J. Appl. Comput. Mech., 6(3) (2020) 394-415
DOI: 10.22055/JACM.2019.29555.1615

ISSN: 2383-4536
jacm.scu.ac.ir

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Shahid Chamran University of Ahvaz

Journal of
Applied and Computational Mechanics

Research Paper

Simple Two Variable Refined Theory for Shear Deformable Isotropic Rectangular Beams

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Received May 15 2019; Revised July 26 2019; Accepted for publication July 27 2019.
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Abstract: In this paper, a displacement-based, variationally consistent, two variable refined theory for shear deformable beams is presented. The beam is assumed to be of linearly elastic, homogeneous, isotropic material and has a uniform rectangular cross-section. In this theory, the beam axial displacement and beam transverse displacement consist of bending components and shearing components. The assumed displacement field of this theory is such that, bending components do not take part in the cross-sectional shearing force, and shearing components do not take part in the cross-sectional bending moment. This theory utilizes linear strain-displacement relations. The displacement functions give rise to the beam transverse shear strain (and hence to the beam transverse shear stress) which varies quadratically through the beam thickness and maintains transverse shear stress-free beam surface conditions. Hence the shear correction factor is not required. Hamilton's principle is utilized to derive governing differential equations and variationally consistent boundary conditions. This theory involves only two governing differential equations of fourth-order. These governing equations are only inertially coupled for the case of dynamics and are decoupled for the case of statics. This theory is simple and has a strong resemblance with the Bernoulli-Euler beam theory. To demonstrate the efficacy of the present theory, illustrative examples pertain to the static bending and free vibrations of shear deformable isotropic rectangular beams are presented.

Keywords: Refined beam theory, Two variable, Shear deformation theory.

1. Introduction

The Bernoulli-Euler beam theory (*BEBT*) is the simplest displacement-based beam theory. The assumed displacement field of the *BEBT* only gives rise to the beam axial strain and does not give rise to the beam transverse shear strain (references [1, 2, 3]). As a result, the *BEBT* can provide reasonably accurate results of the static flexure, stability and lower mode vibrations of longer/slender beams. This is because bending deformation effects are significantly higher as compared to transverse shear deformation effects for such beams.

However, transverse shear deformation effects start to become comparable to bending deformation effects in case of thin beams vibrating at higher modes, beams having a quite large value of the ratio of Young's modulus to the shear modulus and stockier/thick beams (e.g., sandwich beams with soft shear core). The *BEBT* no longer can provide accurate results, and the need for shear deformation beam theories arises for analyzing these cases of beams. The *BEBT* provides underestimated deflections and overestimated buckling loads in case of shear deformable beams, overestimated natural frequencies in case of slender beams vibrating at higher modes as well as shear deformable beams.

In case of shear deformable beams, transverse shear deformation effects cause the transverse deflection to increase, buckling loads and vibrational frequencies to decrease as compared to corresponding values for slender beams. to observe just-mentioned effects of transverse shear on the beam transverse deflection, buckling loads and vibrational frequencies, these quantities must be non-dimensionalized as reported in Shimpi et al. [4]. The significance of effects of transverse shear concerning the beam deformation has been brought out by references [1, 2, 3, 5, 6, 7, 8, 9, 10, 11]. The shear deformation effects are prominent also in case of built-up members (e.g., I-beams) owing to their reduced shear rigidity (references [12, 13, 14]). Built-up members are in general challenging to analyze.

The Timoshenko beam theory (references [3, 5, 6]) is a displacement-based, first-order shear deformation beam theory having two coupled governing differential equations involving two unknown functions. The theory provides corrections concerning the shear deformation to the *BEBT* (Chan et al. [15]). Assumed displacement functions of the Timoshenko beam theory give rise to the constant transverse shear strain (and hence the constant transverse shear stress) through the beam thickness. As a result, the theory does not satisfy transverse shear stress-free beam surface conditions. The theory also requires the shear correction factor. The significance of the shear correction factor concerning the Timoshenko beam theory has been brought out by references [15, 16, 17, 18].

Higher-order shear deformation beam theories are developed to address drawbacks of first-order shear deformation beam theories. Some of the notable work with regard to higher-order beam theories is reported in references [19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35]. These theories provide better estimations of the beam deformation behavior as compared to first-order beam theories. However, higher-order beam theories generally have the number of governing equations involving more number of unknown functions as compared to first-order beam theories. Hence obtaining results of the beam deformation behavior by utilizing higher-order beam theories involve more computational efforts (Wang et al. [36]).

Another interesting methodology concerning the formulation of shear deformation theories consists of splitting of the transverse displacement into sub-components. Well-known work reported by references [37, 38] split the transverse displacement into the bending component and shearing component. Work reported by references [37, 38] involves two governing differential equations with two unknown functions. These governing equations are only inertially coupled for the case of dynamics and are decoupled for the case of statics. Work reported by references [39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50] also utilize similar splitting of the transverse displacement as reported by references [37, 38].

As has been discussed by Bathe [51] in the context of beams, substantial artificial stiffening of lower-order, displacement-based finite elements based on shear deformation beam theories is observed as a thickness-to-length ratio of the beam decreases. As a result, when the beam analyzed is thin, displacements obtained are much smaller than the exact values. This behavior of the beam elements is termed as ‘shear locking’. This conclusion is also valid for lower-order, displacement-based plate and shell elements. It should be noted that finite elements based on the formulations which split the transverse displacement into the bending component and shearing component are generally free from shear locking (references [37, 38]). For example, as reported by Shimpi et al. [38], the finite element based on new first-order shear deformation plate theory-I is free from shear locking as this theory requires continuity of C^0 type for the shearing component and continuity of C^1 type for the bending component. Substantial work on finite elements which do not undergo shear locking has been reported in the literature (references [52, 53, 54]).

The objective of this paper is to present a simple and easy-to-use, variationally consistent, two variables refined theory for shear deformable isotropic rectangular beams. The inspiration for this theory has been taken from the refined plate theory of Shimpi [37]. To demonstrate the efficacy of the present theory, illustrative examples of the static bending and free vibrations of shear deformable isotropic rectangular beams are presented.

As evident through the recent work reported by references [33, 34, 35, 48, 49], the research in the field of predicting the mechanical behavior of shear deformable beams by utilizing shear deformation beam theories is active. The present work needs to be construed in this context.

2. The Present Theory: Assumptions

The inspiration for the present theory has been taken from the refined plate theory of Shimpi [37]. Following are the assumptions made in the present theory:

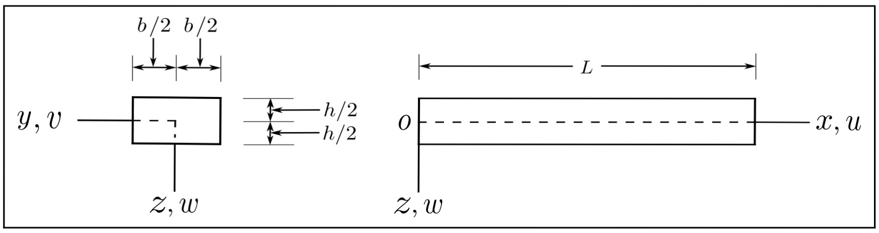

Fig. 1. The beam geometry and co-ordinate system.

1. Assumptions involved concerning the beam under consideration are as follows:
   (a) **The geometry and material of the beam**: The beam under consideration (Fig. 1) has a length L, uniform width b and uniform thickness h. The locus of the center of gravity of the beam cross-section is the elastic axis of the beam. Right-handed Cartesian co-ordinate system *0-x-y-z* is utilized throughout this paper. The elastic axis of the unloaded beam coincides with the x-axis. The y-axis and z-axis are along the width and thickness of the beam cross-section respectively. Time is denoted by symbol t. The beam is made of linearly elastic, homogeneous, isotropic material for which $G = E / [2(1 + \mu)]$ where G, E and μ represent the shear modulus, Young's modulus and Poisson's ratio of the material respectively. The mass density of the material is denoted by symbol ρ.
   (b) **The loading on the beam**: On the beam surface $z = - h / 2$, the beam is subjected to a distributed transverse load of intensity $q(x, t)$. The plane of loading coincides with the zx-plane. If an applied distributed transverse load acts in the positive z-direction, it is assumed to be positive. Beam surfaces $z = \pm h / 2$ are shear stress-free.
   (c) **Boundary conditions of the beam**: At beam ends $x = 0$ and $x = L$, variationally consistent boundary conditions can be prescribed.
2. Assumptions involved concerning beam displacement components are as follows:
   (a) The beam axial displacement $u(x, z, t)$ along the x-direction is made of two components:
      i. Bending component $u_b(x, z, t)$
      ii. Shearing component $u_s(x, z, t)$
   (b) The beam transverse displacement $w(x, t)$ along the z-direction is made of two components:
      i. Bending component $w_b(x, t)$
      ii. Shearing component $w_s(x, t)$

Following points should be noted with regard to above-mentioned beam displacement components:

- The bending component u_b of the beam axial displacement u is analogous to the beam axial displacement of the Bernoulli-Euler beam theory. Hence the expression for u_b can be given as follows:

$$u_b = - z \frac{\partial w_b}{\partial x} \tag{1}$$

- The bending component u_b of the beam axial displacement u together with the bending component w_b of the beam transverse displacement w do not take part in the beam transverse shear strain γ_{zx}.
- The shearing component u_s of the beam axial displacement u and shearing component w_s of the beam transverse displacement w are the only beam displacement components which take part in the beam transverse shear strain γ_{zx}. Displacement components u_s and w_s in conjunction contribute towards γ_{zx} in such a way that, γ_{zx} remains zero at beam surfaces $z = \pm h / 2$ and varies quadratically through the beam thickness. This implies that the beam transverse shear stress τ_{zx} developed also remains zero at beam surfaces $z = \pm h / 2$ and varies quadratically through the beam thickness.
- The contribution of the shearing component u_s of the beam axial displacement u towards the beam axial strain ϵ_x is such that, u_s does not contribute towards the cross-sectional bending moment M_x.

