

M&MoCS



Shahid Chamran
University of Ahvaz

Journal of Applied and Computational Mechanics



Research Paper

Asymptotic Approximations of the Solution for a Traveling String under Boundary Damping

Rajab A. Malookani¹, Sanaullah Dehraj², Sajad H. Sandilo³

¹ Department of Mathematics and Statistics, Faculty of Science, Quaid-e-Awam University of Engineering, Science and Technology
Nawabshah, 67480, Sindh, Pakistan, Email: rajabali@quest.edu.pk

² Department of Mathematics and Statistics, Faculty of Science, Quaid-e-Awam University of Engineering, Science and Technology
Nawabshah, 67480, Sindh, Pakistan, Email: sanaullahdehraj@quest.edu.pk

³ Department of Mathematics and Statistics, Faculty of Science, Quaid-e-Awam University of Engineering, Science and Technology
Nawabshah, 67480, Sindh, Pakistan, Email: s.h.sandilo@quest.edu.pk

Received January 22 2019; Revised May 03 2019; Accepted for publication May 05 2019.

Corresponding author: R.A. Malookani, rajabali@quest.edu.pk

© 2019 Published by Shahid Chamran University of Ahvaz

& International Research Center for Mathematics & Mechanics of Complex Systems (M&MoCS)

Abstract. Transversal vibrations of an axially moving string under boundary damping are investigated. Mathematically, it represents a homogenous linear partial differential equation subject to nonhomogeneous boundary conditions. The string is moving with a relatively (low) constant speed, which is considered to be positive. The string is kept fixed at the first end, while the other end is tied with the spring-dashpot system. The asymptotic approximations for the solution of the equations are obtained by application of two time-scale perturbation technique and the characteristic coordinates method. The vertical displacement of the moving system under boundary damping is computed by using specific initial conditions. It is shown that how the introduced damping at the boundary may affect the vertical displacement of the axially moving system.

Keywords: Axially moving string, Boundary damping, Two time-scale perturbation, Characteristic coordinates.

1. Introduction

Mechanical structures, for instance, chair lifts, conveyor belt systems, pipes transporting liquids and gases, elevator cables, serpentine belts, power transformation lines play a very important role in our lives. These structures are subjected to vibrations due to various causes such as wind, earthquakes and other external excitations. Severe vibrations in the systems may cause discomfort to humans as well as damage to a structure or complete structural failure. Tacoma Narrows suspension bridge is a classic example of structural failure due to heavy wind. Since the vibrations cause collateral damage in the structures, it is important to reduce the vibrations in the mentioned systems. In order to mitigate the structural motion, the dampers are most often used. Both the material and boundary damping are under study for last few decades.

Mechanical structures are mainly categorized into two classes depending on the bending stiffness, that is, axially moving string (without bending stiffness) and Euler-Bernoulli beam (with bending stiffness). The dynamics of these structures can be expressed by mathematical equations such as non-linear second order partial differential equations (string-like equations) or by non-linear fourth order partial differential equations (beam-like equations). Examples of string-like problems were studied in [1-21] and beam-like were studied in [22, 23]. The viscoelastic damping and analysis of structure under internal and external damping were reviewed by Jones [24]. The mathematical equations for axially moving systems are obtained by using the Newton second law of motion [1, 2] and by Hamilton's principle [9, 20, 21]. The mathematical equations governed the vibrations in moving system under viscous damping were derived by Gaiko [9] through the application of extended Hamilton principle.



The solutions of string-like and beam-like equations yield the vibrational behavior of a structure. These mathematical models are often very difficult to solve analytically. The various approximate analytical and numerical methods are used to construct the approximate solutions. Gaiko [5] used the two timescale perturbation technique to find the approximate solutions for (lateral) oscillations in axially moving string under boundary damping. Dormajoyo [8] employed the separation of variables method to find the solution of the damped string at the boundaries. The multiple scales perturbation together with the method of Laplace transform was employed by Gaiko and van Horssen [9] to examine the lateral vibrations in traveling string under boundary damping. The finite difference numerical technique and Galerkin discretization method was utilized by Fung *et al.* [10] to examine the transient response of a viscoelastic traveling string subject to viscous damping. Gahyesh [16] used Routh-Hurwitz criterion to compute the time invariant response of viscoelastic string. The stability of a nonlinear system under the effect of Kelvin-Voigt damping was investigated by Shahruz [17] through Lyapunov stability technique. Chen *et al.* [18] used the finite element method to examine the effect of internal damping at the boundary under few axial string velocities in moving system. The free vertical vibrations in traveling string with the viscoelastic damping via method of multiple scales were examined in [19]. The vertical vibrations of a moving system under the effects of damping were studied in [23] via two time scales perturbation method.

