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A Parameter Uniform Numerical Scheme for Singularly Perturbed Differential-difference Equations with Mixed Shifts

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Abstract. In this paper, we consider a second-order singularly perturbed differential-difference equations with mixed delay and advance parameters. At first, we approximate the model problem by an upwind finite difference scheme on a Shishkin mesh. We know that the upwind scheme is stable and its solution is oscillation free, but it gives lower order of accuracy. So, to increase the convergence, we propose a hybrid finite difference scheme, in which we use the cubic spline difference method in the fine mesh regions and a midpoint upwind scheme in the coarse mesh regions. We establish a theoretical parameter uniform bound in the discrete maximum norm. To check the efficiency of the proposed methods, we consider test problems with delay, advance and the mixed parameters and the results are in agreement with our theoretical findings.

Keywords: Singularly perturbed problem, Differential-difference equation, Mixed shifts, Shishkin mesh, Hybrid scheme, Uniform convergence.

1. Introduction

The study of differential-difference equations involving delay and advance parameters has rapidly grown in recent years. These type of equations when a small perturbation parameter ε is multiplied by the highest order derivative are termed as a singularly perturbed problem (SPP). Such type of models arise in variational problems in control theory [1], in determining the expected time for the generation of action potentials in nerve cells by random synaptic inputs in dendrites. Stein formulated the Poisson process with exponential decay between the inputs, arising due to the distribution representing inputs [2, 3]. Further, Stein's models were studied by Tuckwell [4] and Wilbur et al. [5]. Due to the presence of ε, the problem exhibits boundary or interior layers. It is well known that classical numerical methods (a combination of standard finite difference methods on uniform mesh) fail to produce good approximations for such problems. Hence, one has to look for non-classical methods like fitted mesh and fitted operator methods [6].

In the last few decades, a good number of research articles on non-classical methods are available, which mostly cover the two-point singularly perturbed boundary value problems [7, 8, 9]. But only a few researchers have worked on singularly perturbed differential-difference equations (SPDDE). Lange and Miura [10, 11] gave an asymptotic approach to the solution of SPDDEs for the two classes of singularly perturbed differential-difference equations with small shifts and turning point problems. The problems having layer behaviors were analyzed using Laplace transformation in [12]. A similar type of models was further studied by Kadalbajoo and Sharma [13, 14] using different numerical schemes. They developed a fitted finite difference method to solve a mathematical model arising from neuronal variability on the piecewise uniform mesh [15]. The model was treated numerically after approximating the delay and the advance parameter terms using Taylor series approximation. Rao et al. [16, 17] developed a finite difference method for SPDDEs having layer and oscillatory behaviors. They further approximated the problems with small shifts by developing an exponentially fitted tridiagonal finite difference method [18]. Mohapatra et al. developed a uniformly convergent numerical method for SPDDEs using grid equidistribution

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[19]. Duressa et al. constructed a domain decomposition method for SPDDEs with layer behavior, they introduced a terminal boundary point into the domain to decompose it into inner and the outer regions [20]. Using the idea of domain decomposition, Sirisha et al. [21] introduced the mixed finite difference method. One can develop the idea for the construction of the hybrid difference schemes for SPPs from [22, 23, 24]. There are some related articles available in literature that mostly discuss about the construction of the hybrid schemes such as: In [25, 26, 27] the authors developed weighted essential non-oscillatory scheme (WENO) using discontinuity indicator. For some higher order scheme like as h-adaptive Runga-Kutta discontinuous Galerkin method [28] are developed to solve one-dimensional detonation waves.

In this paper, we consider a second-order SPDDE with both delay and advance parameters. Our main goal is to construct a parameter uniform numerical method to solve the model problem. At first, we approximate the model problem by an upwind finite difference scheme on a Shishkin mesh. We know that the upwind scheme is stable and its solution is oscillation free but it gives first order of accuracy. So, to increase the order of convergence, we propose a hybrid finite difference scheme, in which we use cubic spline difference method in the layer regions and a midpoint upwind scheme in the regular region. The cubic spline method satisfies the discrete maximum norm only in the fine mesh region. Therefore, in the coarse mesh region, the solution given by the cubic spline method is not stable. So, we use the cubic spline method only in the fine region and the midpoint upwind scheme in the coarse mesh region, which provides us not only a stable solution but also the convergence is of second-order parameter uniform. While in [15], the authors had used fitted mesh methods on Shishkin mesh where the convergence of the difference scheme is almost linear. Hence clearly, our proposed method is more efficient and accurate over [15].

This paper is organized as follows: In Section 2, we present a continuous problem. The upwind finite difference scheme and the hybrid finite difference scheme on a piecewise uniform Shishkin mesh are briefly described in Section 3. Error estimates for both the proposed methods is presented in Section 4. For illustration, we have considered some test problems in Section 5. "C" has been used as a generic positive constant throughout this paper, which is independent of ε, mesh size and mesh points. For any continuous function $s(x)$, the supremum norm is defined by: $\|s(x)\| = \max_{x\in\bar{\Omega}} |s(x)|$.

2. Continuous Problem

Consider the following singularly perturbed differential-difference equation involving both the delay and advance parameters:

$$\begin{cases} L_\varepsilon y = \varepsilon y''(x) + a(x)y'(x) + b(x)y(x-\delta) - c(x)y(x) + d(x)y(x+\eta) = f(x), & x\in\Omega=(0,1), \\ y(x)=\gamma(x), \quad -\delta\le x\le 0, \\ y(x)=\psi(x), \quad 1\le x\le 1+\eta. \end{cases} \tag{1}$$