3. In the present theory, the beam axial displacement u and the beam transverse displacement w are small in comparison to the beam thickness. Hence the strains developed are infinitesimal, and linear strain-displacement relations as reported by Shames and Dym [3] are applicable.
4. In general, in comparison with the beam axial stress σ_x, normal stresses σ_y and σ_z are negligible. Similarly in general, in comparison with the beam transverse shear stress τ_{zx}, shear stresses τ_{xy} and τ_{yz} are negligible. Hence stresses σ_y, σ_z, τ_{xy} and τ_{yz} can be ignored.

3. The present theory: Displacement functions, strains, stresses, cross-sectional bending moment, cross-sectional shearing force

3.1. Expressions for displacement functions

Taking inspiration from displacement functions of the refined plate theory (Shimpi [37]) and as a result of assumption 2, it is possible to write expressions for the beam axial displacement u and the beam transverse displacement w of the present theory as follows:

$$u = u_b + u_s = -z \frac{\partial w_b}{\partial x} + h\left[-\frac{5}{3}\left(\frac{z}{h}\right)^3 + \frac{1}{4}\left(\frac{z}{h}\right)\right] \frac{\partial w_s}{\partial x} \tag{2}$$

$$w = w_b + w_s \tag{3}$$

It should be noted that a polynomial function involving cubic and linear terms of the beam thickness co-ordinate is utilized to describe the beam axial displacement (eq. (2)).

3.2. Expressions for strains

As a result of assumption 3, the beam axial strain ϵ_x and the beam transverse shear strain γ_{zx} can be expressed in terms of beam displacement components u and w as follows:

$$\epsilon_x = \frac{\partial u}{\partial x}, \quad \gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \tag{4}$$

Utilizing expressions for beam displacement functions as given by eqs. (2) and (3) into linear strain-displacement relations as given by eq. (4), expressions for ϵ_x and γ_{zx} of the present theory are obtained:

$$\epsilon_x = -z\frac{\partial^2 w_b}{\partial x^2} + h\left[-\frac{5}{3}\left(\frac{z}{h}\right)^3 + \frac{1}{4}\left(\frac{z}{h}\right)\right]\frac{\partial^2 w_s}{\partial x^2} \tag{5}$$

$$\gamma_{zx} = \left[-5\left(\frac{z}{h}\right)^2 + \frac{5}{4}\right]\frac{\partial w_s}{\partial x} \tag{6}$$

In tune with the loading on the beam as discussed in the section 2.1.(b), γ_{zx} satisfies transverse shear strain free beam surface conditions and varies through the beam thickness. It is evident from eq. (6) that in the present theory, γ_{zx} remains zero at beam surfaces $z = \pm h / 2$ and varies quadratically through the beam thickness. Hence as opposed to first-order beam theories, the shear correction factor is not required in the present theory.

3.3. Expressions for stresses

As a result of assumption 4, the beam axial stress σ_x and the beam transverse shear stress τ_{zx} developed can be related to the beam axial strain ϵ_x and the beam transverse shear strain γ_{zx} for the linear elastic, isotropic material as follows:

$$\begin{aligned} \sigma_x &= E\,\epsilon_x \\ \tau_{zx} &= G\,\gamma_{zx} = \frac{E}{2\,(1+\mu)}\gamma_{zx} \end{aligned} \tag{7}$$

Utilizing expressions for strains as given by eqs. (5) and (6) into stress-strain constitutive relations as given by eq. (7), expressions for σ_x and τ_{zx} of the present theory are obtained:

$$\sigma_x = -E\,z\frac{\partial^2 w_b}{\partial x^2} + E\,h\left[-\frac{5}{3}\left(\frac{z}{h}\right)^3 + \frac{1}{4}\left(\frac{z}{h}\right)\right]\frac{\partial^2 w_s}{\partial x^2} \tag{8}$$

$$\tau_{zx} = \frac{E}{2\,(1+\mu)}\left[-5\left(\frac{z}{h}\right)^2 + \frac{5}{4}\right]\frac{\partial w_s}{\partial x} \tag{9}$$

It is evident from eq. (9) that in the present theory, τ_{zx} remains zero at beam surfaces $z = \pm h / 2$ and varies quadratically through the beam thickness.

3.4. Expression for the cross-sectional bending moment

The cross-sectional bending moment M_x in case of a beam is defined as follows:

$$M_x = \int_{y=-\frac{b}{2}}^{y=\frac{b}{2}} \int_{z=-\frac{h}{2}}^{z=\frac{h}{2}} [z\sigma_x]\,dz\,dy \tag{10}$$

Utilizing expression for the beam axial stress σ_x as given by eq. (8) into eq. (10), expression for M_x of the present theory is obtained.

$$M_x = -E\,I\frac{\partial^2 w_b}{\partial x^2} \tag{11}$$

where $I = (b\,h^3) / 12$ is the beam cross-sectional area moment of inertia about the y-axis (Fig. 1). The expression for M_x as given by eq. (11) is same as the corresponding expression of the Bernoulli-Euler beam theory (eq. (6-9a) of Timoshenko and Gere [2]), except for the presence of the subscript in eq. (11).

3.5. Expression for the cross-sectional shearing force

The cross-sectional shearing force V_x in case of a beam is defined as follows:

$$V_x = \int_{y=-\frac{b}{2}}^{y=\frac{b}{2}} \int_{z=-\frac{h}{2}}^{z=\frac{h}{2}} \tau_{zx} \, dz \, dy \tag{12}$$

Utilizing expression for the beam transverse shear stress τ_{zx} as given by eq. (9) into eq. (12), expression for V_x of the present theory is obtained.

$$V_x = \frac{5\,E\,A}{12\,(1+\mu)} \frac{\partial w_s}{\partial x} \tag{13}$$

where $A = b\,h$ is the beam cross-sectional area (Fig. 1).

4. The Present Theory: Governing Differential Equations and Boundary Conditions

In general, the kinetic energy of a shear deformable beam is expressed as follows:

$$\Pi_{\text{Kinetic}} = \int_{x=0}^{x=L} \int_{y=-\frac{b}{2}}^{y=\frac{b}{2}} \int_{z=-\frac{h}{2}}^{z=\frac{h}{2}} \frac{1}{2} \left[\rho \left(\frac{\partial u}{\partial t}\right)^2 + \rho \left(\frac{\partial w}{\partial t}\right)^2 \right] dz\, dy\, dx \tag{14}$$

Utilizing expressions for beam displacement functions as given by eqs. (2) and (3) into eq. (14), expression for Π_{Kinetic} of the present theory is obtained.

$$\begin{aligned} \Pi_{\text{Kinetic}} = {} & \frac{\rho\,A}{2} \int_{x=0}^{x=L} \left(\frac{\partial w_b}{\partial t}\right)^2 dx + \frac{\rho\,I}{2} \int_{x=0}^{x=L} \left[\frac{\partial}{\partial t}\left(\frac{\partial w_b}{\partial x}\right)\right]^2 dx + \frac{\rho\,A}{2} \int_{x=0}^{x=L} \left(\frac{\partial w_s}{\partial t}\right)^2 dx \\ & + \frac{\rho\,I}{168} \int_{x=0}^{x=L} \left[\frac{\partial}{\partial t}\left(\frac{\partial w_s}{\partial x}\right)\right]^2 dx + \rho\,A \int_{x=0}^{x=L} \left(\frac{\partial w_b}{\partial t}\right)\left(\frac{\partial w_s}{\partial t}\right) dx \end{aligned} \tag{15}$$

In general, the strain energy of a shear deformable beam is expressed as follows:

$$\Pi_{\text{Strain}} = \int_{x=0}^{x=L} \int_{y=-\frac{b}{2}}^{y=\frac{b}{2}} \int_{z=-\frac{h}{2}}^{z=\frac{h}{2}} \frac{1}{2} \left[\sigma_x\, \epsilon_x + \tau_{zx}\, \gamma_{zx}\right] dz\, dy\, dx \tag{16}$$

Utilizing expressions for strains as given by eqs. (5), (6) and expressions for stresses as given by eqs. (8), (9) into eq. (16), the expression for Π_{Strain} of the present theory is obtained.

$$\Pi_{\text{Strain}} = \frac{E\,I}{2} \int_{x=0}^{x=L} \left(\frac{\partial^2 w_b}{\partial x^2}\right)^2 dx + \frac{E\,I}{168} \int_{x=0}^{x=L} \left(\frac{\partial^2 w_s}{\partial x^2}\right)^2 dx + \frac{5\,E\,A}{24\,(1+\mu)} \int_{x=0}^{x=L} \left(\frac{\partial w_s}{\partial x}\right)^2 dx \tag{17}$$

In general, the work performed by a distributed transverse load applied on a shear deformable beam is expressed as follows:

$$\Pi_{\text{External work}} = \int_{x=0}^{x=L} q\, w\, dx \tag{18}$$

Utilizing expression for the beam transverse displacement as given by eq. (3) into eq. (18), expression for $\Pi_{\text{External work}}$ of the present theory is obtained.

$$\Pi_{\text{External work}} = \int_{x=0}^{x=L} q\, w_b\, dx + \int_{x=0}^{x=L} q\, w_s\, dx \tag{19}$$

To obtain governing equations along with boundary conditions of the present theory, the Hamilton's principle (Reddy [55]) is utilized.
For a dynamical system, the Lagrangian (L) is stated mathematically as follows:

$$L = \Pi_{\text{Kinetic}} - \Pi_{\text{Strain}} + \Pi_{\text{External work}} \tag{20}$$

Mathematical statement of the Hamilton's principle is as follows:

$$\delta \int_{t=t_1}^{t=t_2} L\, dt = 0 \tag{21}$$

with δ denoting the variational operator.