Gaiko [9] investigated the transversal vibrations in traveling string under boundary damping with the help of multiple scales perturbation method. The authors obtained the approximate-analytic solutions of the problem. However, in his research, the amplitude-response of few modes was computed. This study aims to compute the close form solutions of the governing equations by employing the two time scales perturbation method with the combination of characteristic coordinates method.

2. Mathematical Model

We consider a traveling string with density ρ , the tension T_0 and the damping parameter η which travels with the axial transport speed V . The distance between two supported ends of the moving string is L , where the string is taken to be fixed at the first end, while at the second end, the spring-dashpot system is tied (see Fig. 1). To develop the governing equations of motion for the lateral vibrations string system under boundary damping, we consider the below assumptions:

(i) The transversal vibration $w(x, t)$ is taken into account, where x is the axial coordinate, and t denotes the time coordinate. (ii) It is assumed that the parameters ρ, V, T_0, η are non-zero positive constants. (iii) The bending stiffness and internal damping in the string are neglected. (iv) We also neglect the effect of gravity and other external excitations. Under the above-mentioned assumptions, the application of extended Hamilton's principle [9, 13] yields the following mathematical equations expressing the lateral vibrations of the moving system under boundary damping:

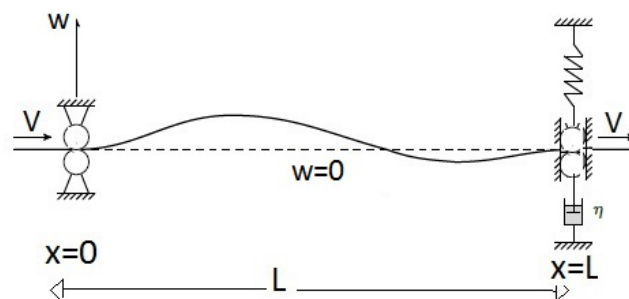


Fig. 1. The physical model of a moving system under boundary damping

$$\rho(w_{tt} + 2Vw_{xt} + V^2w_{xx}) - T_0w_{xx} = 0, \quad 0 < x < L, \quad t \geq 0 \quad (1)$$

The boundary conditions are as follows:

$$\begin{cases} w(0, t) = 0, & t \geq 0, \\ \eta w_t(L, t) + T_0w_x(L, t) - \rho Vw_t(L, t) - \rho V^2w_x(L, t) = 0, & t \geq 0. \end{cases} \quad (2)$$

with the initial conditions as:

$$w(x, 0) = \phi(x), \quad w_t(x, 0) = \psi(x), \quad 0 < x < L, \quad (3)$$

To obtain the dimensionless form of Eqs. (1) to (3), we use the following non-dimensional quantities:

$$\hat{x} = \frac{x}{L}, \quad \hat{w} = \frac{w}{L}, \quad \hat{t} = \frac{tc}{L}, \quad \hat{V} = \frac{V}{c}, \quad \hat{\eta} = \frac{\eta}{T_0c}, \quad \hat{\phi} = \frac{\phi}{L}, \quad \hat{\psi} = \frac{\psi}{c}, \quad (4)$$

where the variable c is the wave speed, $c = \sqrt{T_0 / \rho}$. Thereby, the dimensionless form of Eqs. (1) to (3) can be written as,

$$w_{tt} + 2Vw_{xt} + (V^2 - 1)w_{xx} = 0, \quad 0 < x < 1, \quad t \geq 0 \quad (5)$$

$$\begin{cases} w(0, t) = 0, & t \geq 0, \\ w_x(1, t) = (-\eta w_t(1, t) + Vw_t(1, t) + V^2 w_x(1, t)), & t \geq 0. \end{cases} \quad (6)$$

$$w(x, 0) = \phi(x), \quad w_t(x, 0) = \psi(x), \quad 0 \leq x \leq 1, \quad (7)$$

From Eqs. (5) to (7) and henceforth, we omit the hats expressing the dimensionless quantities. In the section below, the asymptotic approximations for the solution of the Eqs. (5) to (7) will be constructed via the application of two time scales perturbation technique and the characteristic coordinates method.