Here, $0<\varepsilon \ll 1$ is a small singular perturbation parameter, $0\le\delta\ll 1$ and $0\le\eta\ll 1$ are the delay and the advance parameters respectively. The functions $a(x),\ b(x),\ c(x),\ d(x), f(x), \gamma(x)$ and $\psi(x)$ are sufficiently smooth functions on [0,1]. By the Taylor series expansion around the neighborhood of x, we have

$$y(x-\delta) = y(x) - \delta y'(x) + o(\delta^2), \tag{2}$$

$$y(x+\eta) = y(x) + \eta y'(x) + o(\eta^2). \tag{3}$$

Neglecting the second order terms from eq. (2), eq. (3) and putting them in eq. (1), we get the equivalent SPDDE, which takes the form:

$$\begin{cases} L_\varepsilon y = \varepsilon y''(x) + \alpha(x)y'(x) - \beta(x)y(x) = f(x), & x\in\Omega, \\ y(0)=\gamma(0), \quad y(1)=\psi(1) \end{cases} \tag{4}$$

where $\alpha(x)=a(x)-\delta b(x)+\eta d(x),\ \beta(x)=c(x)-b(x)-d(x).$ Assume $\alpha(x)\ge\lambda>0$ for some constant λ and $\beta(x)\ge 0$. Then, the solution of eq. (4) exhibits an exponential boundary layer at $x=0.$

2.1 Properties of the solution and its derivatives

Lemma 2.1.1: *(Minimum principle)* Let w be any smooth function satisfying $w(0)\ge 0$, $w(1)\ge 0$ and $L_\varepsilon w\le 0$, $\forall x\in\Omega$, then $w(x)\ge 0$, $\forall x\in\bar{\Omega}$.

Proof: We shall prove the result by contradiction. Assume that there exists a point $x^*\in\bar{\Omega}$ such that $w(x^*) = \min_{x\in\bar{\Omega}}\{w(x)\}$ and $w(x^*)<0$. Clearly $w'(x^*)=0$ and $w''(x^*)>0$. Now

$$\begin{aligned} L_\varepsilon w(x^*) &= \varepsilon w''(x^*) + \alpha(x^*)w'(x^*) - \beta(x)w(x^*) \\ &= \varepsilon w''(x^*) - \beta(x^*)w(x^*) \\ &\geq 0 \end{aligned}$$

which contradicts our assumption. Hence, $w(x) \geq 0$, $\forall x \in \bar{\Omega}$.

Lemma 2.1.2: The solution y of eq. (4) satisfies the bound: $\|y\| \leq \dfrac{\|f\|}{\lambda} + \max\{y(0), y(1)\}$, $\forall x \in \bar{\Omega}$.

Proof: Let us consider the barrier function $\rho^{\pm}(x) = x\dfrac{\|f\|}{\lambda} + \max\{y(0), y(1)\} \pm y(x)$. Clearly, $\rho^{\pm}(0) \geq 0$, $\rho^{\pm}(1) \geq 0$ and

$$\begin{aligned} L_\varepsilon \rho^{\pm} &= \varepsilon(\rho^{\pm}(x))'' + \alpha(x)(\rho^{\pm}(x))' + \beta(x)(\rho^{\pm}(x)) \\ &= \alpha(x)\frac{\|f\|}{\lambda} + \alpha(x)\max\{y(0), y(1)\} + x\frac{\|f\|}{\lambda} \pm f(x). \end{aligned}$$

Since, $\alpha(x) \geq \lambda > 0$ and $\|f\| \geq f(x)$, we have $\alpha(x)\lambda^{-1}\|f\| \pm f(x) \geq 0$. Thus, we have $L_\varepsilon \rho^{\pm} \geq 0$, $\forall x \in \bar{\Omega}$. Hence, by *Lemma 2.1.1,* we get the required bound.

Lemma 2.1.3: Let y be the solution of eq. (4). Then for $0 \leq k \leq 3$, we have $\left|y^{(k)}(x)\right| \leq C\left(1 + \varepsilon^{k} e^{(-\lambda x/\varepsilon)}\right)$, $\forall x \in \Omega$.

Proof : By the mean value theorem, there exists a point $\xi \in (0, \varepsilon)$ such that $y'(\xi) = \dfrac{y(0) - y(\varepsilon)}{\varepsilon}$. Thus

$$|\varepsilon y'(\xi)| < 2\|y\| \tag{5}$$

By integrating eq. (4) from 0 to ξ , we have

$$\varepsilon y'(\xi) - \varepsilon y'(0) = \int_0^\xi \left(f(z) - \alpha(z)y'(z) + \beta(z)y(z)\right)dz , \tag{6}$$

$$\left|\int_0^\xi (f(z) - \alpha(z)y'(z) + \beta(z)y(z))dz\right| \leq \|f\||\xi| + \int_0^\xi \left(\alpha(z)y'(z)\right)dz + \|\beta\|\|y\||\xi|, \tag{7}$$

also, $\displaystyle\int_0^\xi \left(\alpha(z)y'(z)\right)dz = \alpha(z)y(z)|_0^\xi - \int_0^\xi \alpha'(z)y(z)dz$. Hence,

$$\left|\int_0^z \left(\alpha(z)y'(z)\right)dz\right| \leq 2\|\alpha\|\|y\| + \|\alpha'\|\|y\|\|z\| \tag{8}$$

Using eq. (8) and eq. (7), we get

$$\left|\int_0^z \left(f(z) - \alpha(z)y'(z) + \beta(z)y(z)\right)dz\right| \leq \|f\| + \left(2\|\alpha\| + \|\alpha'\|\|\beta\|\right)\|y\|. \tag{9}$$

Now from eq. (5) and eq. (6), we have $|y'(0)| \leq C\varepsilon^{-1}$. Again from eq. (6) with $\xi = x \in \Omega$, we obtain $|y'(x)| \leq C\varepsilon^{-1}$, $\forall x \in \Omega$. Similarly, using the bounds of y and y' , we get $\left|y^{(k)}(x)\right| \leq C\left(1 + \varepsilon^{k} e^{(-\lambda x/\varepsilon)}\right)$, $\forall x \in \Omega$, $0 \leq k \leq 3$.
The solution of eq. (4) can be decomposed into u and v such that $y = u + v$ where u is the regular component and v is the singular component of y . Now the asymptotic expansion of u is given by $u(x) = u_0 + \varepsilon u_1 + \varepsilon^2 u_2$, where u_0 , u_1 and u_2 satisfy the following system of equation

$$\alpha(x)u'_0 + \beta(x)u_0 = f(x) \ , \ u_0(0) = y_0(0).$$
$$\alpha(x)u'_1 + \beta(x)u_1 = u''_0(x) \ , \ u_1(0) = 0.$$
$$\varepsilon u''_2 + \alpha(x)u'_2 + \beta(x)u_2 = u''_1(x) \ , \ u_2(0) = 0 \ , \ u_1(0) = 0.$$

Thus u is the solution of $L_\varepsilon u(x) = f(x)$ with $u(0) = y(0)$ and $u(1) = u_0(1) + \varepsilon y_1(1)$. And v is the solution of $L_\varepsilon v(x) = 0$ with $v(0) = 0$ and $v(1) = y(1) - u(1)$.

