J. Appl. Comput. Mech., 6(3) (2020) 445-456
DOI: 10.22055/JACM.2019.29960.1649

ISSN: 2383-4536
jacm.scu.ac.ir

M&MoCS

Shahid Chamran University of Ahvaz

Journal of
Applied and Computational Mechanics

Research Paper

On Green and Naghdi Thermoelasticity Model without Energy Dissipation with Higher Order Time Differential and Phase-Lags

Ahmed E. Abouelregal[1,2]

[1] Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt, Email: ahabogal@mans.edu.eg

[2] Department of Mathematics, College of Science and Arts, Jouf University, Al-Qurayyat, Saudi Arabia, Email: ahabogal@gmail.com

Received June 15 2019; Revised August 21 2019; Accepted for publication August 22 2019.
Corresponding author: A.E. Abouelregal (ahabogal@gmail.com)

Abstract. In the present work, a modified model of heat conduction including higher order of time derivative is derived by extending Green and Naghdi theory without energy dissipation. We introduce two phase lag times to include the thermal displacement gradient and the heat flux in the heat conduction and depict microscopic responses more precisely. The constructed model is applied to study thermoelastic waves in a homogeneous and isotropic perfect conducting unbounded solid body containing a spherical cavity. We use the Laplace transform method to analyze the problem. The solutions for the field functions are obtained numerically using the numerical Laplace inversion technique. The results are analyzed in different tables and graphs and compared with those obtained earlier in the contexts of some other theories of thermoelasticity.

Keywords: Thermoelasticity; Green-Naghdi model II; Phase-lags; Higher-order; Spherical cavity.

1. Introduction

Many efforts are being made to remove the contradiction inherent in the classical dynamical coupled theory of thermoelasticity [1], which predicts an infinite speed for the propagation of thermal waves. Thermoelasticity theories that confess finite speeds for thermal waves have raised much interest in the last four decades. These theories involve hyperbolic heat equations and are referred to as generalized thermoelasticity theories. For details about the physical relevance of these theories and a review of the relevant literature, see [2-4].

The extended thermoelasticity model, which provides one time for thermal relaxation in the thermoelastic process, was developed by Lord and Shulman [5] and Green and Lindsay [6] suggested the temperature-rate dependent model of thermoelasticity, which includes two relaxation times. For a review, works of Ignaczak [7] can be mentioned where the two theories are presented and some important results are achieved in this area. The lagging behavior in the thermal transfer was presented by Tzou [8–10] on the so-called dual-phase-lag model of heat conduction. Tzou introduced two-phase lags to both heat flux vector and the temperature gradient and considered a constitutive equation to describe the logging behavior in the heat conduction in solids. Chandrasekharaiah [11] extended the DPL model of heat conduction to a generalized thermoelasticity theory. The stability and the qualitative aspects of the dual-phase-lag model were discussed in details in [12, 13]. These models have been applied to many problems in the field of thermoelasticity as stated in [14-20].

Green and Naghdi [21-23] established generalized thermoelasticity models by proposing three theories which are labeled as GN-I, II, and III models. Green and Naghdi in [23] formulated a new theory of thermoelasticity (GN-II) by including the so-called "thermal-displacement gradient" between the independent constitutive variables. An important characteristic feature of this model, which is not found in other thermoelastic models, is that this theory does not assist thermal energy dissipation. In the context of the linearized version of this theory, theorems on the uniqueness of solutions have been established in [24, 25]. Extending the thermoelastic model introduced by Green–Naghdi [21, 22], Roychoudhuri [26] proposed a three-phase-lag heat (TPL) conduction theory that includes three-phase lags in the heat flux vector, the temperature gradient and in the thermal displacement gradient. El-Karamany, Ezzat [27] proposed three models of thermoelasticity extending Green and Naghdi models. Ciarletta [28] introduced the theory of micropolar thermoelasticity without energy dissipation and Chiriţă and Ciarletta [29] constructed the reciprocal and variational principle in thermoelasticity without energy dissipation. Problems concerning with the generalized thermoelasticity proposed by Green and Naghdi [21-23] were studied by many authors [30-39].

The objective of the present study is to establish a new mathematical model of heat conduction without energy dissipation that includes higher time differential and two phase-lags extending Green and Naghdi [23] model (GN-II). In this model, the Fourier law of heat conduction is replaced using Taylor series expansions to a modification of a new with introducing two different phase lags for the heat flux vector and the thermal displacement gradient and keeping the terms up with suitable higher orders. The established high-order two-phase-lag heat conduction without energy dissipation model (HGN-II) reduces to the previous models of thermoelasticity as special cases.

To verify the accuracy of the current model, we have discussed a thermal heat problem for a perfectly conducting spherical cavity subjected to a continuous point heat source and exposed to a constant magnetic field. Using the Laplace transform and numerical Laplace inversion, the problem is solved. The expressions of the studied variables are calculated under appropriate initial conditions. We deduce some particular cases of interest. The numerical results obtained have been tabulated and graphically illustrated. The results show that the analytic solutions are in good agreement with the numerical solutions. We also investigate the influences of phase-lags of high-order on the considered field variables. The results obtained in this work are found to be comparable with the results in the technical literature. We believe that the analysis of this study will be useful for understanding the basic features of this new model for heat conduction.

2. Derivation of HGN-II model of thermoelasticity with two-phase-lags of high-order

Many theories describing heat conduction have been proposed in recent years. This is to propose a theory where the propagation of heat is demonstrated with a finite propagation speed, in contrast to the classical model using Fourier's law leading to the infinite propagation speed of heat signals. The classical theory of thermoelasticity is based on the principles of thermal conductivity, specifically according to the classical Fourier's law, which relates the heat flux vector $\boldsymbol{q}(\boldsymbol{x},t)$ at a point $\boldsymbol{x}$ at the time t to the temperature gradient $\nabla\theta(\boldsymbol{x},t)$. According to this model, the classical Fourier's law [2, 3]:

$$\boldsymbol{q}(\boldsymbol{x},t) = -K\nabla\theta(\boldsymbol{x},t), \tag{1}$$

which assumes that the thermal disorder spreads with unlimited speed. In Eq. (1) $\theta = T - T_0$ represents the varying temperature in which T is the absolute temperature above the reference temperature T_0 and K denotes the thermal conductivity.

In fact, heat is always found to propagate with a finite speed within the medium. To remove this defect, a thermal wave model of heat transfer has been suggested by Cattaneo and Vernotte [40, 41] based on single phase lagging constitutive relation. This model is based on the constitutive equation:

$$\boldsymbol{q}(\boldsymbol{x},t+\tau_{\mathrm{q}}) = -K\nabla\theta(\boldsymbol{x},t), \tag{2}$$

which captures microscale responses in time. In order to consider the effect of microstructural interactions along with the fast transient effects, Tzou [8, 9] has proposed a dual-phase-lag model in the following form:

$$\boldsymbol{q}(\boldsymbol{x},t+\tau_{\mathrm{q}}) = -K\nabla\theta(\boldsymbol{x},t+\tau_\theta), \tag{3}$$

where τ_{q} and τ_θ are the phase lag of the heat flux and the temperature gradient, respectively. This equation shows that the temperature gradient established across a material volume at the position $\boldsymbol{x}$ at time $t+\tau_\theta$ results in heat flux to flow at a different instant of time $t+\tau_{\mathrm{q}}$. We understand this delay in terms of the microscopic structure of the material.