3. Method of Solution

This section presents two methods to compute the amplitude-response relation for the homogeneous linear partial differential equation under non-homogeneous boundary conditions (5)-(7). The two time scales perturbation technique will be employed together with the method of characteristic coordinates. The reader is referred to Refs. [4, 7, 8, 25, 26] for further details. It is presumed that the axial speed V is low in comparison to the wave speed c , i.e., $V < |c|$. Further, the axial speed V and the boundary damping parameter η are assumed to be small, i.e., $V = O(\varepsilon)$, and $\eta = O(\varepsilon)$, where ε ($0 < \varepsilon \ll 1$) is a small dimensionless parameter. We re-write Eqs. (5) to (7) as follows:

$$w_{tt} - w_{xx} = -2\varepsilon Vw_{xt} + O(\varepsilon^2), \quad 0 < x < 1, \quad t \geq 0, \quad (8)$$

with the boundary conditions:

$$\begin{cases} w(0, t) = 0, & t \geq 0, \\ w_x(1, t) = \varepsilon(-\eta w_t(1, t) + Vw_t(1, t)) + \varepsilon^2 V^2 w_x(1, t), & t \geq 0. \end{cases} \quad (9)$$

and the below initial conditions:

$$w(x, 0) = \phi(x), \quad w_t(x, 0) = \psi(x), \quad 0 \leq x \leq 1, \quad (10)$$

Let's assume the solution of Eq. (8) in the form:

$$w(x, t) = v(x, t, \tau; \varepsilon), \quad (11)$$

where $t = t$ represents the fast timescale and $\tau = \varepsilon t$ denotes the slow timescale. The introduction of the time variables t and τ yield the transformations below:

$$w_t = v_t + \varepsilon v_\tau, \quad w_{tt} = v_{tt} + 2\varepsilon v_{t\tau} + O(\varepsilon^2), \quad w_{xt} = v_{xt} + \varepsilon v_{x\tau} + O(\varepsilon^2) \quad (12)$$

Putting (11) and (12) into equation (8), on can obtain:

$$v_{tt} - v_{xx} = \varepsilon(-2v_{t\tau} - 2Vv_{xt}) + O(\varepsilon^2), \quad 0 < x < 1, \quad t \geq 0, \quad (13)$$

with the BCs:

$$v(0, t, \tau; \varepsilon) = 0, \quad v_x(1, t, \tau; \varepsilon) = \varepsilon(-\eta v_t(1, t, \tau) + Vv_t(1, t, \tau)) + O(\varepsilon^2), \quad (14)$$

and the ICs:

$$v(x, 0, 0; \varepsilon) = \phi(x), \quad v_t(x, 0, 0; \varepsilon) + \varepsilon v_\tau(x, 0, 0; \varepsilon) = \psi(x), \quad (15)$$

Further let's assume that:

$$v(x, t, \tau; \varepsilon) = v_0(x, t, \tau) + \varepsilon v_1(x, t, \tau) + O(\varepsilon^2). \quad (16)$$

This expansion is assumed to be uniformly valid so that the term $\varepsilon v_1/v_0$ is of the order $O(1)$ over the whole range of time t . To find the valid solution on a time scale of the order $O(\varepsilon^{-1})$, it is necessary that v_1 does not contain terms which grow linearly with time.

By Putting Eq. (16) into Eq. (13) to (15), the coefficients of $O(1)$ and $O(\varepsilon)$ of the problem are obtained as follows:

$$O(1): \begin{cases} v_{0tt} - v_{0xx} = 0, & 0 < x < 1, \quad t \geq 0 \\ v_0(0, t, \tau) = 0, \quad v_{0x}(1, t, \tau) = 0, & t \geq 0 \\ v_0(x, 0, 0) = \phi(x), \quad v_{0t}(x, 0, 0) = \psi(x), & 0 < x < 1. \end{cases} \quad (17)$$

$$O(\varepsilon): \begin{cases} v_{1tt} - v_{1xx} = -2v_{0tr} - 2Vv_{0xt}, & 0 < x < 1, \quad t > 0 \\ v_1(0, t, \tau) = 0, \quad v_{1x}(1, t, \tau) = -(\eta - V)v_{0t}(1, t, \tau), & t \geq 0 \\ v_1(x, 0, 0) = 0, \quad v_{1t}(x, 0, 0) = -v_{0\tau}(x, 0, 0), & 0 < x < 1. \end{cases} \quad (18)$$