Lemma 2.1.4: For $0 \le k \le 3$, the regular and the singular components of $y(x)$ satisfies the following bounds:

$$\left|u^{(k)}(x)\right| \le C\left(1 + \varepsilon^{(2-k)} e^{(-\lambda x/\varepsilon)}\right), \tag{10}$$

and

$$\left|v^{(k)}(x)\right| \le C\varepsilon^{-k} e^{(-\lambda x/\varepsilon)}. \tag{11}$$

Proof: From the decomposition of y, clearly we have $\left|u_0^{(k)}\right| \le C$, $0 \le k \le 3$ and $\left|u_1^{(k)}\right| \le C$, $0 \le k \le 3$. Now using *Lemma 2.1.2,* we get the required bound for regular component. For singular component, choose the barrier function $\rho^{\pm}(x) = |v(x)|e^{-(\lambda x/\varepsilon)} \pm v(x)$. Clearly, $\rho^{\pm}(0) \ge 0$, $\rho^{\pm}(1) \ge 0$ and $L\rho^{\pm}(0) \ge 0$. Hence, by applying *Lemma 2.1.1,* one can complete the proof.

3. Discrete Problem

In this section, we construct a piecewise uniform Shishkin mesh on $[0,1]$. Shishkin mesh is a special kind of mesh which differs from other meshes in the choice of transition parameter τ, at which the solution changes abruptly. The transition parameter is defined as:

$$\tau = \min\left\{\frac{1}{2}, \frac{1}{\lambda}\varepsilon \ln N\right\}$$

Now divide each of the subintervals $[0,\tau]$ and $[\tau,1]$ into $N/2$ equal subintervals. Then the Shishkin mesh is given as:

$$\overline{\Omega}^N = \begin{cases} x_i = hi, & i \le N/2, \\ x_i = Hi, & i > N/2, \end{cases}$$

where $h = 2\tau/N$ and $H = 2(1-\tau)/N$.

3.1 Upwind scheme

Let us denote $h_i = x_i - x_{i-1}$, $i = 1, \ldots, N$. For any mesh function $Y = \{Y_i\}$, we define the forward D^+ and the central D^+D^- difference operators as:

$$D^+Y_i = \frac{Y_{i+1} - Y_i}{h_{i+1}}, \qquad D^+D^-Y_i = \frac{2}{h_i + h_{i+1}}\left(\frac{Y_{i+1} - Y_i}{h_{i+1}} - \frac{Y_i - Y_{i-1}}{h_i}\right)$$

Now for the eq. (4), the upwind scheme takes the form:

$$L_u^N Y_i = \begin{cases} \varepsilon D^+D^-Y_i + \alpha(x_i)D^+Y_i - \beta(x_i)Y_i = f(x_i), & x_i \in \overline{\Omega}^N, \\ Y(0) = y(0), \quad Y(1) = y(1) \end{cases} \tag{12}$$

Therefore, on $\overline{\Omega}^N$ we have the following system of equations

$$P_i^- Y_{i-1} - P_i^0 Y_i + P_i^+ Y_{i+1} = P_i^N, \quad i = 1, \ldots, N-1, \tag{13}$$

where

$$P_i^- = \frac{2\varepsilon}{(h_i + h_{i+1})h_i}, \quad P_i^0 = \frac{2\varepsilon}{(h_i + h_{i+1})h_{i+1}} + \frac{2\varepsilon}{(h_i + h_{i+1})h_i} + \frac{\alpha(x_i)}{h_{i+1}} + \beta(x_i),$$

$$P_i^+ = \frac{2\varepsilon}{(h_i + h_{i+1})h_{i+1}} + \frac{\alpha(x_i)}{h_i}, \quad P_i^N = f(x_i)$$

One can easily check that the matrix associated with L_u^N is an M-matrix. For detail analysis of the scheme one may refer [29].

3.2 Cubic spline difference method

First of all, we formulate the cubic spline difference method on the Shishkin mesh. Define an interpolating cubic spline function $E(x)$ corresponding to the values $Y(x_0),\ldots,Y(x_N)$, of a function $Y(x)$ at the points $x_0, x_1, \ldots, x_N$, and it satisfies the following properties:

(i) $E(x)$ coincides with polynomial of degree three on each sub-intervals $[x_{i-1}, x_i]$, $i=1,\ldots,N$.

(ii) $E(x) \in C^2(\overline{\Omega})$.

(iii) $E(x_i) = Y(x_i)$, $i=0,1,\ldots,N$.

The cubic spline function $E(x)$ can be given as:

$$E(x) = \frac{(x_i - x)^3}{6h_i}M_{i-1} + \frac{(x - x_{i-1})^3}{6h_i}M_i + \left(Y(x_{i-1}) - \frac{h_i^2}{6}M_{i-1}\right)\left(\frac{(x_i - x)}{h_i}\right) + \left(Y(x_i) - \frac{h_i^2}{6}M_i\right)\left(\frac{(x - x_{i-1})}{h_i}\right), \quad x \in [x_{i-1}, x_i],$$

$i=1,\ldots,N,$

where $M_i = E''(x_i)$, $i=0,1,\ldots,N$.