4.1. Governing differential equations

The bending component w_b and shearing component w_s are the primary unknowns in the present theory. In order to derive governing differential equations of the present theory, eqs. (15), (17) and (19) are substituted in eq. (21) and then the standard calculus of variation procedure is performed as per Reddy [55].
In the present theory, the first governing equation is obtained by collecting coefficients of δw_b which is as follows:

$$E\, I \frac{\partial^4 w_b}{\partial x^4} - \rho\, I \left[\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 w_b}{\partial x^2}\right)\right] + \rho\, A \frac{\partial^2 w_b}{\partial t^2} + \rho\, A \frac{\partial^2 w_s}{\partial t^2} = q \tag{22}$$

Similarly, the second governing equation is obtained by collecting coefficients of δw_s which is as follows:

$$\frac{E\, I}{84}\frac{\partial^4 w_s}{\partial x^4} - \frac{\rho\, I}{84}\left[\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 w_s}{\partial x^2}\right)\right] - \frac{5\, E\, A}{12\,(1+\mu)}\frac{\partial^2 w_s}{\partial x^2} + \rho\, A \frac{\partial^2 w_b}{\partial t^2} + \rho\, A \frac{\partial^2 w_s}{\partial t^2} = q \tag{23}$$

Following points should be noted with regard to governing differential equations of this theory:
1. Both governing equations are of fourth order and involve only two unknown functions (i.e., bending component w_b of the beam transverse displacement w, shearing component w_s of the beam transverse displacement w).
2. For dynamics problems, eqs. (22) and (23) are only inertially coupled. For statics, eqs. (22) and (23) are decoupled.
3. Governing differential equations applicable for statics problems are obtained by omitting terms involving derivatives of w_b and w_s with respect to time. For statics problems, governing differential equation as given by eq. (22) reduces to the form which is same as governing differential equation of the Bernoulli-Euler beam theory (eq. (6-7) of Timoshenko and Gere [2]), except for the presence of the subscript in eq. (22).

4.2. Boundary conditions

While obtaining governing differential equations by utilizing the Hamilton's principle, variationally consistent boundary conditions of the present theory are obtained as a by-product (Reddy [55]).
At beam ends $x = 0$ and $x = L$, boundary conditions of the present theory are prescribed as follows:
1. Boundary conditions involving the bending component w_b:

$$E\, I \frac{\partial^3 w_b}{\partial x^3} = 0 \quad \text{or} \quad w_b \text{ is specified} \tag{24}$$

$$E\, I \frac{\partial^2 w_b}{\partial x^2} = 0 \quad \text{or} \quad \frac{\partial w_b}{\partial x} \text{ is specified} \tag{25}$$

2. Boundary conditions involving the shearing component w_s:

$$\frac{5\, E\, A}{12\,(1+\mu)}\frac{\partial w_s}{\partial x} - \frac{E\, I}{84}\frac{\partial^3 w_s}{\partial x^3} = 0 \quad \text{or} \quad w_s \text{ is specified} \tag{26}$$

$$\frac{E\, I}{84}\frac{\partial^2 w_s}{\partial x^2} = 0 \quad \text{or} \quad \frac{\partial w_s}{\partial x} \text{ is specified} \tag{27}$$

Furthermore, boundary conditions as given by eqs. (24) through (27) have been categorized into geometric/essential boundary conditions and natural/forced boundary conditions by utilizing the approach as discussed by Reddy [55].

1. Geometric/essential boundary conditions:
   (a) Geometric boundary conditions involving the bending component w_b:

$$w_b \text{ is specified} \tag{28}$$

$$\frac{\partial w_b}{\partial x} \text{ is specified} \tag{29}$$

   (b) Geometric boundary conditions involving the shearing component w_s:

$$w_s \text{ is specified} \tag{30}$$

$$\frac{\partial w_s}{\partial x} \text{ is specified} \tag{31}$$

2. Natural/forced boundary conditions:
   (a) Natural boundary conditions involving the bending component w_b:

$$E\,I\,\frac{\partial^2 w_b}{\partial x^2} = 0 \tag{32}$$

$$E\,I\,\frac{\partial^3 w_b}{\partial x^3} = 0 \tag{33}$$

   (b) Natural boundary conditions involving the shearing component w_s:

$$\frac{5\,E\,A}{12\,(1+\mu)}\,\frac{\partial w_s}{\partial x} - \frac{E\,I}{84}\frac{\partial^3 w_s}{\partial x^3} = 0 \tag{34}$$

$$\frac{E\,I}{84}\,\frac{\partial^2 w_s}{\partial x^2} = 0 \tag{35}$$

This knowledge about boundary conditions is essential especially when analytical solution of a beam problem by utilizing the present theory is difficult to obtain and the use of numerical solution techniques such as Rayleigh-Ritz method or Galerkin method is required to solve such a problem. Following points should be noted with regard to these numerical methods in context of the present theory:

1. The Rayleigh-Ritz method is an 'energy formulation' based numerical solution technique where one operates on the energy functional corresponding to the shear deformation beam theory (Bathe [51]). In other words, it is not essential to have the knowledge of governing differential equations of the beam theory for utilizing this method.
   If one envisages to utilize the Rayleigh-Ritz method to solve a shear deformable beam problem by utilizing the present theory, primary unknowns of the present theory (i.e., the bending component and shearing component of the beam transverse displacement) can be assumed as sums of the products of linearly independent trial functions and unknown multipliers to be determined, over entire domain of the beam (Bathe [51]). Linearly independent trial functions chosen must satisfy only problem-specific geometric/essential boundary conditions and may or may not satisfy natural/forced boundary conditions of this theory. The remaining procedure for determining unknown multipliers by utilizing Rayleigh-Ritz method is same as has been discussed by Bathe [51].
2. The Galerkin method is a 'governing differential equations' based numerical solution technique where one operates on governing differential equations of the shear deformation beam theory (Bathe [51]). In other words, it is not essential to have the knowledge of the energy functional of the beam theory for utilizing this method.
   If one envisages to utilize the Galerkin method to solve a shear deformable beam problem by utilizing the present theory, primary unknowns of the present theory (i.e., the bending component and shearing component of the beam transverse displacement) can be assumed as sums of the products of linearly independent trial functions and unknown multipliers to be determined, over entire domain of the beam (Bathe [51]). Linearly independent trial functions chosen must satisfy problem-specific geometric/essential boundary conditions as well as natural/forced boundary conditions of this theory. The remaining procedure for determining unknown multipliers by utilizing Galerkin method is same as has been discussed by Bathe [51].

Following points should be noted with regard to boundary conditions of this theory:

1. Since eqs. (22) and (23) govern a boundary value problem and both are differential equations of the fourth order,

each of these equations will require specification of two boundary conditions per beam end in order to solve a beam problem. In other words at each end of the beam, it is essential to prescribe four boundary conditions in this theory.

(a) Two of these boundary conditions are prescribed only in terms of the bending component w_b and its derivatives (e.g., conditions (24) and (25) prescribed at the beam end $x = 0$).

(b) Remaining two boundary conditions are prescribed only in terms of the shearing component w_s and its derivatives (e.g., conditions (26) and (27) prescribed at the beam end $x = 0$).

2. Boundary conditions as given by eqs. (24) and (25) are same as corresponding boundary conditions of the Bernoulli-Euler beam theory, except for the presence of the subscript in eqs. (24) and (25).

This completes the formulation of the present theory. Next section describes illustrative examples in order to demonstrate the efficacy of the present theory.

5. Illustrative Examples

Illustrative examples pertaining to the static bending and free vibrations of shear deformable isotropic prismatic rectangular beams will be discussed in this section.

- Illustrative examples for the static bending of the beam

Illustrative examples pertaining to the static bending of simply-supported, clamped-clamped and cantilever shear deformable isotropic prismatic rectangular beams (examples 1 through 3 respectively) will now be presented.

In examples 1 through 3, the geometry and material properties of the beam considered are as discussed in the section 2.1.(a). On the beam surface $z = - h / 2$, the beam is acted upon by a uniformly distributed transverse load of intensity q_o. This load acts in the positive z-direction over the entire beam span. Hence governing equations as given by eqs. (22) and (23) take the following form:

$$E\,I\,\frac{\partial^4 w_b}{\partial x^4} = q_o \tag{36}$$

$$\frac{E\,I}{84}\,\frac{\partial^4 w_s}{\partial x^4} - \frac{5\,E\,A}{12\,(1+\mu)}\,\frac{\partial^2 w_s}{\partial x^2} = q_o \tag{37}$$

Following points should be noted with regard to the beam under consideration in examples 1 through 3:

1. The general solution for the bending component w_b of the beam transverse displacement w:

The general solution of the governing equation as given by eq. (36) is as follows:

$$w_b = \frac{q_o\,L^4}{24\,E\,I}\left(\frac{x}{L}\right)^4 + C_{1b}\left(\frac{x}{L}\right)^3 + C_{2b}\left(\frac{x}{L}\right)^2 + C_{3b}\left(\frac{x}{L}\right) + C_{4b} \tag{38}$$

where C_{1b}, C_{2b}, C_{3b} and C_{4b} in eq. (38) are example-specific integration constants whose values can be obtained by application of boundary conditions pertaining to w_b at beam ends $x = 0$ and $x = L$.

2. The general solution for the shearing component w_s of the beam transverse displacement w:

The general solution of the governing equation as given by eq. (37) is as follows:

$$w_s = -\frac{6\,q_o\,L^2\,(1+\mu)}{5\,E\,A}\left(\frac{x}{L}\right)^2 + C_{1s}\,\alpha\,\exp\left[-\sqrt{\frac{1}{\alpha}}\left(\frac{x}{L}\right)\right] + C_{2s}\,\alpha\,\exp\left[\sqrt{\frac{1}{\alpha}}\left(\frac{x}{L}\right)\right] + C_{3s}\left(\frac{x}{L}\right) + C_{4s} \tag{39}$$

with a non-dimensional entity α defined as follows:

$$\alpha = \frac{(1+\mu)}{420}\left(\frac{h}{L}\right)^2 \tag{40}$$

where C_{1s}, C_{2s}, C_{3s} and C_{4s} in eq. (39) are example-specific integration constants whose values can be obtained by application of boundary conditions pertaining to w_s at beam ends $x = 0$ and $x = L$.

3. The general solution for the beam transverse displacement w:

It is possible to obtain the general solution for w by substituting general solutions for w_b and w_s as given by eqs. (38) and (39) respectively in eq. (3).