Eq. (17) will be solved by employing the method of characteristic coordinates: $\sigma = x - t$ and $\xi = x + t$, $0 < x < 1$, $t \geq 0$. This technique requires a replacement of the initial-boundary value problem to an initial value problem. In addition, the function $v_0(x, t)$ will be extended to all values of x in such a way that it would be 4-periodic in spatial coordinate x , odd at $x = 0$ and even about $x = 1$. The solution of $O(1)$ -problem is obtained by direct integration in the form:

$$v_0(x, t, \tau) = f_0(\sigma, \tau) + g_0(\xi, \tau) \quad (19)$$

with the initial condition, $v_0(x, 0, 0) = f_0(\sigma, 0) + g_0(\sigma, 0) \equiv \phi(\sigma)$ and $v_{0t}(x, 0, 0) = -f_0'(\sigma, 0) + g_0'(\sigma, 0) \equiv \psi(\sigma)$, where the prime symbol indicates the differentiation with respect to the first input variable (which in this case is σ). The odd and/or even and 4-periodic extension in x requires that the functions f_0 and g_0 will fulfill the following conditions: $f_0(\sigma, \tau) = f_0(\sigma + 4, \tau)$, and $f_0(\sigma, \tau) = -g_0(-\sigma, \tau)$, where $-\infty < \sigma < \infty$ and $\tau \geq 0$.

In order to solve Eq. (18) for w_1 , we introduce the following transformation to make the non-homogeneous boundary conditions at $x = 1$ homogeneous, $v_1(x, t, \tau) = y(x, t, \tau) - (\eta - V)F(x)v_{0t}(1, t, \tau)$, where the function $F(x) = x$, for $0 \leq x \leq 1$, which is 4-periodic, odd with respect to $x = 0$ and even with respect to $x = 1$. The Fourier series of $F(x)$ is:

$$F(x) \approx \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{\pi^2 \left(n - \frac{1}{2}\right)^2} \sin\left(\left(n - \frac{1}{2}\right)\pi x\right), \quad (20)$$

In addition, we shall multiply the term v_{0xt} by the function $H(x)$, where $H(x)$ is given as

$$H(x) = \begin{cases} -1, & -2 < x < 0, \\ 1, & 0 < x < 2 \end{cases} \quad (21a)$$

The Fourier series of $H(x)$ is expressed by:

$$H(x) \approx \sum_{n=1}^{\infty} \frac{2}{\pi \left(n - \frac{1}{2}\right)} \sin\left(\left(n - \frac{1}{2}\right)\pi x\right), \quad (21b)$$

with the above transformation, Eq. (18) becomes:

$$y_{tt} - y_{xx} = -2v_{0tr} - 2Vv_{0xt}H(x) + (\eta - V)F(x)v_{0tt}(1, t, \tau), \quad 0 < x < 1, \quad \tau > 0, \quad t > 0, \quad (22)$$

with BCs:

$$\begin{cases} y(0, t, \tau) = 0, \\ y_x(1, t, \tau) = 0, \end{cases} \quad (23)$$

and ICs:

$$\begin{cases} y(x, 0, 0) = (\eta - V)F(x)v_{0t}(1, t, \tau), \\ y_t(x, 0, 0) = -v_{0\tau}(x, 0, 0) + (\eta - V)F(x)v_{0tt}(x, 0, 0). \end{cases} \quad (24)$$

To search for the secular terms in the RHS of Eq. (22), Fourier series representation for f_0 , w_0 and $F(x)$ is helpful.

$$v_0(x, t, \tau) = \sum_{n=1}^{\infty} \left[A_n \sin\left(\left(n - \frac{1}{2}\right)\pi t\right) + B_n \cos\left(\left(n - \frac{1}{2}\right)\pi t\right) \right] \sin\left(\left(n - \frac{1}{2}\right)\pi x\right) \quad (25)$$

$$f_0(\sigma, \tau) = \sum_{n=1}^{\infty} \left[\frac{A_n(\tau)}{2} \cos \left(\left(n - \frac{1}{2} \right) \pi \sigma \right) + \frac{B_n(\tau)}{2} \sin \left(\left(n - \frac{1}{2} \right) \pi \sigma \right) \right] \quad (26)$$

$$F(x) v_{0tt}(1, t, \tau) = -2\pi v_{0t}(x, t, \tau) + G(x, t, \tau), \quad (27)$$

where,

$$G(x, t, \tau) = -2 \sum_{n=1}^{\infty} \sum_{\substack{k=1 \\ k \neq n}}^{\infty} \frac{\left(k - \frac{1}{2} \right)^3}{\left(-\frac{1}{2} \right)^2} (-1)^{k+n+2} \pi^2 \begin{pmatrix} A_k(\tau) \cos \left(\left(k - \frac{1}{2} \right) \pi t \right) \\ -B_k(\tau) \sin \left(\left(k - \frac{1}{2} \right) \pi t \right) \end{pmatrix} \sin \left(\left(n - \frac{1}{2} \right) \pi x \right) \quad (28)$$