From the properties of spline [23], it should satisfy the following condition for $i=1,\ldots,N-1$,

$$\frac{h_i}{6}M_{i-1} + \frac{h_i + h_{i+1}}{3}M_i + \frac{h_{i+1}}{6}M_{i+1} = \frac{Y(x_{i+1}) - Y(x_i)}{h_{i+1}} - \frac{Y(x_i) - Y(x_{i-1})}{h_i}. \tag{14}$$

To obtain an approximation for $y'(x)$ and $y''(x)$, we use the Taylor series approximations for Y about x_i as:

$$Y(x_{i+1}) \approx Y(x_i) + h_{i+1}Y'(x_i) + \frac{h_{i+1}^2}{2}Y''(x_i), \tag{15}$$

$$Y(x_{i-1}) \approx Y(x_i) - h_i Y'(x_i) + \frac{h_i^2}{2}Y''(x_i). \tag{16}$$

Multiplying eq. (16) by h_{i+1}^2 / h_i^2 and then subtracting it from eq. (15), we get an approximation for $Y'(x_i)$. Similarly, multiplying eq. (16) by h_{i+1}/h_i and then adding it to eq. (15), we get an approximation for $Y''(x_i)$ as follows:

$$Y'(x_i) \approx \frac{1}{h_i h_{i+1}(h_i + h_{i+1})}\left(-h_{i+1}^2 Y(x_{i-1}) + (h_{i+1}^2 - h_i^2)Y(x_i) + h_i Y(x_{i+1})\right),$$

$$Y''(x_i) \approx \frac{1}{h_i h_{i+1}(h_i + h_{i+1})}\left(h_{i+1}^2 Y(x_{i-1}) - (h_i + h_{i+1})Y(x_i) + h_i Y(x_{i+1})\right).$$

Using above approximations in $Y'(x_{i+1}) \approx Y'(x_i) + h_{i+1}Y''(x_i)$ and $Y'(x_{i-1}) \approx Y'(x_i) - h_i Y''(x_i)$, we get

$$Y'_j(x_{i+1}) \approx \frac{1}{h_i h_{i+1}(h_i + h_{i+1})}\left(h_{i+1}^2 Y(x_{i-1}) - (h_i + h_{i+1})^2 Y(x_i) + (h_i^2 + 2h_i h_{i+1})Y(x_{i+1})\right)$$

$$Y'_j(x_{i-1}) \approx \frac{1}{h_i h_{i+1}(h_i + h_{i+1})}\left(-(h_{i+1}^2 + 2h_i h_{i+1})Y(x_{i-1}) + (h_i^2 + h_{i+1}^2)^2 Y(x_i) - (h_i^2)Y(x_{i+1})\right)$$

Now for eq. (4) consider the approximation

$$\varepsilon M_j + \alpha(x_j)Y'(x_j) - \beta(x_j)Y(x_j) = f(x_j), \quad j = i, i \pm 1.$$

Putting the above in eq. (14), we have the following system of equation

$$L_{cu}^{N}Y_i = Q_i^{-}y_{i-1} - Q_i^{0}Y_i + Q_i^{+}Y_{i+1} = Q_i^{N}\ ,\ \ i = 1,\dots,N-1$$

where L_{cu}^{N} is the cubic spline operator, and

$$Q_i^{-} = \frac{\alpha(x_{i-1})(h_{i+1}+2h_i)}{2(h_i+h_{i+1})} + \frac{\beta(x_{i-1})h_i}{2} + \frac{\alpha(x_i)h_{i+1}}{h_i} - \frac{\alpha(x_{i+1})h_{i+1}^2}{2h_i(h_i+h_{i+1})} - \frac{3\varepsilon}{h_i},$$

$$Q_i^{0} = \frac{\alpha(x_{i-1})(h_i+h_{i+1})}{2h_{i+1}} + \frac{\alpha(x_i)(h_{i+1}^2-h_i^2)}{h_ih_{i+1}} - \frac{\alpha(x_{i+1})(h_i+h_{i+1})}{2h_i} - \frac{3\varepsilon(h_i+h_{i+1})}{h_ih_{i+1}} + \beta(x)(h_i+h_{i+1}),$$

$$Q_i^{+} = \frac{\alpha(x_{i-1})h_i^2}{2h_{i+1}(h_i+h_{i+1})} - \frac{\alpha(x_i)h_i}{h_{i+1}} - \frac{\alpha(x_{i+1})(h_i+2h_{i+1})}{2(h_i+h_{i+1})} + \frac{\beta(x_{i+1})h_{i+1}}{2} - \frac{3\varepsilon}{h_{i+1}}, \tag{17}$$

$$Q_i^{N} = -\frac{h_if(x_{i-1})}{2} - (h_i+h_{i+1})f(x_i) - \frac{h_{i+1}f(x_{i+1})}{2}.$$

3.3 Hybrid scheme

In this section, we define the hybrid finite difference scheme on a piecewise uniform Shishkin mesh. The discrete problem of model eq. (4) takes the form:

$$L_h^{N}Y_i = \begin{cases} L_{cu}^{N}Y_i = f_i, & \text{for } i \le N/2, \\ L_m^{N}Y_i = f_{i+1/2}, & \text{for } i > N/2, \end{cases} \tag{18}$$

where L_m^{N} is the midpoint upwind finite difference operator [30, 31] and is defined as:

$$L_m^{N}Y_i = \varepsilon D^{+}D^{-}Y_i + \alpha_{i+1/2}D^{+}Y_i - \beta_{i+1/2}Y(x_{i+1/2}) = f_{i+1/2},\ \ x_i \in \overline{\Omega}^{N},$$

and $\bar{z}_{i+1/2} = \bar{z}\left((x_i + x_{i+1})/2\right)$, for any function $\bar{z}$.