Later, Green and Naghdi [23] proposed a new model of thermoelasticity theory without energy dissipation (GN-II) by developing an alternative formulation of heat propagation. In this theory, the gradient of thermal displacement ϑ is considered as a new constitutive variable. Also, the scalar ϑ is interpreted as the counterpart in thermal fields of the mechanical displacement in mechanical fields; it is called the thermal-displacement. The thermal displacement, ϑ satisfies $\dot{\vartheta} = \theta$. The heat conduction law for GN-II model [23] is given by:

$$\boldsymbol{q}(\boldsymbol{x},t) = -K^*\nabla\vartheta(\boldsymbol{x},t), \tag{4}$$

where $K^* > 0$ is the conductivity rate (characteristic of the theory).

We do not expect the classical theories proposed on the macroscopic level, such as heat diffusion assuming Fourier's law to be useful for microscale cases because it describes the microscopic behavior calculated on average on many grains. Microscopic reactions require specific time periods for their completion, ranging from femto-seconds to second or even longer ones. In this regard, the delayed response describes the heat flux vector and the thermal displacement gradient that occurs in different cases of time in the heat transfer process.

Extending the thermoelastic model introduced by Green-Naghdi [23], we introduce a two phase-lags to the heat flux vector $\boldsymbol{q}(\boldsymbol{x},t)$ and the thermal displacement gradient $\vartheta(\boldsymbol{x},t)$. Then, the following generalized constitutive equation for heat conduction is hereby proposed to describe the lagging behavior:

$$\boldsymbol{q}(\boldsymbol{x}, t+\tau_{\mathrm{q}}) = -K^{*}\nabla\vartheta(\boldsymbol{x}, t+\tau_{\vartheta}) \tag{5}$$

The second delay time τ_ϑ may be interpreted, following Tzou [8, 9], as the phase-lag of the thermal displacement gradient. The lagging behavior is closely related to the possibility of explaining the problems related to the applications on the nanoscale: for instance, in the chemical nanotechnologies, transferring energy or the procedure in which the chemical reactions occurring under highly transient conditions require very specific treatment and cannot be treated by the classical macroscopic models. Other areas of interest currently under investigation are those relating to various possible combinations of Taylor series expansion orders for both associates of the constitutive equations.

A refined structure of the lagging response depicted by Eq. (5) can be illustrated by expanding this equation in terms of Taylor's series with respect to time and keeping the terms up to specific orders in τ_q and τ_ϑ:

$$\left(1+\sum_{k=1}^{m}\frac{\tau_q^k}{k!}\frac{\partial^k}{\partial t^k}\right)\boldsymbol{q} = -K^{*}\left(1+\sum_{k=1}^{n}\frac{\tau_\vartheta^k}{k!}\frac{\partial^k}{\partial t^k}\right)\nabla\vartheta \tag{6}$$

We have expanded until order m the heat flux and until order n the thermal displacement gradient. It is worth to mention that a study upon the constitutive equations of type (6) made in the article by Chiriţă [42] shows that for $m \geq 5$ they lead to an unbalanced system and therefore they cannot describe a real situation.

It is also manifest that Eq. (4) can be understood as a special case with respect to the new one modeled by Eq. (6). On taking the time-derivative of this equation and using $\dot{\vartheta} = \theta$, we obtain:

$$\left(1+\sum_{k=1}^{m}\frac{\tau_q^k}{k!}\frac{\partial^k}{\partial t^k}\right)\frac{\partial \boldsymbol{q}}{\partial t} = -K^{*}\left(1+\sum_{k=1}^{n}\frac{\tau_\vartheta^k}{k!}\frac{\partial^k}{\partial t^k}\right)\nabla\theta \tag{7}$$

The increment of the entropy S satisfies the following equations [45, 47]:

$$\operatorname{div}\boldsymbol{q} + Q = -\rho T_0 \dot{S}, \tag{8}$$

$$\rho T_0 S = \rho C_e \theta + \gamma T_0 \operatorname{div}\boldsymbol{u}, \tag{9}$$

where C_e denotes the specific heat at constant strain, $\gamma = (3\lambda + 2\mu)\alpha_t$ represents the stress temperature modulus, in which α_t denotes the thermal expansion coefficient, λ, μ are Lamé's constants, $\boldsymbol{u}$ is the displacement vector, ρ is the density of the medium and Q is the heat supply. The energy balance equation is given from (8) and (9) by:

$$\rho C_e \frac{\partial\theta}{\partial t} + \gamma T_0 \frac{\partial}{\partial t}(\operatorname{div}\boldsymbol{u}) = -\operatorname{div}\boldsymbol{q} + Q \tag{10}$$

Further on differentiation of the energy equation (10) with respect to time and then elimination of $\operatorname{div}\dot{\boldsymbol{q}}$ from Eqs. (7) and (10) leads to the modified heat transport equation with two-phase-lag of higher derivative orders:

$$\left(1+\sum_{k=1}^{m}\frac{\tau_q^k}{k!}\frac{\partial^k}{\partial t^k}\right)\left[\rho C_e \frac{\partial^2\theta}{\partial t^2} + \gamma T_0 \frac{\partial^2}{\partial t^2}(\operatorname{div}\boldsymbol{u}) - \frac{\partial Q}{\partial t}\right] = K^{*}\left(\delta+\sum_{k=1}^{n}\frac{\tau_\vartheta^k}{k!}\frac{\partial^k}{\partial t^k}\right)\nabla^2\theta \tag{11}$$

Equation (11) represents a modified heat equation without energy dissipation with time derivative of higher orders and two phase-lags τ_q and τ_ϑ. The linearized construction permits the transfer of heat flow as thermal waves at a limited speed and the equation of evolution is completely hyperbolic.

In order to compare between our modified model and the previous models, we suggested without loss of generality a new parameter $\delta \in \{0,1\}$. When $\tau_q, \tau_\vartheta \to 0$, and $\delta = 1$ Eq. (11) reduces to the generalized heat conduction equation introduced by Green–Naghdi [16]. Also, when $\delta = 0$, $m = n = 1$, $\tau_\vartheta = 1$ and $K^* = K$, Eq. (11) is transformed into the model of thermoelasticity comprises one relaxation introduced by Lord and Shulman (LS) [5]. The case with $\delta = 0$, $\tau_q = 0$, $\tau_\vartheta = 1$, $n = 1$ and $K^* = K$ corresponds to the classical Fourier's law (CTE).

The additional basic equations of motion, constitutive equations and strain and displacement relations based on the theory of thermoelasticity for a homogeneous and isotropic thermoelastic solid are:

$$\sigma_{ij} = 2\mu e_{ij} + \delta_{ij}[\lambda e_{kk} - \gamma\theta], \tag{12}$$

$$2e_{ij} = u_{j,i} + u_{i,j}, \tag{13}$$

$$\mu u_{i,jj} + (\lambda+\mu)u_{j,ij} - \gamma\theta_{,i} + F_i = \rho\ddot{u}_i. \tag{14}$$

The above system is a fully hyperbolic system in the sense that both equations of motion (14) and the equation of heat transport (11) present in the system are of a hyperbolic-type.