Let us assume that:

$$y(x, t, \tau) = \tilde{y}(\sigma, \xi, \tau), \quad (29)$$

with this assumption, the following transformation is observed:

$$y_{tt} = \tilde{y}_{\sigma\sigma} + \tilde{y}_{\xi\xi} - 2\tilde{y}_{\sigma\xi}, \quad y_{xx} = \tilde{y}_{\sigma\sigma} + \tilde{y}_{\xi\xi} + 2\tilde{y}_{\sigma\xi}, \quad (30)$$

Substituting Eqs. (26) to (29) into Eqs. (22) to (24), we obtain

$$-4\tilde{y}_{\sigma\xi} = -2g_{0\xi\tau} + 2f_{0\sigma\tau} + 2V(g_{0\xi\xi} + f_{0\sigma\sigma})H\left(\frac{\sigma + \xi}{2}\right) + (\eta - V)\left[2\pi(-g_{0\xi} + f_{0\sigma})\right] + (\eta - V)G(\sigma, \xi, \tau), \quad (31)$$

where

$$G(\sigma, \xi, \tau) = -2 \sum_{n=1}^{\infty} \sum_{\substack{k=1 \\ k \neq n}}^{\infty} \frac{\left(k - \frac{1}{2} \right)^3}{\left(n - \frac{1}{2} \right)^2} (-1)^{k+n+2} R_{nk}(\sigma, \xi, \tau), \quad (32)$$

and

$$R_{nk}(\sigma, \xi, \tau) = A_k(\tau) \left[\sin \left(\frac{k-n}{2} \xi - \frac{k+n-1}{2} \sigma \right) + \sin \left(\frac{k-n}{2} \sigma - \frac{k+n-1}{2} \xi \right) \right] + B_k(\tau) \left[\cos \left(\frac{k-n}{2} \xi - \frac{k+n-1}{2} \sigma \right) - \cos \left(\frac{k-n}{2} \sigma - \frac{k+n-1}{2} \xi \right) \right] \quad (33)$$

Integrating Eq. (31) with respect to ξ yields:

$$4\tilde{y}_{\sigma}(\sigma, \xi, \tau) = 4\tilde{y}_{\sigma}(\sigma, \sigma, \tau) + (\xi - \sigma)(2f_{0\sigma\tau} + 2\pi(\eta - V)f_{0\sigma}) + \int_{\sigma}^{\xi} \left[-2g_{0\xi\tau} + 2V(f_{0\sigma\sigma} - g_{0\xi\xi})F\left(\frac{\sigma + \xi}{2}\right) - (\eta - V)g_{0\xi} + (\eta - V)G(\sigma, \xi, \tau) \right] d\xi + S(\sigma, \tau), \quad (34)$$

where $S(\sigma, \tau)$ is the constant of integration. In order to vanish the secular terms in Eq. (34), the functions f_0 and g_0 will have to satisfy the following equations:

$$2f_{0\sigma\tau} + 2\pi(\eta - V)f_{0\sigma} = 0, \quad (35)$$

$$-2g_{0\xi\tau} - 2\pi(\eta - V)g_{0\xi} = 0, \quad (36)$$

It is observed that Eqs. (35) and (36) are equivalent. The partial differential equation (34) under the general initial conditions:

$$f_{0\sigma}(\sigma, 0) = \frac{1}{2}(\phi'(\sigma) - \psi(\sigma)), \quad (37)$$

is solved as:

$$f_{0\sigma}(\sigma, \tau) = \frac{1}{2}(\phi(\sigma) - \psi(\sigma)) \exp(-\pi(\eta - V)\tau), \quad (38)$$

The solution of Eq. (38) is obtained by integration with respect to σ :

$$f_0(\sigma, \tau) = \frac{1}{2} \left[\phi(\sigma) - \int_0^\sigma \psi(\sigma) d\sigma \right] \exp(-\pi(\eta - V)\tau). \quad (39)$$