Thus, we have the system of linear equations as:

$$R_i^{-}Y_{i-1} - R_i^{0}Y_i + R_i^{+}Y_{i+1} = R_i^{N}\ ,\ \ i = 1,\dots,N-1,$$

where

$$R_i^{-} = \begin{cases} \dfrac{\alpha(x_{i-1})(h_{i+1}+2h_i)}{2(h_i+h_{i+1})} + \dfrac{\beta(x_{i-1})h_i}{2} + \dfrac{\alpha(x_i)h_{i+1}}{h_i} - \dfrac{\alpha(x_{i+1})h_{i+1}^2}{2h_i(h_i+h_{i+1})} - \dfrac{3\varepsilon}{h_i}, & \text{for } i \le N/2, \\ \dfrac{3\varepsilon}{(h_i+h_{i+1})h_i}, & \text{for } i > N/2, \end{cases}$$

$$R_i^{0} = \begin{cases} \dfrac{\alpha(x_{i-1})(h_i+h_{i+1})}{2h_{i+1}} + \dfrac{\alpha(x_i)(h_{i+1}^2-h_i^2)}{h_ih_{i+1}} + \beta(x_i)(h_i+h_{i+1}) - \dfrac{\alpha(x_{i+1})(h_i+h_{i+1})}{2h_i} - \dfrac{3\varepsilon(h_i+h_{i+1})}{h_ih_{i+1}}, & \text{for } i \le N/2, \\ \dfrac{2\varepsilon}{(h_i+h_{i+1})h_{i+1}} + \dfrac{2\varepsilon}{(h_i+h_{i+1})h_i} + \dfrac{\alpha(x_i)}{h_{i+1}} - \dfrac{\beta(x_i)}{2}, & \text{for } i > N/2, \end{cases}$$

$$R_i^+ = \begin{cases} \frac{\alpha(x_{i-1})h_i^2}{2h_{i+1}(h_i+h_{i+1})} - \frac{\alpha(x_i)h_i}{h_{i+1}} - \frac{3\varepsilon}{h_{i+1}} - \frac{\alpha(x_{i+1})(h_i+2h_{i+1})}{2(h_i+h_{i+1})} + \frac{\beta(x_{i+1})h_{i+1}}{2}, & for\ i \le N/2, \\ \frac{2\varepsilon}{(h_i+h_{i+1})h_{i+1}} + \frac{\alpha(x_i)}{h_{i+1}} + \frac{\beta(x_i)}{2}, & for\ i > N/2, \end{cases}$$

$$R_i^N = \begin{cases} -\frac{h_i f(x_{i-1})}{2} - (h_i+h_{i+1})f(x_i) - \frac{h_{i+1}f(x_{i+1})}{2}, & for\ i \le N/2, \\ \frac{f_i+f_{i+1}}{2}, & for\ i > N/2. \end{cases}$$

One can easily check that the matrix associated with L_h^N is an M-matrix. For the analysis of the above operator one may refer [29].

Lemma 3.4: (*Discrete minimum principle*) Suppose the mesh function $W(x_i)$ satisfies $W(x_0) \ge 0$, $W(x_N) \ge 0$ and $L_\varepsilon W(x_i) \le 0$, $\forall x_i \in \overline{\Omega}^N$, then $W(x_i) \ge 0$, $\forall x_i \in \overline{\Omega}^N$.

4. Error Estimation

4.1 Error analysis for upwind scheme

Lemma 4.1.1: The error associated to the regular component satisfies the following bound:

$$|u - U| \le CN^{-1}. \tag{19}$$

Proof: From the truncation error analysis of the operator defined in eq. (12), we have

$$L_u^N(u-U) = \varepsilon\left(\frac{d^2}{dx^2} - D^+D^-\right)u + \alpha(x_i)\left(\frac{d}{dx} - D^+\right)u. \tag{20}$$

Then by the standard local truncation error estimates and *Lemma 2.1.4,* we get

$$\left|L_u^N(u-U)(x_i)\right| \le \frac{\varepsilon}{3}2h\|u'''\| + \frac{\alpha(x_i)}{2}h\|u''\| \le CN^{-1}. \tag{21}$$

Let us choose the barrier function $\rho^\pm(x_i) = CN^{-1} \pm (u-U)(x_i)$. Now using *Lemma 3.4,* we get the required bound.

Lemma 4.1.2: We have the following estimate for the singular component:

$$|v - V| \le CN^{-1}\ln N \tag{22}$$

Proof: By the standard local truncation error estimates and *Lemma 2.1.3,* we have

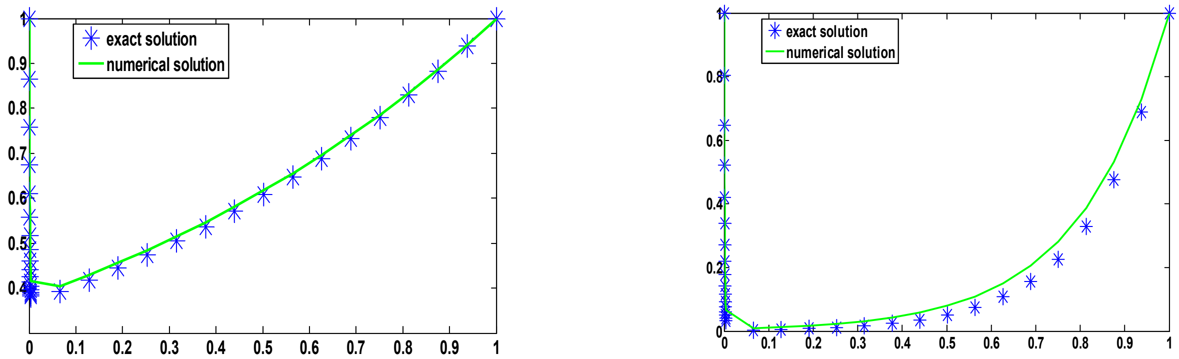
$$\left|L_u^N(v-V)(x_i)\right| \le C\varepsilon^{-2}2he^{\lambda x_{i-1}/\varepsilon} \le C\varepsilon^{-2}N^{-1}e^{\lambda x_{i-1}/\varepsilon} \tag{23}$$

We consider the barrier function $\rho^\pm(x_i) = CN^{-1}\ln N \pm (v-V)(x_i)$. Therefore, by using *Lemma 3.4,* we can obtain the required bound.