3. Application of the modified heat conduction equation

Taking into account the new linear theory of thermoelasticity with the higher order time differential and two-phase-lag, a suitable initial value problem is considered to study the accuracy of the introduced model. To achieve this goal, we consider a spherical cavity in a homogeneous, isotropic unbounded perfect conductive thermoelastic body penetrated by an initial magnetic field $\boldsymbol{H}$. For spherically symmetric interactions, the displacement vector possesses only the radi al component $u_r = u(r,t)$, where r is the radial distance in spherical coordinates (r,φ,Θ) measured from the origin. Also, we shall consider a one-dimensional disturbance of the medium, so that all the field variables depend only on the distance r and the time t.

The strain component takes the form:

$$e_{rr} = \frac{\partial u}{\partial r}\ , e_{\varphi\varphi} = e_{\Theta\Theta} = \frac{u}{r},\ e_{r\Theta} = e_{r\varphi} = e_{\Theta\varphi} = 0. \tag{15}$$

The dilatation is given by:

$$e = \frac{\partial u}{\partial r} + \frac{2u}{r} \tag{16}$$

The radial stress σ_{rr} and the hoop stress $\sigma_{\varphi\varphi}$ given in Eq. (12) are given by:

$$\begin{aligned}\sigma_{rr} &= 2\mu\frac{\partial u}{\partial r} + \lambda\left(\frac{\partial u}{\partial r} + \frac{2u}{r}\right) - \gamma\theta,\\ \sigma_{\varphi\varphi} &= 2\mu\frac{u}{r} + \lambda\left(\frac{\partial u}{\partial r} + \frac{2u}{r}\right) - \gamma\theta.\end{aligned} \tag{17}$$

The motion equation in the spherical coordinates is expressed as:

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{2}{r}(\sigma_{rr} - \sigma_{\varphi\varphi}) + F_r = \rho\frac{\partial^2 u}{\partial t^2}, \tag{18}$$

where F_r is the Lorentz force.
The Maxwell's electromagnetic field equations for a homogeneous and electrically conducting thermoelastic solid can be retrieved as [43].

$$\begin{gathered}\nabla\times\boldsymbol{h} = \boldsymbol{J} + \varepsilon_0\frac{\partial\boldsymbol{E}}{\partial t},\ \ \nabla\times\boldsymbol{E} = -\mu_0\frac{\partial\boldsymbol{h}}{\partial t},\quad \boldsymbol{E} = -\mu_0\left(\frac{\partial\boldsymbol{u}}{\partial t}\times\boldsymbol{H}\right),\\ \boldsymbol{h} = \nabla\times(\boldsymbol{u}\times\boldsymbol{H}),\ \ \nabla\cdot\boldsymbol{h} = 0.\end{gathered} \tag{19}$$

where the induced magnetic field is $\boldsymbol{h}$, induced electric field is $\boldsymbol{E}$, μ_0 denotes the magnetic permeability and ε_0 is the electric permeability. Assume that the initial magnetic field permeated into the medium has components $\boldsymbol{H} = (0,0,H_0)$ hence, from the relations (19), we can get:

$$\boldsymbol{E} = \left(0,\mu_0 H_0\frac{\partial\boldsymbol{u}}{\partial t},0\right), \boldsymbol{h} = -\left(0,0,H_0\left(\frac{\partial u}{\partial r} + \frac{2u}{r}\right)\right),\ \boldsymbol{J} = \mu_0 H_0\left(\frac{\partial e}{\partial r} - \varepsilon_0\mu_0\frac{\partial^2 \mathrm{u}}{\partial t^2}\right). \tag{20}$$

The Lorentz force can be obtained from Eq. (20) in the form:

$$F_r = \mu_0(\boldsymbol{J}\times\boldsymbol{h})_r = \mu_0 H_0^2\left(\frac{\partial e}{\partial r} - \varepsilon_0\mu_0\frac{\partial^2 \mathrm{u}}{\partial t^2}\right) \tag{21}$$

The constitutive relations (17) together with the motion equation (18) yields:

$$(\lambda + 2\mu + \mu_0 H_0^2)\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r} + \frac{2u}{r}\right) - \gamma\frac{\partial\theta}{\partial r} = (\rho + \varepsilon_0\mu_0^2 H_0^2)\frac{\partial^2 u}{\partial t^2} \tag{22}$$

The modified equation of heat conduction without heat source can be deduced as:

$$\left(1 + \sum_{k=1}^{m}\frac{\tau_q^k}{k!}\frac{\partial^k}{\partial t^k}\right)\left[\rho C_e\frac{\partial^2\theta}{\partial t^2} + \gamma T_0\frac{\partial^2 e}{\partial t^2} - \frac{\partial Q}{\partial t}\right] = K^*\left(1 + \sum_{r=k}^{n}\frac{\tau_\vartheta^k}{k!}\frac{\partial^k}{\partial t^k}\right)\nabla^2\theta \tag{24}$$

To solve the problem, we use the following non-dimensional parameters:

$$\begin{gathered}\{r',u'\} = c_0\eta\{r,u\},\ \{t',\tau'_q,\tau'_\vartheta\} = c_0^2\eta\{t,\tau_q,\tau_\vartheta\},\ h' = \frac{h}{H_0}, K^{*\prime} = \frac{K^*}{c_0^2\rho C_E},\\ \{E',J'\} = \frac{\{E,J\}}{c_0\mu_0 H_0},\ \theta' = \frac{\gamma\theta}{\lambda+2\mu},\ \sigma'_{ij} = \frac{\sigma_{ij,}}{\lambda+2\mu},\ \eta = \frac{\rho C_E}{K},\ \ c_0 = \sqrt{\frac{\lambda+2\mu}{\rho}}.\end{gathered} \tag{25}$$

In view of Eq. (25), the non-dimensional form of Eqs. (17), (20), (23) and (24) reduce to:

$$\left(1 + \frac{a_0^2}{c_0^2}\right)\nabla^2 u - \frac{\partial\theta}{\partial r} = \left(1 + \frac{a_0^2}{c^2}\right)\frac{\partial^2 u}{\partial t^2}, \tag{26}$$

$$\left(1 + \sum_{k=1}^{m}\frac{\tau_q^k}{k!}\frac{\partial^k}{\partial t^k}\right)\left[\frac{\partial^2\theta}{\partial t^2} + \varepsilon\frac{\partial^2 e}{\partial t^2} - \frac{\partial Q}{\partial t}\right] = K^*\left(1 + \sum_{k=1}^{n}\frac{\tau_\vartheta^k}{k!}\frac{\partial^k}{\partial t^k}\right)\nabla^2\theta, \tag{27}$$

$$\begin{aligned}\sigma_{rr} &= (\beta^2 - 2)e + 2\frac{\partial u}{\partial r} - \beta^2\theta,\\ \sigma_{\varphi\varphi} &= (\beta^2 - 2)e + 2\frac{u}{r} - \beta^2\theta,\end{aligned} \tag{28}$$

$$E = \frac{\partial u}{\partial t},\qquad h = -e,\qquad J = \frac{\partial e}{\partial r} - V^2\frac{\partial^2 u}{\partial t^2}, \tag{29}$$

where

$$\beta^2=\frac{\lambda+2\mu}{\mu},\varepsilon=\frac{\gamma^2 T_0}{\rho C_E(\lambda+2\mu)},a_0=\frac{\mu_0 H_0^2}{\rho},c^2=\frac{1}{\mu_0\varepsilon_0},V=\frac{c_0}{c},\ \nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}. \quad (30)$$

We have dropped the primes in Eqs. (26)-(29) for convenience and clarity of the problem.
We assume that initially the medium is at rest, both thermally and mechanically, so that initial conditions of the problem are taken as:

$$u(r,0)=\frac{\partial u(r,0)}{\partial r}=0,\quad \theta(r,0)=\frac{\partial\theta(r,0)}{\partial r}=0. \quad (31)$$

We also suppose that the non-dimensional point heat source Q near the original and situated along the radial direction, causing thermoelastic interactions (at time $t>0$) and is specified by [44]:

$$Q(r,t)=Q_0\delta(r)H(t) \quad (32)$$

Here, Q_0 is constant, $\delta(r)$ is the Dirac delta function and $H(t)$ is the Heaviside unit step function.