4. Effect of Initial Conditions

In this section, the transverse displacement of the system will be computed under the following initial conditions:

$$\phi(x) = \sin(\pi x), \quad \psi(x) = 0. \quad (40)$$

Thus, the functions $\phi(\sigma)$ and $\psi(\sigma)$ will then become: $\phi(\sigma) = \sin(\pi\sigma)$, and $\psi(\sigma) = 0$. The application of the mentioned initial displacement and initial velocity in Eq. (39) yields:

$$f_0(\sigma, \tau) = \sin(\pi\sigma) \exp(-\pi(\eta - V)\tau), \quad (41)$$

Hence, we shall obtain the first order approximation v_0 in the form:

$$v_0(\sigma, \xi, \tau) = f_0(\sigma, \tau) - f_0(-\xi, \tau) \equiv [\sin(\pi\sigma) + \sin(\pi\xi)] \exp(-\pi(\eta - V)\tau) \quad (42)$$

where the function $g_0(\xi, \tau)$ is: $g_0(\sigma, \tau) = -f_0(-\sigma, \tau)$. Thus, the valid solution of Eqs. (5) to (7) on long timescales of order ε^{-1} becomes:

$$w(x, t) = v_0(x, t, \tau) + O(\varepsilon) \\ \equiv [\sin(\pi(x+t)) + \sin(\pi(x-t))] \exp(-\pi(\eta - V)\tau) + O(\varepsilon). \quad (43)$$

Next, the following initial conditions are taken into consideration:

$$\phi(x) = \sin(\pi x), \quad \psi(x) = 0.5 \sin(\pi x). \quad (44)$$

The solution of the Eq. (5) to (7) becomes:

$$w(x, t) = \left[\left\{ 0.5 \sin(\pi\sigma) + \frac{0.25}{\pi} \cos(\pi\sigma) \right\} + \left\{ 0.5 \sin(\pi\xi) - \frac{0.25}{\pi} \cos(\pi\xi) \right\} \right] \exp(-\beta\tau) + O(\varepsilon), \quad (45)$$

in which $\beta = \pi(\eta - V)$, $\xi = x + t$, $\sigma = x - t$.

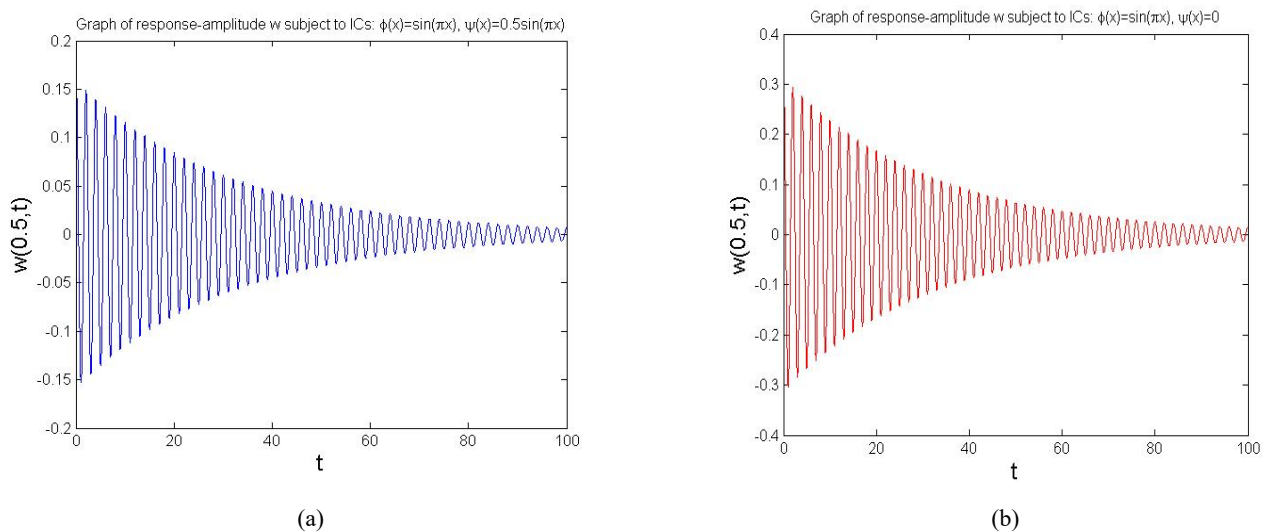


Fig. 2. The transversal displacement $w(x, t)$ versus time t (a) ICs: $\phi(x) = \sin(\pi x)$, $\psi(x) = 0$ and (b) ICs:

$\phi(x) = \sin(\pi x)$, $\psi(x) = 0.5 \sin(\pi x)$ with $V = 2$, $\eta = 3$.