Example 1: Simply-supported beam (*SS* beam) carrying a uniformly distributed transverse load

Beam ends $x = 0$ and $x = L$ are simply-supported in this example. The beam boundary conditions pertaining to w_b and

w_s for the *SS* beam are as follows:

1. Boundary conditions at beam end $x = 0$:
   (a) Boundary conditions pertaining to w_b:

$$\left[w_b\right]_{x=0} = 0 \quad \text{and} \quad \left[E\,I\,\frac{\partial^2 w_b}{\partial x^2}\right]_{x=0} = 0 \tag{41}$$

   (b) Boundary conditions pertaining to w_s:

$$\left[w_s\right]_{x=0} = 0 \quad \text{and} \quad \left[\frac{E\,I}{84}\,\frac{\partial^2 w_s}{\partial x^2}\right]_{x=0} = 0 \tag{42}$$

2. Boundary conditions at beam end $x = L$:
   (a) Boundary conditions pertaining to w_b:

$$\left[w_b\right]_{x=L} = 0 \quad \text{and} \quad \left[E\,I\,\frac{\partial^2 w_b}{\partial x^2}\right]_{x=L} = 0 \tag{43}$$

   (b) Boundary conditions pertaining to w_s:

$$\left[w_s\right]_{x=L} = 0 \quad \text{and} \quad \left[\frac{E\,I}{84}\,\frac{\partial^2 w_s}{\partial x^2}\right]_{x=L} = 0 \tag{44}$$

Integration constants C_{1b} through C_{4b} present in the general solution of w_b as given by eq. (38) can be obtained by substituting eq. (38) in conditions (41) and (43), followed by solving the obtained set of simultaneous linear equations. The obtained expression for w_b in case of the *SS* beam is as follows:

$$w_b = \frac{q_o\,L^4}{24\,E\,I}\left[\left(\frac{x}{L}\right)^4 - 2\left(\frac{x}{L}\right)^3 + \left(\frac{x}{L}\right)\right] \tag{45}$$

Integration constants C_{1s} through C_{4s} present in the general solution of w_s as given by eq. (39) can be obtained by substituting eq. (39) in conditions (42) and (44), followed by solving the obtained set of simultaneous linear equations. The obtained expression for w_s in case of the *SS* beam is as follows:

$$w_s = -\frac{6\,q_o\,L^2\,(1+\mu)}{5\,E\,A}\left\{\left(\frac{x}{L}\right)^2 - \frac{2\,\alpha\,\exp\left[\sqrt{\frac{1}{\alpha}}\left(1-\frac{x}{L}\right)\right]}{\left\{1 + \exp\left[\sqrt{\frac{1}{\alpha}}\right]\right\}} - \frac{2\,\alpha\,\exp\left[\sqrt{\frac{1}{\alpha}}\left(\frac{x}{L}\right)\right]}{\left\{1 + \exp\left[\sqrt{\frac{1}{\alpha}}\right]\right\}} - \left(\frac{x}{L}\right) + 2\,\alpha\right\} \tag{46}$$

To obtain expression for the beam transverse displacement w for *SS* beam, resulting expressions for w_b and w_s for the *SS* beam as given by eqs. (45) and (46) respectively are substituted back in eq. (3).
For the *SS* beam, w is maximum at $x = L / 2$. The expression for maximum w for the *SS* beam arrived at by utilizing the present theory is as follows:

$$\left[w\right]_{x=L/2} = \frac{5\,q_o\,L^4}{384\,E\,I} - \frac{6\,q_o\,L^2\,(1+\mu)}{5\,E\,A}\left\{-\frac{4\,\alpha\,\exp\left[\sqrt{\frac{1}{4\,\alpha}}\right]}{\left\{1 + \exp\left[\sqrt{\frac{1}{\alpha}}\right]\right\}} - \frac{1}{4} + 2\,\alpha\right\} \tag{47}$$

Numerical results associated with the static bending of the *SS* beam are tabulated in Tables 1 through 3 and are presented in section titled 'Numerical results'.

Example 2: Clamped-clamped beam (*CC* beam) carrying a uniformly distributed transverse load

Beam ends $x = 0$ and $x = L$ are clamped in this example. The beam boundary conditions pertaining to w_b and w_s for the *CC* beam are as follows:

1. Boundary conditions at beam end $x = 0$:
   (a) Boundary conditions pertaining to w_b:

$$[w_b]_{x=0} = 0 \quad \text{and} \quad \left[\frac{\partial w_b}{\partial x}\right]_{x=0} = 0 \tag{48}$$

(b) Boundary conditions pertaining to w_s:

$$[w_s]_{x=0} = 0 \quad \text{and} \quad \left[\frac{EI}{84}\frac{\partial^2 w_s}{\partial x^2}\right]_{x=0} = 0 \tag{49}$$

2. Boundary conditions at beam end $x = L$:
   (a) Boundary conditions pertaining to w_b:

$$[w_b]_{x=L} = 0 \quad \text{and} \quad \left[\frac{\partial w_b}{\partial x}\right]_{x=L} = 0 \tag{50}$$

   (b) Boundary conditions pertaining to w_s:

$$[w_s]_{x=L} = 0 \quad \text{and} \quad \left[\frac{EI}{84}\frac{\partial^2 w_s}{\partial x^2}\right]_{x=L} = 0 \tag{51}$$

Integration constants C_{1b} through C_{4b} present in the general solution of w_b as given by eq. (38) can be obtained by substituting eq. (38) in conditions (48) and (50), followed by solving the obtained set of simultaneous linear equations. The obtained expression for w_b in case of the *CC* beam is as follows:

$$w_b = \frac{q_o L^4}{24 EI}\left[\left(\frac{x}{L}\right)^4 - 2\left(\frac{x}{L}\right)^3 + \left(\frac{x}{L}\right)^2\right] \tag{52}$$

Integration constants C_{1s} through C_{4s} present in the general solution of w_s as given by eq. (39) can be obtained by substituting eq. (39) in conditions (49) and (51), followed by solving the obtained set of simultaneous linear equations. The obtained expression for w_s in case of the *CC* beam is as follows:

$$w_s = -\frac{6 q_o L^2 (1+\mu)}{5EA}\left\{\left(\frac{x}{L}\right)^2 - \frac{2\alpha \exp\left[\sqrt{\frac{1}{\alpha}}\left(1-\frac{x}{L}\right)\right]}{\left\{1+\exp\left[\sqrt{\frac{1}{\alpha}}\right]\right\}} - \frac{2\alpha \exp\left[\sqrt{\frac{1}{\alpha}}\left(\frac{x}{L}\right)\right]}{\left\{1+\exp\left[\sqrt{\frac{1}{\alpha}}\right]\right\}} - \left(\frac{x}{L}\right) + 2\alpha\right\} \tag{53}$$

To obtain expression for the beam transverse displacement w for *CC* beam, resulting expressions for w_b and w_s for the *CC* beam as given by eqs. (52) and (53) respectively are substituted back in eq. (3).
For the *CC* beam, w is maximum at $x = L / 2$. The expression for maximum w for the *CC* beam arrived at by utilizing the present theory is as follows:

$$[w]_{x=L/2} = \frac{q_o L^4}{384 EI} - \frac{6 q_o L^2 (1+\mu)}{5EA}\left\{-\frac{4\alpha \exp\left[\sqrt{\frac{1}{4\alpha}}\right]}{\left\{1+\exp\left[\sqrt{\frac{1}{\alpha}}\right]\right\}} - \frac{1}{4} + 2\alpha\right\} \tag{54}$$

Numerical results associated with the static bending of the *CC* beam are tabulated in Tables 4 through 6 and are presented in section titled 'Numerical results'.

Example 3: Cantilever beam (*FC* beam) carrying a uniformly distributed transverse load

The beam end $x = 0$ is free and beam end $x = L$ is clamped in this example. The beam boundary conditions pertaining to w_b and w_s for the *FC* beam are as follows:

1. Boundary conditions at beam end $x = 0$:
   (a) Boundary conditions pertaining to w_b:

$$\left[EI\frac{\partial^2 w_b}{\partial x^2}\right]_{x=0} = 0 \quad \text{and} \quad \left[EI\frac{\partial^3 w_b}{\partial x^3}\right]_{x=0} = 0 \tag{55}$$

(b) Boundary conditions pertaining to w_s:

$$\left[\frac{\partial w_s}{\partial x}\right]_{x=0} = 0 \quad \text{and} \quad \left[\frac{5\,E\,A}{12\,(1+\mu)}\frac{\partial w_s}{\partial x} - \frac{E\,I}{84}\frac{\partial^3 w_s}{\partial x^3}\right]_{x=0} = 0 \tag{56}$$

2. Boundary conditions at beam end $x = L$:
 (a) Boundary conditions pertaining to w_b:

$$\left[w_b\right]_{x=L} = 0 \quad \text{and} \quad \left[\frac{\partial w_b}{\partial x}\right]_{x=L} = 0 \tag{57}$$

 (b) Boundary conditions pertaining to w_s:

$$\left[w_s\right]_{x=L} = 0 \quad \text{and} \quad \left[\frac{E\,I}{84}\frac{\partial^2 w_s}{\partial x^2}\right]_{x=L} = 0 \tag{58}$$

Integration constants C_{1b} through C_{4b} present in the general solution of w_b as given by eq. (38) can be obtained by substituting eq. (38) in conditions (55) and (57), followed by solving the obtained set of simultaneous linear equations. The obtained expression for w_b in case of the *FC* beam is as follows:

$$w_b = \frac{q_o\,L^4}{24\,E\,I}\left[\left(\frac{x}{L}\right)^4 - 4\left(\frac{x}{L}\right) + 3\right] \tag{59}$$

Integration constants C_{1s} through C_{4s} present in the general solution of w_s as given by eq. (39) can be obtained by substituting eq. (39) in conditions (56) and (58), followed by solving the obtained set of simultaneous linear equations. The obtained expression for w_s in case of the *FC* beam is as follows:

$$w_s = -\frac{6\,q_o\,L^2\,(1+\mu)}{5\,E\,A}\left\{\left(\frac{x}{L}\right)^2 - \frac{2\,\alpha\,\exp\left[\sqrt{\frac{1}{\alpha}}\left(1-\frac{x}{L}\right)\right]}{\left\{1+\exp\left[\sqrt{\frac{4}{\alpha}}\right]\right\}} - \frac{2\,\alpha\,\exp\left[\sqrt{\frac{1}{\alpha}}\left(1+\frac{x}{L}\right)\right]}{\left\{1+\exp\left[\sqrt{\frac{4}{\alpha}}\right]\right\}} - 1 + 2\,\alpha\right\} \tag{60}$$

To obtain expression for the beam transverse displacement w for *FC* beam, resulting expressions for w_b and w_s for the *FC* beam as given by eqs. (59) and (60) respectively are substituted back in eq. (3).
For the *FC* beam, w is maximum at $x = 0$. The expression for maximum w for the *FC* beam arrived at by utilizing the present theory is as follows:

$$\left[w\right]_{x=0} = \frac{q_o\,L^4}{8\,E\,I} - \frac{6\,q_o\,L^2\,(1+\mu)}{5\,E\,A}\left\{-\frac{4\,\alpha\,\exp\left[\sqrt{\frac{1}{\alpha}}\right]}{\left\{1+\exp\left[\sqrt{\frac{4}{\alpha}}\right]\right\}} - 1 + 2\,\alpha\right\} \tag{61}$$

Numerical results associated with the static bending of the *FC* beam are tabulated in Tables 7 through 9 and are presented in section titled 'Numerical results'.