4. Transform solution

Applying the Laplace transform technique, Eqs. (26)-(29) can be transformed into the forms:

$$\left(1+\frac{a_0^2}{c_0^2}\right)\left(\frac{d^2\bar{u}}{dr^2}+\frac{1}{r}\frac{d\bar{u}}{dr}-\frac{\bar{u}}{r^2}\right)-s^2\left(1+\frac{a_0^2}{c^2}\right)\bar{u}=\frac{d\bar{\theta}}{dr}, \quad (33)$$

$$s\ell_q\bar{\theta}+s\varepsilon\ell_q e-\frac{\ell_q Q_0}{s^2}\delta(r)=\ell_\vartheta\nabla^2\bar{\theta}, \quad (34)$$

$$\bar{\sigma}_{rr}=(\beta^2-2)\bar{e}+2\frac{d\bar{u}}{dr}-\beta^2\bar{\theta},$$
$$\bar{\sigma}_{\varphi\varphi}=(\beta^2-2)\bar{e}+2\frac{\bar{u}}{r}-\beta^2\bar{\theta}, \quad (35)$$

$$\bar{E}=s\,\bar{u},\qquad h=-\bar{e},\qquad J=\frac{\partial\bar{e}}{\partial r}-V^2 s^2\bar{u}\,, \quad (36)$$

where

$$\ell_q=s\left(1+\sum_{k=1}^{m}\frac{\tau_q^k}{k!}s^k\right),\qquad \ell_\vartheta=K^*\left(1+\sum_{k=1}^{n}\frac{\tau_\vartheta^k}{k!}s^k\right). \quad (37)$$

Introducing the thermoelastic potential function ψ, defined by:

$$\bar{u}=\frac{d\bar{\psi}}{dr} \quad (38)$$

Hence, we can write Eqs. (33) and (34) in the following forms:

$$\alpha_1\nabla^2\bar{\psi}-\alpha_2\bar{\psi}=\bar{\theta}, \quad (39)$$

$$\alpha_4\nabla^2\bar{\psi}-\alpha_3\delta(r)=\ell_\vartheta\nabla^2\bar{\theta}-s\ell_q\bar{\theta}, \quad (40)$$

where

$$\alpha_1=1+\frac{a_0^2}{c_0^2},\ \alpha_2=s^2\left(1+\frac{a_0^2}{c^2}\right),\alpha_3=\frac{\ell_q Q_0}{s^2},\alpha_4=s\varepsilon\ell_q. \quad (41)$$

Elimination of $\bar{\theta}$ from Eqs. (39) and 40) yields:

$$\left[\alpha_1\ell_\vartheta\nabla^4-\{\alpha_2\ell_\vartheta+\alpha_1 s\ell_q+\alpha_4\}\nabla^2+\alpha_2 s\ell_q\right]\bar{\psi}=-\alpha_3\delta(r) \quad (42)$$

Equation (42) may be rewritten in the form:

$$(\nabla^2-m_1^2)(\nabla^2-m_2^2)\bar{\psi}=-\alpha_5\delta(r), \quad (43)$$

where $\alpha_5=\alpha_3/\alpha_1\ell_\vartheta$ and m_1^2 and m_2^2 are the roots with positive real parts of the biquadratic equation:

$$\alpha_1\ell_\vartheta m^4-\{\alpha_2\ell_\vartheta+\alpha_1 s\ell_q+\alpha_4\}m^2+\alpha_2 s\ell_q=0. \quad (44)$$

Using the well-known Helmholtz equation [45]:

$$(\nabla^2-m^2)\left(\frac{e^{-mr}}{r}\right)=-4\pi\delta(r), \quad (45)$$

and imposing the regularity condition that $\psi\to 0$ as $r\to\infty$, the general solutions of (45) can be expressed as:

$$\bar{\psi}=\frac{\alpha_5}{4\pi r(m_1^2-m_2^2)}\left[e^{-m_1 r}-e^{-m_2 r}\right] \quad (46)$$

Substituting from Eq. (46) into Eqs. (38) and (39), the solution of the temperature $\bar{\theta}$ can be written as:

$$\bar{\theta} = \frac{\alpha_5}{4\pi\ (m_1^2 - m_2^2)}[(\alpha_1 m_1^2 - \alpha_2)e^{-m_1 r} - (\alpha_1 m_2^2 - \alpha_2)e^{-m_2 r}], \tag{47}$$

$$\bar{u} = -\frac{\alpha_5}{4\pi\ (m_1^2 - m_2^2)}\left[\left(m_1 + \frac{1}{r}\right)e^{-m_1 r} - \left(m_2 + \frac{1}{r}\right)e^{-m_2 r}\right]. \tag{48}$$

By substituting the expressions of $\bar{u}$ and $\bar{\theta}$ into Eq. (35), we obtain:

$$\begin{aligned}\bar{\sigma}_{rr} = -\frac{\alpha_5 e^{-(m_1+m_1)r}}{4\pi r^2(m_1^2 - m_2^2)}&[e^{m_1 r}\left(4 + 4m_2 r + \alpha_2 r^2\beta^2 + r^2\beta^2 m_2^2(1-\alpha_1)\right)\\ &-[e^{m_2 r}\left(4 + 4m_1 r + \alpha_2 r^2\beta^2 + r^2\beta^2 m_1^2(1-\alpha_1)\right)],\end{aligned} \tag{49}$$

$$\begin{aligned}\bar{\sigma}_{\varphi\varphi} = \frac{\alpha_5 e^{-(m_1+m_1)r}}{4\pi r^2(m_1^2 - m_2^2)}&[e^{m_1 r}\left(2 + 2m_2 r + r^2 m_2^2(\beta^2 - 2) - r^2\beta^2(\alpha_2 - \alpha_1 m_2^2)\right)\\ -[e^{m_2 r}&\left(2 + 2m_1 r + r^2 m_1^2(\beta^2 - 2) - r^2\beta^2(\alpha_2 - \alpha_1 m_1^2)\right)].\end{aligned} \tag{50}$$

In addition, the induced electric and magnetic fields can be deduced as:

$$\bar{h} = \frac{\alpha_5}{4\pi r(m_1^2 - m_2^2)}[m_1^2 e^{-m_1 r} - m_2^2 e^{-m_2 r}], \tag{51}$$

$$\bar{E} = -\frac{s\alpha_5}{4\pi r(m_1^2 - m_2^2)}\left[\left(m_1 + \frac{1}{r}\right)e^{-m_1 r} - \left(m_2 + \frac{1}{r}\right)e^{-m_2 r}\right]. \tag{52}$$

In this work an accurate and efficient numerical method based on a Fourier series expansion [46] is used to obtain the inversion of the Laplace transforms. In this technique, any function $\bar{g}(r,s)$ in Laplace domain can be inverted to the time domain $g(r,t)$ as:

$$g(r,t) = \frac{e^{\omega t}}{t}\left(\frac{1}{2}\bar{g}(r,\omega) + Re\sum_{n=1}^{N_f}\bar{g}\left(r, \omega + \frac{in\pi}{t}\right)(-1)^n\right), \tag{53}$$

where N_f is a finite number of terms, Re is the real part and i is imaginary number unit. For faster convergence, numerous numerical experiments have shown that the value of ω satisfies the relation $\omega t \cong 4.7$ [42].