5. Parameter Effects

The effect of various parameters, such as, the string speed V , the damping parameter η on the transverse displacement of a traveling string system is analyzed. Figs. 2 and 3 show the vertical displacement $w(x,t)$ of the system versus time t . It can clearly be seen that the motion of the system in terms of vertical displacement tending to zero speedily as the value of the damping coefficient grows.

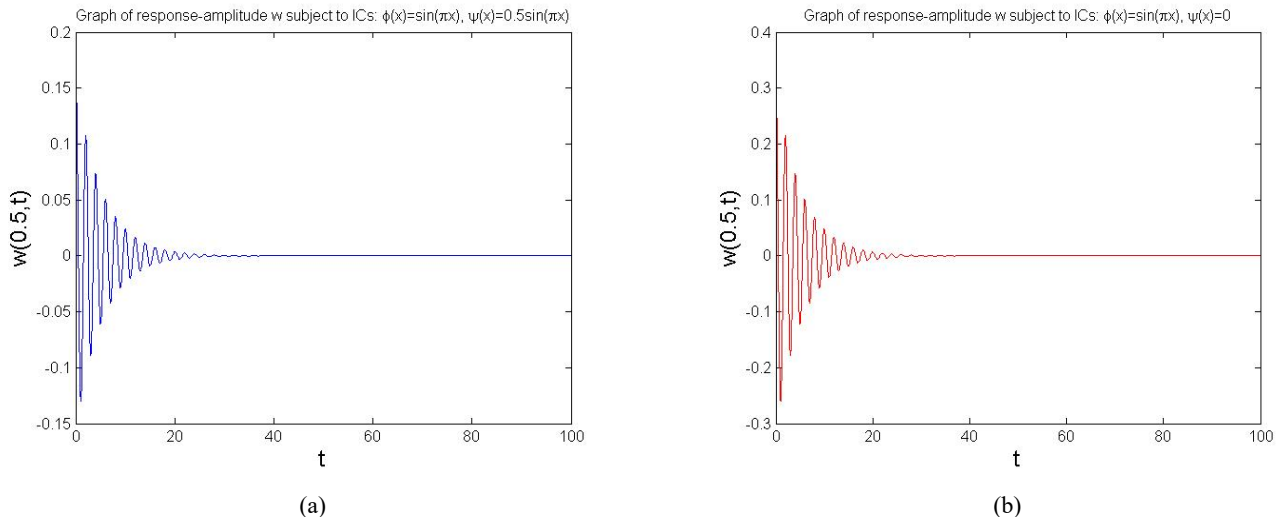


Fig. 3. The transversal displacement $w(x,t)$ versus time t (a) ICs: $\phi(x) = \sin(\pi x)$, $\psi(x) = 0$ and (b) ICs: $\phi(x) = \sin(\pi x)$, $\psi(x) = 0.5\sin(\pi x)$ with $V = 2$, $\eta = 10$.

6. Conclusion

In this article, the transversal vibrations of a moving string system under the effect of damping at boundary were studied. The vertical vibrations at the first end of the string were considered to be zero, while at the second end, the vertical vibrations were taken to be non-zero, (that is, the spring-dashpot system is tied). The string speed between a pair of pulleys was assumed to be positive and constant. Mathematically, this problem was modeled in terms of homogeneous linear partial differential equations under nonhomogeneous boundary conditions. The asymptotic approximations for the solutions of equations were obtained through the two time scales perturbation and characteristic coordinate method. The vertical displacement of the moving system was also obtained under two particular initial conditions. The effects of boundary damping were clearly seen in the vertical displacement of the system. It was shown that the motion of the traveling string in terms of vertical displacement is damped out by increasing the damping in the system.

Acknowledgments

The authors would like to express the deepest appreciation to unknown reviewers for their valuable comments and suggestions which helped us to improve our manuscript.

Conflict of Interest

The author(s) declared no potential conflicts of interest with respect to the research, authorship and publication of this article.

Funding

The author(s) received no financial support for the research, authorship and publication of this article.

References

- [1] Miranker, W.L., The wave equation in a medium in motion. *IBM Journal of Research and Development*, 4(1), 1960, 36-42.
- [2] Swope, R.D., Ames, W.F., Vibrations of a Moving Threadline. *Journal of the Franklin Institute*, 275(1), 1963, 36-55.
- [3] Malookani, R.A., van Horssen, W.T., On resonances and the applicability of Galerkin's truncation method for an axially moving string with time-varying velocity. *Journal of Sound and Vibration*, 344, 2015, 1-17.
- [4] Malookani, R.A., van Horssen, W.T., On the asymptotic approximation of the solution of an equation for a non-constant

axially moving string. *Journal of Sound and Vibration*, 367, 2016, 203-218.