Theorem 4.1.3: Let y_i and Y_i be respective solutions of eq. (4) and the discrete problem eq. (12). Then, we have the following estimation:

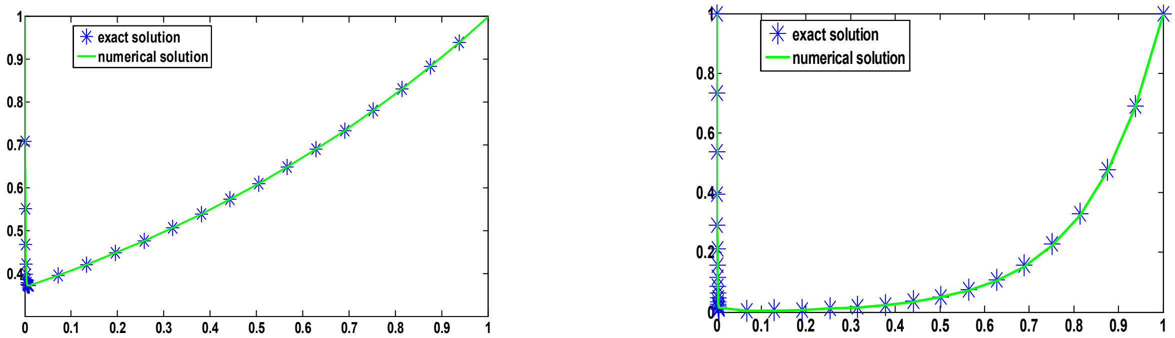
$$\left|y_i - Y_i\right| \le CN^{-1}\ln N \tag{24}$$

Proof: Using *Lemma 4.1.1* and *Lemma 4.1.2* one can prove the required estimate.



(a) For Example 1 (b) For Example 3

Fig. 1. Solutions using upwind scheme for *Example 1* and *Example 3* with $\varepsilon = 10^{-3}$ and $N = 32$.



(a) For Example 2 (b) For Example 3

Fig. 2. Solutions using hybrid scheme for *Examples 2 md 3* with $\varepsilon = 10^{-3}$ and $N = 32$.

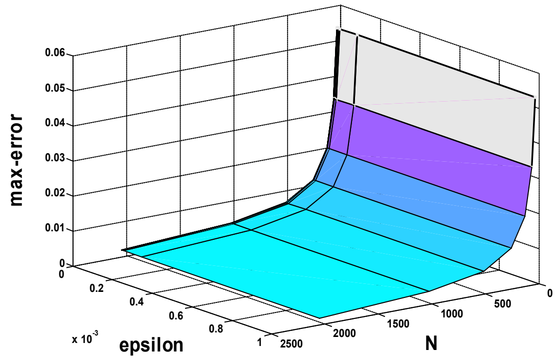


Fig. 3. E_ε^N generated by the upwind scheme for *Example 1.*

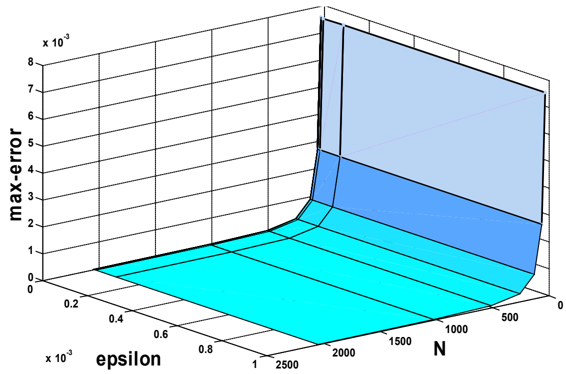


Fig. 4. E_ε^N generated by the hybrid scheme for *Example 2.*

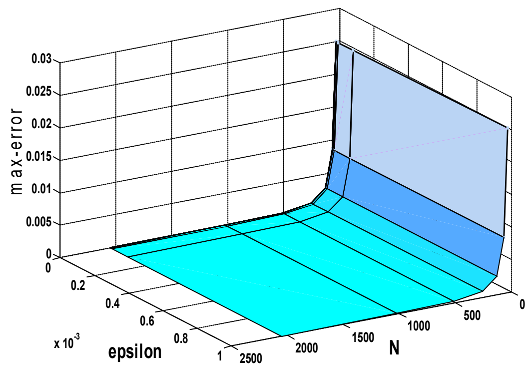


Fig. 5. E_ε^N generated by the hybrid scheme for *Example 3.*

4.2 Error analysis for hybrid scheme

Lemma 4.2.1: For the regular component, we have the following estimate:

$$|u-U| \le CN^{-2}. \tag{25}$$

Proof: From the truncation error analysis, we have

$$\left|L_h^N (u-U)(x_i)\right| \le \varepsilon h_i \|u'''\| + Ch_i^2 (\|u'''\| - \|u''\|). \tag{26}$$

Now from eq. (10), we get $\left|L_h^N (u-U)\right| \le CN^{-2}$. Let us consider the barrier function $\rho^{\pm}(x_i) = CN^{-2}(x_i) \pm (u-U)(x_i)$. Clearly, $\rho^{\pm}(x_0) \ge 0$ and $\rho^{\pm}(x_N) \ge 0$. Also $L_h^N \rho^{\pm}(x_i) \ge 0,$ for $i = 1,\dots,N-1$. So, by *Lemma 3.4,* we get the required estimate.

Lemma 4.2.2: At each mesh point $x_i \in \overline{\Omega}^N$, we have the following estimate for the singular component

$$|v-V| \le CN^{-2}(\ln N)^2 \tag{27}$$

Proof: From the truncation error analysis, we have the following estimate:

$$\left|L_h^N (v-V)\right| \le \varepsilon h_i^2 \left\|v^{(4)}\right\| \le CN^{-2}(\ln N)^2. \tag{28}$$

Now taking the barrier function $\rho^{\pm}(x_i) = \rho^{\pm}(x_i) \pm \|(v-V)(x_i)\|,$ for $i = 1,\dots,N-1$, we have $\rho^{\pm}(x_0) \ge 0$, $\rho^{\pm}(x_N) \ge 0$ and $L_h^N \rho^{\pm}(x_i) \ge 0$. So, by *Lemma 3.4,* we get the required bound.