5. Numerical results and verification

To illustrate and validate the accuracy of the theoretical results obtained in the previous section and to compare different theories of thermoelasticity that were developed earlier, some numerical results are given to the field variables covered in the study. The results depict the variations the displacement u, temperature θ as well as thermal stresses σ_{rr} and $\sigma_{\varphi\varphi}$ in the context of the modified model of heat conduction including higher order of time derivative. We prepare some reference results in tabular form to make comparisons with the results of other researchers. Also, our results will be presented in tabular form to help other researchers compare their results and validate their accuracy. In addition, some comparisons of the field quantities are shown in figures to explain the effects of the higher expansion orders m, n and the two-phase-lag parameters τ_q and τ_ϑ. For the computational purpose, copper has been taken into account. The physical values are taken as [47, 48]:

$$C_E = 383.1\frac{\text{J}}{(\text{kg K})},\quad T_0 = 293\ (\text{K}),\quad \alpha_t = 1.78\times10^{-5}\left(\frac{1}{\text{K}}\right),\quad K = 386\frac{\text{W}}{(\text{m K})},$$

$$\lambda = 7.76\times10^{10}\left(\frac{\text{N}}{\text{m}^2}\right),\ \mu = 3.86\times10^{10}\left(\frac{\text{N}}{\text{m}^2}\right), \rho = 8954\left(\frac{\text{kg}}{\text{m}^3}\right), c = 425\left(\frac{\text{m}}{\text{s}}\right)$$

$$\varepsilon_0 = 8.854\times10^{-12}\left(\frac{\text{C}^2}{\text{Nm}^2}\right),\quad \mu_0 H_0 = 1\ (\text{Tesla}),\quad \mu_0 = 1.256\times10^{-6}\left(\frac{\text{Ns}^2}{\text{C}^2}\right).$$

$$\tau_q = 0.04\ \text{sec}, \tau_\vartheta = 0.03\ \text{sec}, a_0 = 1.02,\quad \varepsilon = 0.0168,\ Q_0 = 10,\ a = 1.$$

Taking into account the physical data above, we evaluated the numerical results of the field variables using Mathematica software. As aforementioned in the literature, most researchers explained the thermoelastic problems without presenting any tabulated results, but often in the form of graphs. The obtained results for the field quantities for various high orders m and n and the instantaneous time t are depicted in reference Tables with radius r. Tables (1-4) represent the distributions of the temperature θ, the displacement u as well as the thermal stresses σ_{rr} and $\sigma_{\varphi\varphi}$ for different models of thermoelasticity (CTE, LS, GN-II, and HGN). The Tables show that the values of the field variables in the recommended generalized thermoelasticity models are restricted in a specific bounded area close to the surface, outside this region the distributions of the variables are absent. This means that heat propagation speeds are limited.

Many researchers in this field have neglected the upper expansion orders of the time derivatives and retaining terms up to the first and second orders [8, 11, 49-53]. From Tables (1-4) it can be found that the physical quantities depend not only on the time t and radial space r, but also on the higher expansion order parameters m and n. This means that higher derivatives in time must be considered and cannot be neglected in studies. According to this phenomenon, the delayed behavior of thermal conductivity in solids should not be ignored, especially when the times elapsed during the interim process are too small or the heat flow is too high [42].

Table 1. Effect of the higher order Taylor expansions on the temperature θ

r	CTE	LS	GN	HGN					
				$m=1,n=1$	$m=2,n=1$	$m=2,n=2$	$m=3,n=2$	$m=4,n=2$	$m=4,n=3$
1.0	0.40155	0.233612	0.206969	0.208987	0.208654	0.212442	0.214088	0.214293	0.214314
1.1	0.234559	0.133424	0.121355	0.139991	0.15464	0.141015	0.141898	0.142008	0.142019
1.2	0.142106	0.0784585	0.0736269	0.0950443	0.0913761	0.0947302	0.0951565	0.0952091	0.0952144
1.3	0.0875364	0.0464836	0.0452998	0.0648221	0.0702469	0.0638026	0.0639564	0.0639751	0.063977
1.4	0.0541531	0.0273217	0.0279035	0.0441033	0.0415541	0.0427571	0.0427519	0.0427509	0.0427508
1.5	0.0333289	0.015695	0.0170304	0.0297422	0.0306872	0.0282966	0.0282036	0.0281917	0.0281905
1.6	0.0202243	0.00864005	0.0101895	0.0197392	0.018256	0.0183281	0.0181918	0.0181746	0.0181729
1.7	0.011971	0.00440464	0.00589355	0.0127729	0.0121202	0.0114708	0.0113191	0.0113	0.0112981
1.8	0.00680478	0.00191813	0.0032214	0.00794451	0.00699313	0.00678833	0.00663777	0.00661891	0.00661703
1.9	0.00361318	0.000513157	0.00158883	0.00462951	0.00370613	0.00363232	0.0034925	0.00347501	0.00347327
2.0	0.00168383	0.000230546	0.000619783	0.00238763	0.00169804	0.00154784	0.00142359	0.00140809	0.00140653

Table 2. Effect of the higher order Taylor expansions on the displacement u

r	CTE	LS	GN	HGN					
				$m=1,n=1$	$m=2,n=1$	$m=2,n=2$	$m=3,n=2$	$m=4,n=2$	$m=4,n=3$
1.0	0.186858	0.111646	0.0953463	0.104506	0.111724	0.108145	0.109308	0.109453	0.109468
1.1	0.130156	0.0771082	0.0667825	0.0709333	0.0753158	0.0728845	0.0735811	0.0736679	0.0736766
1.2	0.0896681	0.0524655	0.0461962	0.0472499	0.0497471	0.0481388	0.0485307	0.0485794	0.0485843
1.3	0.0610868	0.0351558	0.0315521	0.0308675	0.0321745	0.0311344	0.0313362	0.0313612	0.0313637
1.4	0.0411372	0.0231826	0.0212679	0.0197527	0.0203482	0.0196901	0.0197795	0.0197906	0.0197917
1.5	0.027369	0.0150277	0.0141379	0.012358	0.0125558	0.0121486	0.0121761	0.0121795	0.0121798
1.6	0.0179755	0.00956073	0.00925902	0.0075373	0.00753297	0.00728713	0.00728389	0.00728343	0.00728339
1.7	0.0116421	0.00595593	0.0059653	0.00446195	0.00437086	0.00422661	0.00421066	0.00420865	0.00420845
1.8	0.00742451	0.00362078	0.00377321	0.00254623	0.00243159	0.00234984	0.00233095	0.00232858	0.00232835
1.9	0.00465267	0.00213723	0.00233645	0.00138479	0.00127767	0.0012334	0.00121623	0.00121409	0.00121388
2.0	0.00285688	0.00121517	0.00141042	0.000702942	0.000615761	0.000593273	0.000579587	0.000577885	0.000577715