- Illustrative example for free vibrations of the beam

Illustrative example pertaining to free vibrations of the simply-supported shear deformable isotropic prismatic rectangular beam (example 4) will now be presented.
In example 4, the geometry and material properties of the beam considered are as discussed in the section 2.1.(a). As the beam is in the state of free vibrations (i.e., when $q(x, t)$ is zero), governing equations as given by eqs. (22) and (23) reduce to the following form:

$$E\,I\,\frac{\partial^4 w_b}{\partial x^4} - \rho\,I\left[\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 w_b}{\partial x^2}\right)\right] + \rho\,A\,\frac{\partial^2 w_b}{\partial t^2} + \rho\,A\,\frac{\partial^2 w_s}{\partial t^2} = 0 \tag{62}$$

$$\frac{E\,I}{84}\frac{\partial^4 w_s}{\partial x^4} - \frac{\rho\,I}{84}\left[\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 w_s}{\partial x^2}\right)\right] - \frac{5\,E\,A}{12\,(1+\mu)}\frac{\partial^2 w_s}{\partial x^2} + \rho\,A\,\frac{\partial^2 w_b}{\partial t^2} + \rho\,A\,\frac{\partial^2 w_s}{\partial t^2} = 0 \tag{63}$$

Example 4: Free vibrations of simply-supported beam (*SS* beam)

For the *SS* beam, boundary conditions at beam ends $x = 0$ and $x = L$ are the same as boundary conditions utilized in the example 1 and are given by eqs. (41) through (44).
Following functions are assumed for the bending component w_b of the beam transverse displacement w and shearing component w_s of the beam transverse displacement w which satisfy boundary conditions (41) through (44):

$$\begin{Bmatrix} w_b \\ w_s \end{Bmatrix} = \begin{Bmatrix} W_{bn} \\ W_{sn} \end{Bmatrix} \sin(\omega_n t) \sin\left(\frac{n \pi x}{L}\right) \tag{64}$$

where W_{bn} and W_{sn} are constants, n is the beam vibration mode parameter and can take values $n = 1, 2, 3\ldots$ and ω_n is the circular frequency of the beam vibration associated with the mode parameter n.
Substituting assumed expressions for w_b and w_s as given by eq. (64) in governing equations (62) and (63), following equations are obtained:

$$\left[\begin{bmatrix} k_{11n} & k_{12n} \\ k_{21n} & k_{22n} \end{bmatrix} - (\omega_n)^2 \begin{bmatrix} m_{11n} & m_{12n} \\ m_{21n} & m_{22n} \end{bmatrix}\right] \begin{Bmatrix} W_{bn} \\ W_{sn} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{65a}$$

where

$$\begin{aligned} k_{11n} &= \frac{n^4 \pi^4 E I}{L^4}, \quad k_{12n} = k_{21n} = 0, \quad k_{22n} = \frac{n^4 \pi^4 E I}{84 L^4} + \frac{5 n^2 \pi^2 E A}{12(1+\mu) L^2} \\ m_{11n} &= \frac{n^2 \pi^2 \rho I}{L^2} + \rho A, \quad m_{12n} = m_{21n} = \rho A, \quad m_{22n} = \frac{n^2 \pi^2 \rho I}{84 L^2} + \rho A \end{aligned} \tag{65b}$$

It should be noted that eq. (65) is of the standard eigenvalue form. Free vibration frequencies of the *SS* beam are obtained by solving eq. (65). Numerical results associated with free vibrations of *SS* beam are tabulated in Tables 10 through 12 and are presented in section titled 'Numerical results'.

6. Numerical Results

In this section, numerical results in tabular form pertaining to the static bending and free vibrations of shear deformable isotropic prismatic rectangular beams are presented.

6.1. Numerical results for static bending of the beam (examples 1 through 3)

The non-dimensional beam transverse displacement $\overline{w}$, non-dimensional beam axial stress $\overline{\sigma}_x$ and non-dimensional beam transverse shear stress $\overline{\tau}_{zx}$ are defined as follows:

$$\overline{w} = \frac{E I}{q_o L^4} w \tag{66}$$

$$\overline{\sigma}_x = \frac{b}{q_o} \sigma_x \tag{67}$$

$$\overline{\tau}_{zx} = \frac{b}{q_o} \tau_{zx} \tag{68}$$

Numerical results pertaining to $\overline{w}$, $\overline{\sigma}_x$ and $\overline{\tau}_{zx}$ for examples 1 through 3 obtained by utilizing the present theory for various values of beam thickness-to-length ratio h / L are presented in Tables 1 through 9. These results are compared with corresponding results obtained by utilizing the two-dimensional theory of elasticity (plane stress), single variable beam theory, Levinson beam theory, Timoshenko beam theory and Bernoulli-Euler beam theory so as to display the efficacy of the proposed theory.
While obtaining results which are presented in Tables 1 through 9, following points are considered:

1. The Poisson's ratio μ is assumed to be 0.3.
2. The beam thickness-to-length ratio h / L is assumed to have values 0.01, 0.05, 0.10 and 0.15.
3. Numerical results for $\overline{w}$ by the Levinson beam theory and single variable beam theory, numerical results for $\overline{\sigma}_x$

and $\overline{\tau}_{zx}$ by the single variable beam theory are as reported by Shimpi et al. [4].

4. Numerical results for $\overline{w}$, $\overline{\sigma}_x$ and $\overline{\tau}_{zx}$ obtained by utilizing the two-dimensional theory of elasticity (plane stress), Timoshenko beam theory and Bernoulli-Euler beam theory have been calculated by the present authors.
(a) For this purpose, expressions available in corresponding references reported in tables are utilized.
(b) For calculating results by utilizing the Timoshenko beam theory, the shear correction factor is taken as 5 / 6.
5. In Tables 1 through 9, percentage difference quoted in parentheses against results reported by the particular theory (either by the present theory, single variable beam theory, Levinson beam theory, Timoshenko beam theory or Bernoulli-Euler beam theory, as the case may be) is calculated as follows:

$$\%\ \text{difference} = \left(\frac{\text{The result reported by the particular theory}}{\text{The corresponding result of two-dimensional theory of elasticity (plane stress)}} - 1 \right) \times 100 \tag{69}$$

Table 1. Non-dimensional transverse displacement ($\overline{w}$) pertaining to example 1 (*SS* beam, Fig. 1) obtained by utilizing the present theory and its comparison, $\mu = 0.3$.

	Non-dimensional transverse displacement at $x = L/2$, $\overline{w} = w E I / (q_o L^4)$ (Values in parentheses indicate the percentage difference)$			
Theory	$h/L = 0.01$	$h/L = 0.05$	$h/L = 0.10$	$h/L = 0.15$
Present	0.01302 (0.00 %)	0.01310 (0.00 %)	0.01335 (0.23 %)	0.01375 (0.44 %)
Bernoulli-Euler [3]	0.01302 (0.00 %)	0.01302 (-0.61 %)	0.01302 (-2.25 %)	0.01302 (-4.89 %)
Timoshenko [3]	0.01302 (0.00 %)	0.01310 (0.00 %)	0.01335 (0.23 %)	0.01375 (0.44 %)
Levinson [4]	0.01302 (0.00 %)	0.01310 (0.00 %)	0.01335 (0.23 %)	—
Single variable theory [4]	0.01302 (0.00 %)	0.01310 (0.00 %)	0.01335 (0.23 %)	—
Theory of elasticity [3]	0.01302	0.01310	0.01332	0.01369

$ % difference is calculated by utilizing eq. (69).
— represents unavailability of the result in the cited reference.

Table 2. Non-dimensional axial stress ($\overline{\sigma}_x$) pertaining to example 1 (*SS* beam, Fig. 1) obtained by utilizing the present theory and its comparison, $\mu = 0.3$.

	Non-dimensional axial stress at $x = L/2$, $z = h/2$, $\overline{\sigma}_x = \sigma_x b / q_0$ (Values in parentheses indicate the percentage difference)$			
Theory	$h/L = 0.01$	$h/L = 0.05$	$h/L = 0.10$	$h/L = 0.15$
Present	7500.26 (0.00 %)	300.26 (0.02 %)	75.26 (0.08 %)	33.59 (0.18 %)
Bernoulli-Euler [3]	7500.00 (0.00 %)	300.00 (-0.07 %)	75.00 (-0.27 %)	33.33 (-0.60 %)
Timoshenko [3]	7500.00 (0.00 %)	300.00 (-0.07 %)	75.00 (-0.27 %)	33.33 (-0.60 %)
Single variable theory [4]	7500.00 (0.00 %)	300.26 (0.02 %)	75.26 (0.08 %)	—
Theory of elasticity [3]	7500.20	300.20	75.20	33.53

$ % difference is calculated by utilizing eq. (69).
— represents unavailability of the result in the cited reference.

Table 3. Non-dimensional transverse shear stress ($\overline{\tau}_{zx}$) pertaining to example 1 (*SS* beam, Fig. 1) obtained by utilizing the present theory and its comparison, $\mu = 0.3$.