Theorem 4.2.3: Let y_i and Y_i be respective solutions of eq. (4) and the discrete problem eq. (18). Then, we have the following estimate:

$$\left|y_i - Y_i\right| \le CN^{-2}(\ln N)^2. \tag{29}$$

Proof: From the triangular inequality, we have $\left|y_i - Y_i\right| \le |u_i - U_i| + |v_i - V_i|$. From *Lemma 4.2.1* and *Lemma 4.2.2*, we get the desired result.

5. Numerical Results

In this section, we consider three different test problems to check the efficiency of the upwind scheme and the hybrid scheme. The exact solution of the continuous problem eq. (4) is given by

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}, \tag{30}$$

where

$$m_1 = \frac{-\alpha(x) + \sqrt{(\alpha(x))^2 + 4\varepsilon\beta(x)}}{2\varepsilon}, \quad m_2 = \frac{-\alpha(x) - \sqrt{(\alpha(x))^2 + 4\varepsilon\beta(x)}}{2\varepsilon}, \quad c_1 = \frac{e^{m_2} - 1}{e^{m_2} - e^{m_1}}, \quad c_2 = \frac{1 - e^{m_1}}{e^{m_2} - e^{m_1}}. \tag{31}$$

For any N, the maximum point-wise error E_ε^N is defined by $E_\varepsilon^N = \max_{0<i\le N} \|y_i - Y_i\|$, where Y_i is obtained by the proposed method. And the corresponding rate of convergence R_ε^N is defined by $R_\varepsilon^N = \log_2(E_\varepsilon^N / E_\varepsilon^{2N})$.

Example 1: Consider the following test problem:

$$\varepsilon y''(x) + y'(x) + 2y(x-\delta) - 3y(x) = 0, \quad y(0) = 1, \ y(1) = 1. \tag{32}$$

Example 2: Consider the following test problem:

$$\varepsilon y''(x) + y'(x) - 3y(x) + 2y(x+\eta) = 0, \quad y(0) = 1, \ y(1) = 1 \tag{33}$$

Example 3: Consider the following test problem:

$$\varepsilon y''(x) + y'(x) - 2y(x-\delta) - 5y(x) + y(x+\eta) = 0, \quad y(0) = 1, \ y(1) = 1 \tag{34}$$

Table 1. E_ε^N and R_ε^N generated for *Example 1*, with $\delta = 0.1\varepsilon$.

N	$\varepsilon = 10^{-3}$		$\varepsilon = 10^{-8}$	
	Upwind scheme	Hybrid scheme	Upwind scheme	Hybrid scheme
32	5.349e-2 0.671	4.722e-2 1.577	5.375e-2 0.672	4.722e-2 1.578
64	3.359e-2 0.747	1.581e-2 1.633	3.373e-2 0.747	1.581e-2 1.633
128	2.002e-2 0.783	5.098e-3 1.655	2.009e-2 0.783	5.096e-3 1.656
256	1.163e-2 0.820	1.618e-3 1.655	1.167e-2 0.820	1.617e-3 1.656
512	6.588e-3 0.844	5.135e-4 1.697	6.612e-3 0.844	5.130e-4 1.698
1024	3.669e-3 0.862	1.582e-4 1.724	3.683e-3 0.863	1.580e- 4 1.726

Table 2. E_ε^N and R_ε^N generated for *Example 1*, with $\delta = 0.5\varepsilon$.

N	$\varepsilon = 10^{-3}$		$\varepsilon = 10^{-8}$	
	Upwind scheme	Hybrid scheme	Upwind scheme	Hybrid scheme
32	5.350e-2 0.671	4.717e-2 1.578	5.375e-2 0.672	4.722e-2 1.578
64	3.359e-2 0.747	1.579e-2 1.633	3.373e-2 0.747	1.581e-2 1.633
128	2.001e-2 0.783	5.092e-3 1.655	2.009e-2 0.783	5.093e-3 1.656
256	1.162e-2 0.820	1.616e-3 1.655	1.167e-2 0.820	1.617e-3 1.656
512	6.586e-3 0.844	5.129e-4 1.697	6.612e-3 0.844	5.130e-4 1.698
1024	3.668e-3 0.862	1.580e-4 1.724	3.683e-3 0.862	1.580e-4 1.725

Fig. 1 and Fig. 2 depicts the exact and numerical solution of *Example 1, Example 2* and *Example 3* generated by the upwind scheme and the hybrid scheme. In Fig. 3, the maximum point-wise error E_ε^N for Example *1* obtained by using the upwind scheme, are plotted as a function of ε and N. Note that the error decreases for all values of ε with increasing the values of N. Similar results are represented for the hybrid scheme in Fig. 4 and Fig. 5 for *Example 2* and *Example 3.*

Table 1 and Table 2 represents the maximum point-wise error and corresponding rate of convergence for $\varepsilon = 10^{-3}$ and $\varepsilon = 10^{-8}$ of the upwind scheme and the hybrid scheme for *Example 1* with $\delta = 0.1\varepsilon$ and $\delta = 0.5\varepsilon$. Similar results are shown in Table 3 and Table 4 for *Example 2* with $\eta = 0.1\varepsilon$ and $\eta = 0.5\varepsilon$. Table 5 and Table 6 represents the similar results for *Example 3* for $\delta = \eta = 0.1\varepsilon$ and $\delta = \eta = 0.5\varepsilon$ respectively. These results show the parameter-uniform nature of the upwind scheme and the hybrid scheme. Clearly, these results are in agreement with the theoretical uniform bounds. Table 7, represents the comparisons of maximum pointwise error obtain using the hybrid scheme (eq. 18) with the result given in [15]. One can observe that the errors obtained by our proposed scheme is significantly less than the errors obtained in [15] for smaller values of ε keeping N fixed, which clearly shows the efficiency of the our proposed scheme over [15].

Table 3. E_ε^N and R_ε^N generated for *Example 2*, with $\eta = 0.1\varepsilon$.