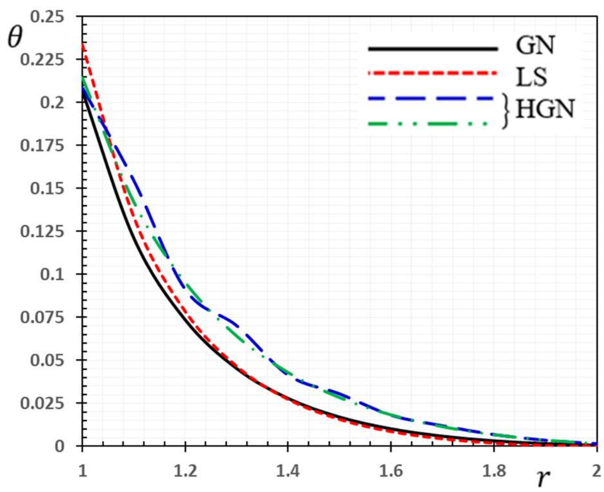

Fig. 1. Variation of temperature θ versus radius r for different models of thermoelasticity

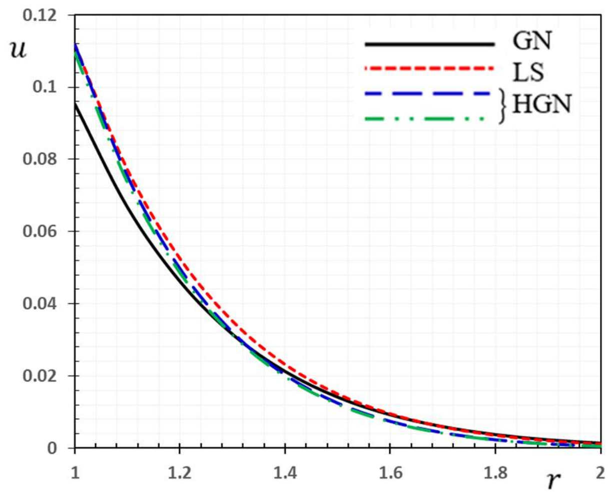

Fig. 2. Variation of displacement u versus radius r for different models of thermoelasticity

When expansion orders are less than or equal to four, the relevant models can be compatible with thermodynamics, provided that appropriate assumptions are made at the time of delay [28, 29, 42, 54]. All of these expansion orders are able to account for higher-order effects in τ_q and τ_ϑ linked to the thermal lagging phenomena and are closely related to the number of heat carriers involved. From the proposed Tables (1-4), we find that the results obtained confirm this phenomenon. It is enough to put $m=4$, $n=2$, for close and accurate results. When the values of the orders m, n, are increased, it seems that the wave front is propagated faster than those for small orders (see Tables 1-4). In fact, all tables that show the HGN model gives accurate results for all values of m and n. Two-phase lag models are practically applicable to modeling problems of nanoscale thermal transport problems for semiconductor devices. Therefore, semiconductors can be classified by their ability to heat conduction [43].

Table 1 indicates the variation of temperature θ for range $1 \le r \le 2$, for the four models of thermoelasticity. Through the tables provided we note that the variations of temperature are observed to be very sensitive to the parameters of the higher order of the time derivatives m and n. It has been is well verified that in the classical theory of coupled thermoelasticity (CTE), the heat conduction equation represents the infinite speed of propagation of the thermal wave, which contrasts physical observations. The hyperbolic models of thermoelasticity (LS, GN-II, and HGN) give significantly different results than the parabolic theory (CTE). The phenomenon of limited speeds of propagation was observed. This is expected because the thermal wave propagates with a finite speed. This observation is consistent with the theoretical results acquired by other investigations.

Table 3. Effect of the higher order Taylor expansions on the radial stress σ_{rr}

r	CTE	LS	GN	HGN					
				$m=1, n=1$	$m=2, n=1$	$m=2, n=2$	$m=3, n=2$	$m=4, n=2$	$m=4, n=3$
1.0	-1.52626	-0.893985	-0.781114	-0.814208	-0.864058	-0.83416	-0.841781	-0.84273	-0.842825
1.1	-0.898936	-0.517833	-0.46172	-0.459655	-0.482714	-0.466241	-0.46973	-0.470164	-0.470207
1.2	-0.551859	-0.311323	-0.283872	-0.268331	-0.278449	-0.26911	-0.270624	-0.270812	-0.270831
1.3	-0.344097	-0.189207	-0.1769	-0.157625	-0.161359	-0.156035	-0.156582	-0.156649	-0.156656
1.4	-0.215299	-0.114762	-0.11038	-0.0918797	-0.0926044	-0.0895746	-0.0896673	-0.0896784	-0.0896795
1.5	-0.134247	-0.0689127	-0.0684675	-0.0526249	-0.0520826	-0.0503738	-0.0502769	-0.0502646	-0.0502634
1.6	-0.0830196	-0.0406994	-0.041995	-0.0293392	-0.0283975	-0.027452	-0.0272983	-0.027279	-0.0272771
1.7	-0.0507092	-0.0234797	-0.025346	-0.0157333	-0.0147933	-0.014287	-0.0141375	-0.0141188	-0.014117
1.8	-0.0304647	-0.0131156	-0.0149704	-0.0079646	-0.0071844	-0.00692702	-0.00680511	-0.00678995	-0.00678844
1.9	-0.0179105	-0.00699911	-0.00859068	-0.00367059	-0.0030876	-0.00296737	-0.00287783	-0.00286673	-0.00286562
2.0	-0.0102327	-0.00348384	-0.00473741	-0.00140432	-0.00100145	-0.0009536	-0.000892882	-0.00088538	-0.00088463

Table 4. Effect of the higher order Taylor expansions on the hoop stress $\sigma_{\varphi\varphi}$

r	CTE	LS	GN	HGN					
				$m=1, n=1$	$m=2, n=1$	$m=2, n=2$	$m=3, n=2$	$m=4, n=2$	$m=4, n=3$
1.0	-0.815989	-0.46821	-0.423626	-0.407567	-0.402979	-0.409904	-0.412355	-0.412661	-0.412691
1.1	-0.4719	-0.262118	-0.245346	-0.25383	-0.282056	-0.251725	-0.252651	-0.252765	-0.252776
1.2	-0.28136	-0.150033	-0.145962	-0.160082	-0.169775	-0.156197	-0.156352	-0.15637	-0.156372
1.3	-0.168788	-0.0854891	-0.0870439	-0.100604	-0.0992513	-0.096275	-0.0960598	-0.0960322	-0.0960294
1.4	-0.100619	-0.0476983	-0.0513521	-0.0622994	-0.0588253	-0.0581939	-0.0578283	-0.0577822	-0.0577776
1.5	-0.0590795	-0.0256416	-0.0296637	-0.0375984	-0.0344153	-0.0340215	-0.0336241	-0.0335743	-0.0335693
1.6	-0.033874	-0.0129761	-0.0165875	-0.0217977	-0.0194082	-0.0188504	-0.0184816	-0.0184354	-0.0184308
1.7	-0.0187577	-0.00590754	-0.00883097	-0.0118511	-0.0100562	-0.00952412	-0.00921008	-0.00917091	-0.00916699
1.8	-0.00985884	-0.00213291	-0.00434252	-0.00574315	-0.00425669	-0.00397152	-0.00371952	-0.00368816	-0.00368502
1.9	-0.00475739	-0.000254054	-0.00183678	-0.00212758	-0.000894239	-0.00082381	-0.000630843	-0.00060688	-0.00060448
2.0	-0.00194137	0.000569791	-0.000510777	-0.000104331	0.000844436	0.000822682	0.000964407	0.000981967	0.000983722