	Non-dimensional transverse shear stress at $x = 0$, $z = 0$, $\overline{\tau}_{zx} = \tau_{zx} b / q_0$ (Values in parentheses indicate the percentage difference)$			
Theory	$h/L = 0.01$	$h/L = 0.05$	$h/L = 0.10$	$h/L = 0.15$
Present	74.92 (-0.11 %)	14.92 (-0.53 %)	7.42 (-1.07 %)	4.92 (-1.60 %)
Bernoulli-Euler [3]	75.00* (0.00 %)	15.00* (0.00 %)	7.50* (0.00 %)	5.00* (0.00 %)
Timoshenko [3]	50.00# (-33.33 %)	10.00# (-33.33 %)	5.00# (-33.33 %)	3.33# (-33.33 %)
Single variable theory [4]	75.00 (0.00 %)	15.00 (0.00 %)	7.50 (0.00 %)	—
Theory of elasticity [3]	75.00	15.00	7.50	5.00

$ % difference is calculated by utilizing eq. (69).

* As the Bernoulli-Euler beam theory neglects the effects of beam transverse shear deformation, it is not possible to obtain $\bar{\tau}_{zx}$ by utilizing a constitutive relation between the shear stress and shear strain. For the Bernoulli-Euler beam theory, $\bar{\tau}_{zx}$ has been obtained by utilizing the equilibrium approach (Shames and Dym [3]).

\# For the Timoshenko beam theory, $\bar{\tau}_{zx}$ remains constant through the beam thickness because it is a first-order theory. Whereas for the present theory, $\bar{\tau}_{zx}$ remains zero at beam surfaces $z = \pm h/2$ and varies quadratically through the beam thickness.

— represents unavailability of the result in the cited reference.

Table 4. Non-dimensional transverse displacement ($\bar{w}$) pertaining to example 2 (*CC* beam, Fig. 1) obtained by utilizing the present theory and its comparison, $\mu = 0.3$.

	Non-dimensional transverse displacement at $x = L/2$, $\bar{w} = w E I/(q_o L^4)$ (Values in parentheses indicate the percentage difference)$			
Theory	$h/L = 0.01$	$h/L = 0.05$	$h/L = 0.10$	$h/L = 0.15$
Present	0.00261 (0.00 %)	0.00269 (-0.74 %)	0.00293 (-2.66 %)	0.00334 (-5.11 %)
Bernoulli-Euler [2, 3]	0.00260 (-0.38 %)	0.00260 (-4.06 %)	0.00260 (-13.62 %)	0.00260 (-26.14 %)
Timoshenko [3, 21]	0.00261 (0.00 %)	0.00269 (-0.74 %)	0.00293 (-2.66 %)	0.00334 (-5.11 %)
Levinson [4]	0.00261 (0.00 %)	0.00271 (0.00 %)	0.00301 (0.00 %)	—
Single variable theory [4]	0.00261 (0.00 %)	0.00271 (0.00 %)	0.00301 (0.00 %)	—
Theory of elasticity [56]	0.00261	0.00271	0.00301	0.00352

$ % difference is calculated by utilizing eq. (69).

— represents unavailability of the result in the cited reference.

Table 5. Non-dimensional axial stress ($\bar{\sigma}_x$) pertaining to example 2 (*CC* beam, Fig. 1) obtained by utilizing the present theory and its comparison, $\mu = 0.3$.

	Non-dimensional axial stress at $x = 0$, $z = h/2$, $\bar{\sigma}_x = \sigma_x b/q_0$ (Values in parentheses indicate the percentage difference)$			
Theory	$h/L = 0.01$	$h/L = 0.05$	$h/L = 0.10$	$h/L = 0.15$
Present	-5000.00 (0.01 %)	-200.00 (0.29 %)	-50.00 (1.15 %)	-22.22 (2.63 %)
Bernoulli-Euler [2, 3]	-5000.00 (0.01 %)	-200.00 (0.29 %)	-50.00 (1.15 %)	-22.22 (2.63 %)
Timoshenko [3, 21]	-5000.00 (0.01 %)	-200.00 (0.29 %)	-50.00 (1.15 %)	-22.22 (2.63 %)
Single variable theory [4]	-4999.35 (0.00 %)	-199.35 (-0.04 %)	-49.35 (-0.16 %)	—
Theory of elasticity [56]	-4999.43	-199.43	-49.43	-21.65

$ % difference is calculated by utilizing eq. (69).

— represents unavailability of the result in the cited reference.

Table 6. Non-dimensional transverse shear stress ($\bar{\tau}_{zx}$) pertaining to example 2 (*CC* beam, Fig. 1) obtained by utilizing the present theory and its comparison, $\mu = 0.3$.

	Non-dimensional transverse shear stress at $x = 0$, $z = 0$, $\bar{\tau}_{zx} = \tau_{zx} b/q_0$ (Values in parentheses indicate the percentage difference)$			
Theory	$h/L = 0.01$	$h/L = 0.05$	$h/L = 0.10$	$h/L = 0.15$
Present	74.92 (-0.11 %)	14.92 (-0.53 %)	7.42 (-1.07 %)	4.92 (-1.60 %)
Bernoulli-Euler [2, 3]	75.00* (0.00 %)	15.00* (0.00 %)	7.50* (0.00 %)	5.00* (0.00 %)
Timoshenko [3, 21]	50.00# (-33.33 %)	10.00# (-33.33 %)	5.00# (-33.33 %)	3.33# (-33.33 %)
Single variable theory [4]	75.00 (0.00 %)	15.00 (0.00 %)	7.50 (0.00 %)	—
Theory of elasticity [56]	75.00	15.00	7.50	5.00

$ % difference is calculated by utilizing eq. (69).

* As the Bernoulli-Euler beam theory neglects the effects of beam transverse shear deformation, it is not possible to obtain $\bar{\tau}_{zx}$ by utilizing a constitutive relation between the shear stress and shear strain. For the Bernoulli-Euler beam theory, $\bar{\tau}_{zx}$ has been obtained by utilizing the equilibrium approach (Shames and Dym [3]).

\# For the Timoshenko beam theory, $\bar{\tau}_{zx}$ remains constant through the beam thickness because it is a first-order theory. Whereas for the present theory, $\bar{\tau}_{zx}$ remains zero at beam surfaces $z = \pm h/2$ and varies quadratically through the beam thickness.

— represents unavailability of the result in the cited reference.

Table 7. Non-dimensional transverse displacement ($\overline{w}$) pertaining to example 3 (*FC* beam, Fig. 1) obtained by utilizing the present theory and its comparison, $\mu = 0.3$.

	Non-dimensional transverse displacement at $x = 0$, $\overline{w} = w E I / (q_o L^4)$ (Values in parentheses indicate the percentage difference)[$]			
Theory	$h / L = 0.01$	$h / L = 0.05$	$h / L = 0.10$	$h / L = 0.15$
Present	0.12501 (-0.01 %)	0.12533 (-0.15 %)	0.12630 (-0.60 %)	0.12793 (-1.32 %)
Bernoulli-Euler [2, 3]	0.12500 (-0.02 %)	0.12500 (-0.41 %)	0.12500 (-1.62 %)	0.12500 (-3.58 %)
Timoshenko [3, 21]	0.12501 (-0.01 %)	0.12533 (-0.15 %)	0.12630 (-0.60 %)	0.12793 (-1.32 %)
Levinson [4]	0.12502 (0.00 %)	0.12549 (-0.02 %)	0.12695 (-0.09 %)	—
Single variable theory [4]	0.12502 (0.00 %)	0.12549 (-0.02 %)	0.12695 (-0.09 %)	—
Theory of elasticity [57]	0.12502	0.12552	0.12706	0.12964

$ % difference is calculated by utilizing eq. (69).
— represents unavailability of the result in the cited reference.

Table 8. Non-dimensional axial stress ($\overline{\sigma}_x$) pertaining to example 3 (*FC* beam, Fig. 1) obtained by utilizing the present theory and its comparison, $\mu = 0.3$.

	Non-dimensional axial stress at $x = L$, $z = h / 2$, $\overline{\sigma}_x = \sigma_x b / q_0$ (Values in parentheses indicate the percentage difference)[$]			
Theory	$h / L = 0.01$	$h / L = 0.05$	$h / L = 0.10$	$h / L = 0.15$
Present	-30000.00 (0.00 %)	-1200.00 (0.02 %)	-300.00 (0.07 %)	-133.33 (0.15 %)
Bernoulli-Euler [2, 3]	-30000.00 (0.00 %)	-1200.00 (0.02 %)	-300.00 (0.07 %)	-133.33 (0.15 %)
Timoshenko [3, 21]	-30000.00 (0.00 %)	-1200.00 (0.02 %)	-300.00 (0.07 %)	-133.33 (0.15 %)
Single variable theory [4]	-29999.74 (0.00 %)	-1199.74 (-0.01 %)	-299.74 (-0.02 %)	—
Theory of elasticity [57]	-29999.80	-1199.80	-299.80	-133.13

$ % difference is calculated by utilizing eq. (69).
— represents unavailability of the result in the cited reference.

Table 9. Non-dimensional transverse shear stress ($\overline{\tau}_{zx}$) pertaining to example 3 (*FC* beam, Fig. 1) obtained by utilizing the present theory and its comparison, $\mu = 0.3$.

	Non-dimensional transverse shear stress at $x = L$, $z = 0$, $\overline{\tau}_{zx} = \tau_{zx} b / q_0$ (Values in parentheses indicate the percentage difference)[$]			
Theory	$h / L = 0.01$	$h / L = 0.05$	$h / L = 0.10$	$h / L = 0.15$
Present	-149.92 (-0.05 %)	-29.92 (-0.27 %)	-14.92 (-0.53 %)	-9.92 (-0.80 %)
Bernoulli-Euler [2, 3]	-150.00* (0.00 %)	-30.00* (0.00 %)	-15.00* (0.00 %)	-10.00* (0.00 %)
Timoshenko [3, 21]	-100.00[#] (-33.33 %)	-20.00[#] (-33.33 %)	-10.00[#] (-33.33 %)	-6.67[#] (-33.33 %)
Single variable theory [4]	-150.00 (0.00 %)	-30.00 (0.00 %)	-15.00 (0.00 %)	—
Theory of elasticity [57]	-150.00	-30.00	-15.00	-10.00

$ % difference is calculated by utilizing eq. (69).
* As the Bernoulli-Euler beam theory neglects the effects of beam transverse shear deformation, it is not possible to obtain $\overline{\tau}_{zx}$ by utilizing a constitutive relation between the shear stress and shear strain. For the Bernoulli-Euler beam theory, $\overline{\tau}_{zx}$ has been obtained by utilizing the equilibrium approach (Shames and Dym [3]).
\# For the Timoshenko beam theory, $\overline{\tau}_{zx}$ remains constant through the beam thickness because it is a first-order theory. Whereas for the present theory, $\overline{\tau}_{zx}$ remains zero at beam surfaces $z = \pm h / 2$ and varies quadratically through the beam thickness.
— represents unavailability of the result in the cited reference.