N	$\varepsilon = 10^{-3}$		$\varepsilon = 10^{-8}$	
	Upwind scheme	Hybrid scheme	Upwind scheme	Hybrid scheme
32	5.349e-2 0.671	4.724e-2 1.577	5.375e-2 0.672	4.722e-2 1.578
64	3.359e-2 0.747	1.582e-2 1.633	3.373e-2 0.747	1.581e-2 1.633
128	2.001e-2 0.783	5.101e-3 1.655	2.009e-2 0.783	5.096e-3 1.656
256	1.163e-2 0.820	1.619e-3 1.655	1.167e-2 0.820	1.617e-3 1.656
512	6.589e-3 0.844	5.138e-04 1.697	6.612e-3 0.844	5.130e-4 1.698
1024	3.670e-3 0.862	1.583e-4 1.724	3.683e-3 0.863	1.580e-4 1.726

Table 4. E_ε^N and R_ε^N generated for *Example 2*, with $\eta = 0.5\varepsilon$.

N	$\varepsilon = 10^{-3}$		$\varepsilon = 10^{-8}$	
	Upwind scheme	Hybrid scheme	Upwind scheme	Hybrid scheme
32	5.348e-2 0.671	4.730e-2 1.577	5.375e-2 0.672	4.722e-2 1.578
64	3.360e-2 0.747	1.585e-2 1.634	3.373e-2 0.747	1.581e-2 1.633
128	2.002e-2 0.783	5.107e-3 1.655	2.009e-2 0.783	5.096e-3 1.656
256	1.163e-2 0.820	1.621e-3 1.656	1.167e-2 0.820	1.617e-3 1.656
512	6.590e-3 0.844	5.143e-4 1.697	6.612e-3 0.844	5.130e-4 1.698
1024	3.671e-3 0.863	1.585e-4 1.724	3.683e-3 0.863	1.580e-4 1.726

Table 5. E_ε^N and R_ε^N generated for *Example 3*, with $\delta = \eta = 0.1\varepsilon$.

N	$\varepsilon = 10^{-3}$		$\varepsilon = 10^{-8}$	
	Upwind scheme	Hybrid scheme	Upwind scheme	Hybrid scheme
32	7.564e-2 0.651	7.491e-2 1.574	7.610e-2 0.649	7.450e-2 1.578
64	4.816e-2 0.721	2.515e-2 1.635	4.850e-2 0.722	2.495e-2 1.633
128	2.922e-2 0.772	8.096e-3 1.654	2.939e-2 0.772	8.041e-3 1.656
256	1.711e-2 0.808	2.571e-3 1.656	1.721e-2 0.808	2.551e-3 1.656
512	9.772e-3 0.835	8.155e-4 1.698	9.830e-3 0.835	8.095e-4 1.698
1024	5.475e-3 0.856	2.513e-04 1.724	5.507e-3 0.856	2.494e-4 1.725

Table 6. E_ε^N and R_ε^N generated for *Example 3*, with $\delta = \eta = 0.5\varepsilon$.

N	$\varepsilon = 10^{-3}$		$\varepsilon = 10^{-8}$	
	Upwind scheme	Hybrid scheme	Upwind scheme	Hybrid scheme
32	7.574e-2 0.651	7.509e-2 1.573	7.610e-2 0.649	7.450e-2 1.578
64	4.821e-2 0.721	2.522e-2 1.635	4.850e-2 0.722	2.495e-2 1.633
128	2.926e-2 0.772	8.116e-3 1.654	2.939e-2 0.772	8.041e-3 1.656
256	1.713e-2 0.808	2.578e-3 1.657	1.721e-2 0.808	2.551e-3 1.656
512	9.784e-3 0.836	8.174e-4 1.698	9.830e-3 0.836	8.095e-4 1.698
1024	5.481e-3 0.856	2.519e-4 1.724	5.507e-3 0.856	2.494e-4 1.725

Table 7. Comparison of E_ε^N generated for *Example 3*, with $\delta = \eta = 0.5\varepsilon$.

N		$\varepsilon = 10^{-2}$		$\varepsilon = 10^{-4}$		$\varepsilon = 10^{-6}$	
		Result in [15]	Our result	Result in [15]	Our result	Result in [15]	Our result
8	E_ε^N	0.1872710	0.309510	0.2061414	0.3181652	0.2063460	0.3182940
16	E_ε^N	0.1069782	0.1177974	0.1204841	0.1715907	0.1206323	0.1715239
32	E_ε^N	0.0590411	0.0801672	0.0698994	0.0745653	0.0700229	0.0745073
64	E_ε^N	0.0307968	0.0276215	0.0372137	0.0249776	0.0372876	0.0249506
128	E_ε^N	0.0156796	0.0087590	0.0193277	0.0080492	0.0193712	0.0080418
256	E_ε^N	0.0079907	0.0028049	0.0098423	0.0025544	0.0098657	0.0025518

6. Conclusion

In this paper, we considered an SPDDE of second order. The original model problem was transformed into an equivalent singularly perturbed differential equation by using Taylor series expansion. Then, we solved the model problem by using the upwind and the hybrid method on Shishkin mesh. Here, the hybrid method consists of a cubic spline approximation in the fine mesh region and a midpoint upwind scheme in the coarse mesh region. Error analysis was carried out for both schemes. We showed that the upwind scheme gives almost first-order convergence irrespective of the perturbation parameter and the hybrid scheme gives almost second-order convergence on the piecewise uniform Shishkin mesh. The numerical result provided the efficiency and performance of the theoretical findings.

Conflict of Interest

The authors declared no potential conflicts of interest with respect to the research, authorship and publication of this article.

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Nomenclature

SPP	Singular perturbation problem	D^+, D^+D^-	Difference operators
SPDDE	Singularly perturbed differential-difference equations	Ω	Continuous spatial domain
$\lVert . \rVert$	Standard supremum norm	$\overline{\Omega}^N$	Discrete spatial domain
ε	Perturbation parameter	$L_\varepsilon, L_u^N, L_h^N, L_{cu}^N, L_m^N$	Differential operators
δ, η	Delay and advance parameter	E_ε^N	Maximum pointwise error
$\gamma(x), \psi(x)$	Boundary data	R_ε^N	Order of convergence
h, H, h_i	Mesh size	N	Number of mesh intervals

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