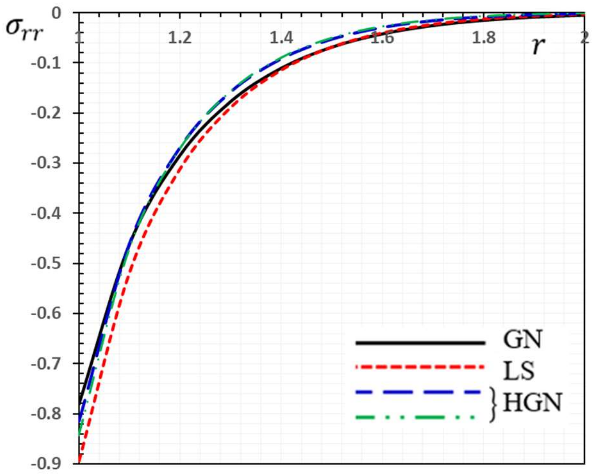

Fig. 3. Variation of radial stress σ_{rr} versus radius r for different models of thermoelasticity

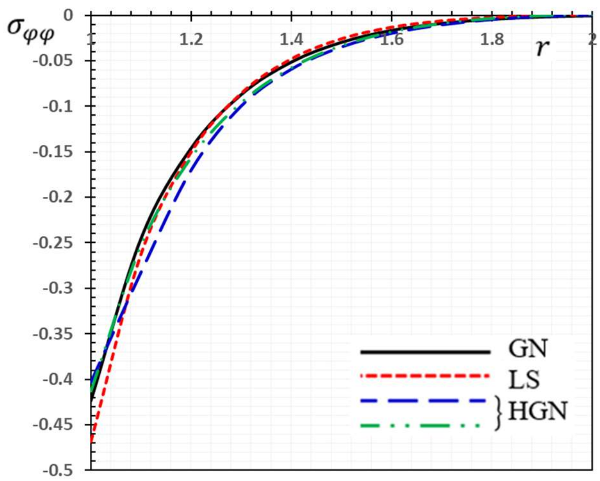

Fig. 4. Variation of hoop stress $\sigma_{\varphi\varphi}$ versus radius r for different models of thermoelasticity

Figure 1 is plotted to show the variation of the temperature θ against the space coordinate r in the LS, GN-II, and HGN models. It is clear from Fig. 1 that variation of temperature θ in all models decreases with the distance r. It is also observed that the magnitude of temperature θ for the LS model is greater than that of the GN-II, and HGN model. The graph corresponding to the GN-II model decreasing faster than that corresponding to the graph of the HGN model.

The variation of the radial displacement u in the range $1 \le r \le 2$, for the four theories of thermoelasticity is given in Table 2. It was found that the values of the parameters m and n play a significant role in changing the displacement value. As seen from Table 2, displacement attains maximum value on the boundary of the cavity $r = 1$ and decays sharply in the range $1 \le r \le 2$. As the distance r increases, the magnitude of displacement decreases tends to a steady state continuously and finally diminishes to zero value. Also, the displacement profile is larger for the CTE model than that of the HGN model, which is larger than the GN-II model. When the values of m and n increase, the greatest value of the displacement increases and then stabilizes at certain values of these parameters ($m = 3$ and $n = 2$).

Fig. 2 depicts the variation of the displacement u versus the distance r. This figure shows the behaviors are similar to those in Fig. 1. Also, it show the significant effect of the values of the parameters m and n, where we can see that the increase in the parameters m and n cause the increase in the displacement field.

Tables 3 and 4 show the variations of thermal stresses σ_{rr} and $\sigma_{\varphi\varphi}$ against the radial distance r for different high expansion orders m and n. It is observed that the stresses are compressive at the boundary of the cavity. As seen from Tables, thermal stresses attain the maximum values on the boundary of the cavity r for all the models. With the increase of radius r, the magnitude of thermal stresses decay sharply and diminishes to zero values. From Tables 3 and 4, it is shown that the magnitude of thermal stresses for the HGN model is larger than that of the GN-II model. We also, observed that the stress fields were affected by the higher order of time derivative m and n where increasing the parameters values led to a decrease in the amplitudes of the thermal waves.

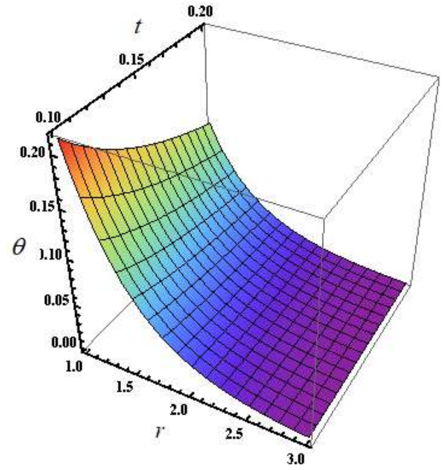

Fig. 5. The temperature θ for different time instant t and distance r

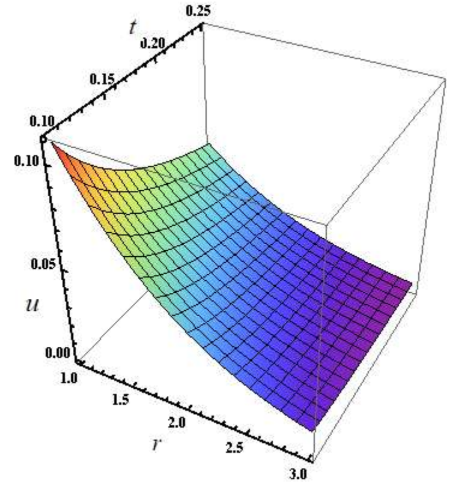

Fig. 6. The displacement u for different time instant t and distance r

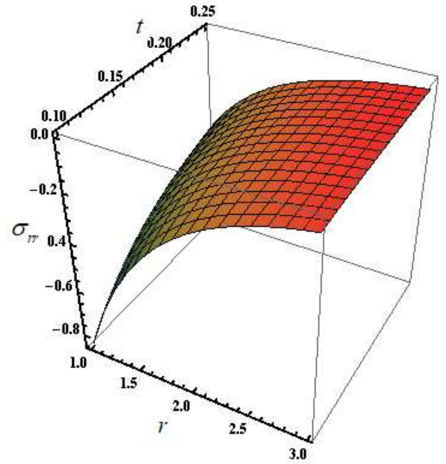

Fig. 7. The radial stress σ_{rr} for different time instant t and distance r

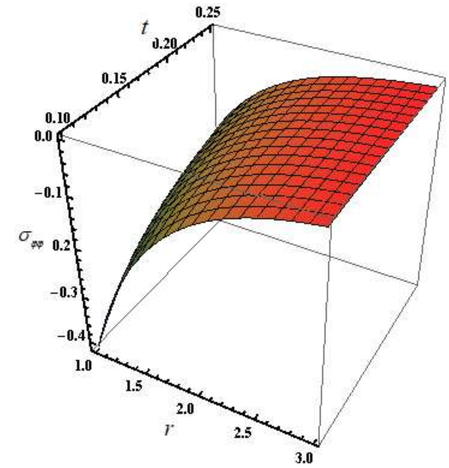

Fig. 8. The hoop stress $\sigma_{\varphi\varphi}$ for different time instant t and distance r