6.2. Numerical results for free vibrations of the beam (example 4)

The non-dimensional natural frequency of beam vibrations $\overline{\omega}_n$ is defined as follows:

$$\overline{\omega}_n = \sqrt{\frac{\rho A L^4}{E I}}\, \omega_n \tag{70}$$

Numerical results pertaining to $\bar{\omega}_n$ for the simply-supported shear deformable isotropic prismatic rectangular beam obtained by utilizing the present theory for various values of the beam thickness-to-length ratio h / L and beam vibration mode parameter n are presented in Tables 10 through 12. These results are compared with corresponding results obtained by utilizing the higher-order trigonometric theory of Touratier [60] adapted for beams by Sayyad [61], single variable beam theory, Timoshenko beam theory and Bernoulli-Euler beam theory so as to display the efficacy of the proposed theory.

Following points are considered while obtaining results which are presented in Tables 10 through 12:

1. The Poisson's ratio μ is assumed to be 0.3.
2. The beam vibration mode parameter n is assumed to have values 1, 2 and 3.
3. The beam thickness-to-length ratio h / L is assumed to have values 0.01, 0.05, 0.10 and 0.15.
4. Numerical results for $\bar{\omega}_n$ by the single variable beam theory are as reported by Shimpi et al. [4].
5. Numerical results for $\bar{\omega}_n$ obtained by utilizing higher-order trigonometric theory of Touratier, Timoshenko beam theory and Bernoulli-Euler beam theory have been calculated by the present authors.
   (a) In case of the Bernoulli-Euler beam theory, ω_n is calculated by utilizing eq. (12) of Timoshenko [5] wherein effects of rotary inertia are taken into consideration.
   (b) In case of the Timoshenko beam theory, ω_n is calculated by utilizing eq. (10) of Timoshenko [5]. The shear correction factor is taken as 5 / 6.
   (c) In case of the higher-order trigonometric theory of Touratier, ω_n is calculated by utilizing eqs. (14), (30) through (36) of Sayyad [61].
6. In Tables 10 through 12, the percentage difference quoted in parentheses against results reported by the particular theory (either by the present theory, single variable beam theory, Timoshenko beam theory or Bernoulli-Euler beam theory, as the case may be) is calculated as follows:

$$\%\ \text{difference} = \left(\frac{\text{The result reported by the particular theory}}{\text{The corresponding result of higher-order trigonometric theory of Touratier}} - 1 \right) \times 100 \tag{71}$$

Table 10. Non-dimensional natural frequencies of the beam vibration ($\bar{\omega}_n$) pertaining to the example 4 (the *SS* beam, Fig. 1) obtained by utilizing the present theory and their comparison, the beam vibration mode parameter $n = 1$ and $\mu = 0.3$.

	Non-dimensional natural frequencies of the beam vibration, $\bar{\omega}_n = \omega_n \sqrt{\rho A L^4 / E I}$ (Values in parentheses indicate the percentage difference)$			
Theory	$h / L = 0.01$	$h / L = 0.05$	$h / L = 0.10$	$h / L = 0.15$
Present	9.8679 (0.00 %)	9.8281 (0.00 %)	9.7075 (0.00 %)	9.5182 (0.00 %)
Bernoulli-Euler [5]	9.8692 (0.01 %)	9.8595 (0.32 %)	9.8293 (1.25 %)	9.7795 (2.74 %)
Timoshenko [5]	9.8679 (0.00 %)	9.8281 (0.00 %)	9.7075 (0.00 %)	9.5180 (-0.01 %)
Single variable theory [4]	9.8679 (0.00 %)	9.8281 (0.00 %)	9.7075 (0.00 %)	—
Touratier [61]	9.8679	9.8282	9.7077	9.5186

$ % difference is calculated by utilizing eq. (71).
— represents unavailability of the result in the cited reference.

Table 11. Non-dimensional natural frequencies of the beam vibration ($\bar{\omega}_n$) pertaining to the example 4 (the *SS* beam, Fig. 1) obtained by utilizing the present theory and their comparison, the beam vibration mode parameter $n = 2$ and $\mu = 0.3$.

	Non-dimensional natural frequencies of the beam vibration, $\bar{\omega}_n = \omega_n \sqrt{\rho A L^4 / E I}$ (Values in parentheses indicate the percentage difference)$			
Theory	$h / L = 0.01$	$h / L = 0.05$	$h / L = 0.10$	$h / L = 0.15$
Present	39.4517 (0.00 %)	38.8301 (0.00 %)	37.0981 (-0.01 %)	34.7431 (-0.02 %)
Bernoulli-Euler [5]	39.4719 (0.05 %)	39.3171 (1.25 %)	38.8446 (4.70 %)	38.0937 (9.63 %)
Timoshenko [5]	39.4517 (0.00 %)	38.8299 (0.00 %)	37.0962 (-0.01 %)	34.7354 (-0.04 %)
Touratier [61]	39.4517	38.8308	37.1009	34.7491

$ % difference is calculated by utilizing eq. (71).

Table 12. Non-dimensional natural frequencies of the beam vibration ($\bar{\omega}_n$) pertaining to the example 4 (the *SS* beam, Fig. 1) obtained by utilizing the present theory and their comparison, the beam vibration mode parameter n = 3 and μ = 0.3.

Theory	Non-dimensional natural frequencies of the beam vibration, $\bar{\omega}_n = \omega_n \sqrt{\rho A L^4 / E I}$ (Values in parentheses indicate the percentage difference)$			
	$h / L = 0.01$	$h / L = 0.05$	$h / L = 0.10$	$h / L = 0.15$
Present	88.6914 (0.00 %)	85.6634 (0.00 %)	78.1719 (-0.02 %)	69.5629 (-0.04 %)
Bernoulli-Euler [5]	88.7936 (0.12 %)	88.0158 (2.74 %)	85.7108 (9.63 %)	82.2414 (18.18 %)
Timoshenko [5]	88.6914 (0.00 %)	85.6619 (-0.01 %)	78.1547 (-0.04 %)	69.5062 (-0.12 %)
Touratier [61]	88.6915	85.6671	78.1855	69.5908

$ % difference is calculated by utilizing eq. (71).

7. Discussion on Numerical Results

In this section, the discussion on numerical results pertaining to the static bending (examples 1 through 3, Tables 1 through 9) and free vibrations (example 4, Tables 10 through 12) of shear deformable isotropic prismatic rectangular beams is presented.

7.1. Discussion on numerical results of the static bending of the beam (examples 1 through 3)

Tables 1 through 9 present numerical results for the non-dimensional beam transverse displacement $\bar{w}$, non-dimensional beam axial stress $\bar{\sigma}_x$ and non-dimensional beam transverse shear stress $\bar{\tau}_{zx}$ pertaining to the *SS*, *CC* and *FC* beams. With regard to these numerical results, following observations can be made:

1. Observations with regard to numerical results pertaining to the *SS* beam (example 1) presented in Tables 1 through 3:
   In case of the present theory, the maximum percentage difference involved in predicting $\bar{w}$ is 0.44 % (for h / L = 0.15), in predicting $\bar{\sigma}_x$ is 0.18 % (for h / L = 0.15) and in predicting $\bar{\tau}_{zx}$ is -1.60 % (for h / L = 0.15). The percentage difference involved in predicting $\bar{w}$, $\bar{\sigma}_x$ and $\bar{\tau}_{zx}$ for remaining beam thickness-to-length ratios (i.e., h / L = 0.01, h / L = 0.05 and h / L = 0.10) are reasonably low.
   Hence for the case of the *SS* beam, results obtained by utilizing the present theory have an excellent agreement with corresponding exact results obtained by utilizing the two-dimensional theory of elasticity for thin beams and shear deformable beams. Also for case of the *SS* beam, results obtained by utilizing the present theory have an excellent agreement with corresponding results of the Levinson beam theory and single variable beam theory as reported by Shimpi et al. [4] for thin beams and shear deformable beams.
   (a) Observations with regard to $\bar{w}$:
      i. The Bernoulli-Euler beam theory neglects the beam transverse shear deformation effects. Hence it provides underestimated results for $\bar{w}$ in case of shear deformable beams.
      ii. Even though results for $\bar{w}$ arrived at by utilizing the Timoshenko beam theory match exactly with corresponding results obtained by utilizing the present theory, following should be noted:
         A. As the Timoshenko beam theory is a first-order beam theory, the accuracy of $\bar{w}$ obtained by utilizing the Timoshenko beam theory depends upon the accuracy of the shear correction factor utilized for obtaining the results.
         B. The just-mentioned difficulty does not arise in case of the present theory as it does not need the shear correction factor.
      iii. Even though results for $\bar{w}$ of the Levinson beam theory and single variable beam theory as reported by Shimpi et al. [4] have an excellent agreement with corresponding results obtained by utilizing the present theory, the Levinson beam theory and single variable beam theory are variationally inconsistent theories, whereas the present beam theory is a variationally consistent theory.
   (b) Observations with regard to $\bar{\sigma}_x$:
      Results for $\bar{\sigma}_x$ obtained by utilizing the present theory, Bernoulli-Euler beam theory, Timoshenko beam theory, single variable beam theory and two-dimensional theory of elasticity vary by a very small margin for the beam thickness-to-length ratio ranging from h / L = 0.01 to h / L = 0.15. This observation is not surprising as even the elementary beam theory and classical plate theory can predict bending stresses for shear deformable beams and plates respectively, which are in very favorable agreement with corresponding results obtained by utilizing the theory of elasticity (references [37, 58, 59]).
   (c) Observations with regard to $\bar{\tau}_{zx}$:
      i. Even though results for $\bar{\tau}_{zx}$ obtained by utilizing the Bernoulli-Euler beam theory match exactly with

corresponding exact results obtained by utilizing the two-dimensional theory of elasticity, following should be noted:

A. The Bernoulli-Euler beam theory neglects the beam transverse shear deformation effects. Hence it is not possible to obtain the beam transverse shear stress in case of the Bernoulli-Euler beam theory by utilizing a constitutive relation between the shear stress and shear strain.

