decomposition MethodKluwer. Boston, MA. 1994.
[34] Abboui K, Cherruault Y. New ideas for proving convergence of decomposition methods. *Computational and Applied Mathematics*. 29(7), 1995, 103-5.
[35] Adomian G, Rach R. Noise terms in decomposition solution series. *Computers & Mathematics with Applications*. 24(11), 1992, 61-4.
[36] Babolian E, Biazar J. On the order of convergence of Adomian method. *Applied Mathematics and Computation*. 130(2-3), 2002, 383-7.
[37] Cherruault Y. Convergence of Adomian's method. *Kybernetes*. 18(2), 1989, 31-8.
[38] Cherruault Y, Adomian G, Abbaoui K, Rach R. Further remarks on convergence of decomposition method. *International Journal of Bio-Medical Computing*. 38(1), 1995, 89-93.
[39] Hosseini MM, Nasabzadeh H. On the convergence of Adomian decomposition method. *Applied Mathematics and Computation*. 182(1), 2006, 536-43.
[40] Hsu J-C, Lai H-Y. An innovative eigenvalue problem solver for free vibration of uniform Timoshenko beams by using the Adomian modified decomposition method. *Journal of Sound and Vibration*. 325(1), 2009, 451-70.
[41] Hsu J-C, Lai H-Y, Chen Co-K. Free vibration of non-uniform Euler–Bernoulli beams with general elastically end constraints using Adomian modified decomposition method. *Journal of Sound and Vibration*. 318(4), 2008, 965-81.
[42] Mao Q, Pietrzko S. Design of shaped piezoelectric modal sensor for beam with arbitrary boundary conditions by using Adomian decomposition method. *Journal of Sound and Vibration*. 329(11), 2010, 2068-82.
[43] Mao Q, Pietrzko S. Free vibration analysis of stepped beams by using Adomian decomposition method. *Applied Mathematics and Computation*. 217(7), 2010, 3429-41.
[44] Mao Q, Pietrzko S. Free vibration analysis of a type of tapered beams by using Adomian decomposition method. *Applied Mathematics and Computation*. 219(6), 2012, 3264-71.
[45] Jamali M, Shojaee T, Kolahchi R, Mohammadi B. Buckling analysis of nanocomposite cut out plate using domain decomposition method and orthogonal polynomials. *Steel and Composite Structures*. 22(3), 2016, 691-712.
[46] Karlicic D, Murmu T, Adhikari S, McCarthy M. Non-local structural mechanics: John Wiley & Sons; 2015.
[47] Wang Q, Han Q, Wen B. Estimate of material property of carbon nanotubes via nonlocal elasticity. *Advances in Theoretical and Applied Mechanics*. 1(1), 2008, 1-10.
[48] Wang Q, Wang C. The constitutive relation and small scale parameter of nonlocal continuum mechanics for modelling carbon nanotubes. *Nanotechnology*. 18(7), 2007, 075702.
[49] Duan W, Wang C, Zhang Y. Calibration of nonlocal scaling effect parameter for free vibration of carbon nanotubes by molecular dynamics. *Journal of Applied Physics*. 101(2), 2007, 024305.
[50] Li C, Lai S, Yang X. On the nano-structural dependence of nonlocal dynamics and its relationship to the upper limit of nonlocal scale parameter. *Applied Mathematical Modelling*. 69, 2019, 127-41.
[51] Shodja H, Ahmadpoor F, Tehranchi A. Calculation of the additional constants for fcc materials in second strain gradient elasticity: behavior of a nano-size Bernoulli-Euler beam with surface effects. *Journal of Applied Mechanics*. 79(2), 2012, 021008.
[52] Timoshenko SP, Woinowsky-Krieger S. Theory of plates and shells: McGraw-hill; 1959.
[53] Reddy JN. Mechanics of laminated composite plates and shells: theory and analysis: CRC press; 2004.
[54] Wazwaz A-M. A reliable modification of Adomian decomposition method. *Applied Mathematics and Computation*. 102(1), 1999, 77-86.
[55] Wazwaz A-M. Partial differential equations and solitary waves theory: Springer Science & Business Media; 2010.
[56] Leissa AW. Vibration of Plates, Office of Technology Utilization. National Aeronautics and Space Administration, Washington, DC. 1969.

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