- [5] Gaiko, N.V., van Horssen, W.T., On the transverse, low frequency vibrations of a traveling string with boundary damping. *Journal of Vibration and Acoustics*, 137(4), 2015, 041004-041014.
- [6] Sandilo, S.H., van Horssen, W.T., On a cascade of autoresonances of an elevator cable system. *Nonlinear Dynamics*, 80(3), 2015, 1613-1630.
- [7] Chikwendu, C., Kevorkian, J., A perturbation method for hyperbolic equations with Small nonlinearities. *SIAM Journal on Applied Mathematics*, 22(2), 1972, 235-258.
- [8] Darmawijoyo, van Horssen, W.T., Celement, PH., On a Rayleigh wave equation with boundary damping. *Nonlinear Dynamics*, 33, 2003, 399-429.
- [9] Gaiko, N.V., *Transverse waves and vibrations in axially moving continua*. PhD thesis, TU Delft, 2017.
- [10] Fung, R.F., Huang, J.S., Chen, Y.C., The transient amplitude of the viscoelastic traveling string: an integral constitutive law. *Journal of Sound and Vibration*, 201, 1997, 153-167.
- [11] Mahalingam, S., Transverse vibrations of power transmission chains. *British Journal of Applied Physics*, 8(4), 1957, 145-148.
- [12] Sack, R.A., Transverse oscillations in traveling strings. *British Journal of Applied Physics*, 5(6), 1954, 224-226.
- [13] Maitlo, A.A., Sandilo, S.H., Shaikh, A.H., Malookani, R.A., Qureshi, S., On aspects of viscous damping for an axially transporting string. *Science International*, 28(4), 2016, 3721-3727.
- [14] Suweken, G., van Horssen, W.T., On the transversal vibrations of a conveyor belt with a low and time-varying velocity. Part I: the string-like case. *Journal of Sound and Vibration*, 264(1), 2003, 117-133.
- [15] Ponomareva, S.V., van Horssen, W.T., On the transversal vibrations of an axially moving string with a time-varying velocity. *Nonlinear Dynamics*, 50(1-2), 2007, 315-323.
- [16] Ghayesh, M.H., Nonlinear transversal vibration and stability of an axially moving viscoelastic string supported by a partial viscoelastic guide. *Journal of Sound and Vibration*, 314, 2008, 757-774.
- [17] Shahruz, S.M., Stability of a nonlinear axially moving string with the Kelvin-Voigt damping. *Journal of Vibration and Acoustics*, 131(1), 2009, 014501.
- [18] Chen, W.E., Ferguson, N.S., Analysis of energy dissipation in an elastic moving string with a viscous damper at one end. *Journal of Sound and Vibration*, 333(9), 2014, 2556-2570.
- [19] Zhang, L., Zu, J.W., Nonlinear vibrations viscoelastic moving belts, Part 1: Free Vibration analysis. *Journal of Sound and Vibration*, 216(1), 1998, 75-91.
- [20] Archibald, F.R., Emslie, A.G., The Vibration of a String Having a Uniform Motion along Its Length. *Journal of Applied Mechanics*, 25(3), 1958, 347-348.
- [21] Thurman, A.L., Mote, C.D., Free, periodic, nonlinear oscillation of an axially moving strip. *Journal of Applied Mechanics*, 36, 1969, 83-91.
- [22] Sandilo, S.H., van Horssen, W.T., On boundary damping for an axially moving tensioned beam. *Journal of Vibration and Acoustics*, 134(1), 2012, 011005.
- [23] Sandilo, S.H., Malookani, R.A., Shaikh A.H., On oscillations of an axially translating tensioned beam under viscous damping. *Science International*, 28(4), 2016, 4123-4127.
- [24] Jones, D.I., *Handbook of viscoelastic vibration damping*, Wiley, New York, 2001.
- [25] Nayfeh, A.H., *Perturbation Methods*. John Wiley and Sons, New York, 2000.
- [26] Kevorkian, J., Cole, J.D., *Multiple Scale and Singular Perturbation*, Springer, 1996.



© 2019 by the authors. Licensee SCU, Ahvaz, Iran. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution-NonCommercial 4.0 International (CC BY-NC 4.0 license) (<http://creativecommons.org/licenses/by-nc/4.0/>).