B. The beam transverse shear stress in case of the Bernoulli-Euler beam theory can only be obtained by utilizing the equilibrium approach as discussed by Shames and Dym [3].

C. The just-mentioned difficulty does not arise in case of the present theory. The beam transverse shear stress in case of the present theory can readily be obtained by utilizing a constitutive relation between the shear stress and shear strain.

ii. As the Timoshenko beam theory is a first-order beam theory, $\bar{\tau}_{zx}$ obtained in its case remains constant through the beam thickness.

A. $\bar{\tau}_{zx}$ obtained by utilizing the Timoshenko beam theory does not satisfy transverse shear stress free beam surface conditions on beam surfaces $z = \pm h / 2$.

B. However, $\bar{\tau}_{zx}$ obtained by utilizing the present theory remains zero on beam surfaces $z = \pm h / 2$ and varies quadratically through the beam thickness.

C. The accuracy of $\bar{\tau}_{zx}$ obtained by utilizing the Timoshenko beam theory also depends upon the accuracy of shear correction factor utilized for obtaining the results.

D. The just-mentioned difficulty does not arise in case of the present theory as it does not need the shear correction factor.

iii. As is the case with the present theory, $\bar{\tau}_{zx}$ of the single variable beam theory of Shimpi et al. [4] remains zero on beam surfaces $z = \pm h / 2$ and varies quadratically through the beam thickness. However, even though results for $\bar{\tau}_{zx}$ of the single variable beam theory have an excellent agreement with corresponding results obtained by utilizing the present theory, the single variable beam theory is a variationally inconsistent theory, whereas the present beam theory is a variationally consistent theory.

2. Observations with regard to numerical results pertaining to the *CC* beam (example 2) presented in Tables 4 through 6:
In case of the present theory, the maximum percentage difference involved in predicting $\bar{w}$ is -5.11 % (for $h / L = 0.15$), in predicting $\bar{\sigma}_x$ is 2.63 % (for $h / L = 0.15$) and in predicting $\bar{\tau}_{zx}$ is -1.60 % (for $h / L = 0.15$). The percentage difference involved in predicting $\bar{w}$, $\bar{\sigma}_x$ and $\bar{\tau}_{zx}$ for remaining beam thickness-to-length ratios (i.e., $h / L = 0.01$, $h / L = 0.05$ and $h / L = 0.10$) are reasonably low.
Hence for the case of the *CC* beam, results obtained by utilizing the present theory have very good agreement with corresponding exact results obtained by utilizing the two-dimensional theory of elasticity for thin beams and shear deformable beams. Also for case of the *CC* beam, results obtained by utilizing the present theory have an excellent agreement with corresponding results of the Levinson beam theory and single variable beam theory as reported by Shimpi et al. [4] for thin beams and shear deformable beams.

It should be noted that observations 7.1.1.(a), 7.1.1.(b) and 7.1.1.(c) made with regard to $\bar{w}$, $\bar{\sigma}_x$ and $\bar{\tau}_{zx}$ respectively for the *SS* beam are also valid for the *CC* beam.

3. Observations with regard to numerical results pertaining to the *FC* beam (example 3) presented in Tables 7 through 9:
In case of the present theory, the maximum percentage difference involved in predicting $\bar{w}$ is -1.32 % (for $h / L = 0.15$), in predicting $\bar{\sigma}_x$ is 0.15 % (for $h / L = 0.15$) and in predicting $\bar{\tau}_{zx}$ is -0.80 % (for $h / L = 0.15$). The percentage difference involved in predicting $\bar{w}$, $\bar{\sigma}_x$ and $\bar{\tau}_{zx}$ for remaining beam thickness-to-length ratios (i.e., $h / L = 0.01$, $h / L = 0.05$ and $h / L = 0.10$) are reasonably low.
Hence for the case of the *FC* beam, results obtained by utilizing the present theory have an excellent agreement with corresponding exact results obtained by utilizing the two-dimensional theory of elasticity for thin beams and shear deformable beams. Also for case of the *FC* beam, results obtained by utilizing the present theory have an excellent agreement with corresponding results of the Levinson beam theory and single variable beam theory as reported by Shimpi et al. [4] for thin beams and shear deformable beams.

It should be noted that observations 7.1.1.(a), 7.1.1.(b) and 7.1.1.(c) made with regard to $\bar{w}$, $\bar{\sigma}_x$ and $\bar{\tau}_{zx}$ respectively for the *SS* beam are also valid for the *FC* beam.

7.2. Discussion on numerical results of free vibrations of the beam (example 4)

Tables 10 through 12 present numerical results for non-dimensional natural frequencies of beam vibration $\bar{\omega}_n$ pertaining to the *SS* beam. With regard to these numerical results, following observations can be made:

1. In case of the present theory, the maximum percentage difference involved in predicting $\bar{\omega}_n$ is -0.04 % (for $n = 3$, $h /$

L = 0.15). The percentage difference involved in predicting $\bar{\omega}_n$ for remaining beam vibration mode parameters (i.e., n = 1 and n = 2) and for remaining beam thickness-to-length ratios (i.e., h / L = 0.01, h / L = 0.05 and h / L = 0.10) are reasonably low. $\bar{\omega}_n$ obtained by utilizing the present theory also have an excellent agreement with corresponding results of the single variable beam theory as reported by Shimpi et al. [4] for n = 1 and h / L = 0.01, h / L = 0.05 and h / L = 0.10.

Hence results obtained by utilizing the present theory have an excellent agreement with corresponding results reported by the single variable beam theory and with corresponding results obtained by utilizing the higher-order trigonometric theory of Touratier adapted for beams.

2. Even though results for $\bar{\omega}_n$ of the single variable beam theory as reported by Shimpi et al. [4] have an excellent agreement with corresponding results obtained by utilizing the present theory for n = 1 and h / L = 0.01, h / L = 0.05 and h / L = 0.10, the single variable beam theory is a variationally inconsistent theory, whereas the present beam theory is a variationally consistent theory.
3. For higher values of the beam vibration mode parameter and beam thickness-to-length ratio, $\bar{\omega}_n$ obtained by utilizing the present theory match better with corresponding results obtained by utilizing the higher-order trigonometric theory of Touratier adapted for beams as compared to corresponding $\bar{\omega}_n$ calculated by utilizing the Timoshenko beam theory.

   For example: For n = 3 and h / L = 0.15, the percentage difference involved in predicting $\bar{\omega}_n$ by the present theory is -0.04 %. Whereas this percentage difference is -0.12 % in case of the Timoshenko beam theory.
4. As is well known, the Bernoulli-Euler beam theory provides overestimated $\bar{\omega}_n$ for thin beams vibrating at higher modes as well as for shear deformable beams.

   For example: For n = 3 and h / L = 0.15, the percentage difference involved in predicting $\bar{\omega}_n$ by the Bernoulli-Euler beam theory is 18.18 %.

8. Concluding Remarks

In this paper, a displacement-based, variationally consistent, refined theory for shear deformable isotropic rectangular beams was presented. Important features of the present theory are summarized as follows:

1. Features with regard to displacement functions of the present theory are as follows:
   (a) The beam axial displacement and beam transverse displacement consist of bending components and shearing components.
   (b) The bending component of the beam axial displacement is analogous to the beam axial displacement of the Bernoulli-Euler beam theory.
   (c) The bending component of the beam axial displacement along with the bending component of the beam transverse displacement do not take part in the beam transverse shear strain, and hence in the cross-sectional shearing force.
   (d) The contribution of the shearing component of the beam axial displacement towards the beam axial strain is such that, it does not contribute toward the cross-sectional bending moment.
   (e) The shearing component of the beam axial displacement and shearing component of the beam transverse displacement are the only displacement components which contribute toward the beam transverse shear strain and hence toward the cross-sectional shearing force.
      i. These displacement components give rise to the beam transverse shear strain (and hence to the beam transverse shear stress) which varies quadratically through the beam thickness and maintains transverse shear stress-free beam surface conditions.
      ii. Hence unlike first-order beam theories, the present theory does not require the shear correction factor.
2. Governing differential equations and variationally consistent boundary conditions of the present theory are obtained by utilizing the Hamilton's principle.
   (a) The present theory has only two governing differential equations of fourth order and involves only two unknown functions.
   (b) For dynamics problems, these governing equations are only inertially coupled. For statics problems, these governing equations are decoupled.
3. Some of the expressions involved in the present theory have a strong resemblance with corresponding expressions of the Bernoulli-Euler beam theory.
   (a) Expression of the cross-sectional bending moment as given by eq. (11) is same as the corresponding expression of the Bernoulli-Euler beam theory, except for the presence of the subscript in eq. (11).
   (b) For statics problems, the governing differential equation as given by eq. (22) reduces to the form which is same as the governing differential equation of the Bernoulli-Euler beam theory, except for the presence of the subscript in eq. (22).
   (c) Boundary conditions as given by eqs. (24) and (25) are same as corresponding boundary conditions of the

Bernoulli-Euler beam theory, except for the presence of the subscript in eqs. (24) and (25).

4. Numerical results pertaining to the static bending and free vibrations of shear deformable isotropic prismatic rectangular beams are obtained by utilizing the present theory. To demonstrate the efficacy of the present theory, these results are compared with corresponding results obtained from other theories present in the literature. The results are found to be accurate.

In conclusion, the present theory is a simple and yet accurate shear deformation beam theory, involving only two unknown functions and having a strong resemblance with the Bernoulli-Euler beam theory.

Conflict of Interest

The authors declared no potential conflicts of interest with respect to the research, authorship and publication of this article.

Funding

The authors received no financial support for the research, authorship and publication of this article.

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