Also, Figs. 3 and 4 show that there are significant differences in the magnitude of thermal stresses σ_{rr} and $\sigma_{\varphi\varphi}$ between the HGN theory and the theory of GN-II. We observed also that the values of variances of thermal stresses are decreased with increasing time. From these figures, it is seen that the new generalized HGN model is more accurate than the LS and CTE models. In the absence of the two-phase lag effect, the results corresponding to the HGN models correspond to the results of the existing literature. From Figs. 3 and 4, it is observed that the effects of delay time parameters and it's higher order derivatives on the field variables are significant for thermal stress fields σ_{rr} and $\sigma_{\varphi\varphi}$. The value of radial stress σ_{rr} is more for the HGN model in comparison to GN-II and LS models due to the two-phase delay. The peak value of $\sigma_{\varphi\varphi}$ predicted by the CTE model is greater than the values of the other models.

In another case, a numerical value of the field variables is computed for different values of distance ($1 \le r \le 2$) and dimensionless time ($0.1 \le t \le 0.2$) and the values are plotted (3D plots) in different Figs. (5-8) to observe the nature of the variations of the field quantities under the HTPL model at different times. The HGN theory with $m = 3, n = 2$ is used in all figures. It is obvious that the field variables are sensitive with the instant of time included. Fig. 5 is plotted to study the effect of the dimensionless time on the temperature θ versus the distance r. Fig. 6 depicts the variation of the displacement u versus the distance r and instant time t. Figs.7 and 8 display the variations of thermal stresses σ_{rr} and $\sigma_{\varphi\varphi}$with distance r and time t. It has been found from Fig. 5 to 8 that the distributions of the temperature θ, the displacement u as well as the thermal stresses σ_{rr} and $\sigma_{\varphi\varphi}$ attains their limiting values and satisfies the enjoined initial conditions recommended in the model.

6. Conclusions

Many researchers have proposed many theories describing heat conduction in recent years. The current work was a theoretical proposition where the diffusion of heat is propagated at a limited speed, unlike the classical models depend on the Fourier's law that leads to an infinite propagation velocity of heat signals. Recently, great interest has been established to investigate the equation of heat conduction and various Taylor approximations [8-13, 18]. In this paper, we could consider several types of Taylor approximations to recover (in particular) Green and Naghdi model without energy dissipation [23]. As used in the theories proposed by Tzou [8, 10] and Choudhuri [26], we extended the Green and Naghdi model (GN-II) [23]. Thus, the HGN model is an alternative construction of the GN-II model.

In addition, the modified model was constructed by considering the Taylor series expansions of both sides of the Fourier's heat and keep in mind the terms up to convenient orders in the phase-lag of heat flux τ_{q} and the phase-lag of thermal displacement gradient τ_{ϑ}. Also, the LS model and GN theory were compared to the new higher-order model (HGN) to investigate the influence of the higher-order time derivative on the field variables. In previous proposed models [8, 11, 26, 50, 53] authors only took into account the first terms of the Taylor series expansion of time order whereas, in our model, the number of terms was determined by the convergence and the stability in the results.

The results showed that: the higher-order time derivative order parameter has a significant effect on the responses, it may tend to weaken the effect of thermal propagation. The added parameter m and n keeps the dimensions of order heat conduction equation consistent. Qualitative comparison with the results in previous proposed models showed that parameters m and n play a pronounced role and affect the velocity of thermal wave.

Conflict of Interest

The author declared no potential conflicts of interest with respect to the research, authorship and publication of this article.

Funding

The author received no financial support for the research, authorship and publication of this article.

References

[1] Biot, M., Thermoelasticity and Irreversible Thermodynamics, *J. Appl. Phys.*, 27, 1956, 240–253.
[2] Hetnarski, R.B., Ignaczak, J., Generalized Thermoelasticity, *J. Thermal Stresses*, 22, 1999, 451–470.
[3] Hetnarski, R.B. and Ignaczak, J., Nonclassical Dynamical Thermoelasticity, *Inter. J. Solids Struct.*, 37, 2000, 215–224.
[4] Chandrasekharaiah, D.S., Hyperbolic Thermoelasticity, A Review of Recent Literature, *Appl. Mech. Rev.*, 51, 1998, 705–729.
[5] Lord, H., Shulman, Y., A Generalized Dynamical Theory of Thermoelasticity, *J. Mech. Phys. Solids*, 15, 1967, 299.
[6] Green, A.E., Lindsay, K.A., Thermoelasticity, *J. Elasticity*, 2, 1972, 1–7.
[7] Ignaczak, J., *Generalized Thermoelasticity and its Applications, in R. B. Hetnarski (ed.), Mechanical and Mathematical Methods, Thermal Stresses III*, North Holland, 1989.
[8] Tzou, D.Y., A Unified Field Approach for Heat Conduction from Macro- to Micro-Scales, *ASME J. Heat Transfer*, 117, 1995, 8–16.
[9] Tzou, D.Y., The Generalized Lagging Response in Small-Scale and High-Rate Heating, *Int. J. Heat Mass Transfer*, 38, 1995, 3231–3240.
[10] Tzou, D.Y., Experimental Support for the Lagging Behavior in Heat Propagation, *AIAA J. Thermophys. Heat Transfer*, 9, 1995, 686–693.
[11] Chandrasekharaiah, D.S., Hyperbolic Thermoelasticity: A Review of Recent Literature, *Appl. Mech. Rev.*, 51, 1998, 705–729.
[12] Quintanilla, R., Racke, R., A Note on Stability of Dual Phase-Lag Heat Conduction, *Int. J. Heat. Mass Transf.*, 49, 2006, 1209–1213.
[13] Quintanilla, R., Racke, R., Qualitative Aspects in Dual Phase-Lag Heat Conduction, *Proc. Royal Soc. A.*, 463, 659–674, 2007.
[14] Zenkour, A.M., Abouelregal, A.E., Effects of Phase-Lags in a Thermoviscoelastic Orthotropic Continuum with a Cylindrical Hole and Variable Thermal Conductivity, *Arch. Mech.*, 67, 2015, 457–475.

[15] Zenkour, A.M., Mashat, D.S., Abouelregal, A.E., The Effect of Dual-Phase-Lag Model on Reflection of Thermoelastic Waves in a Solid Half Space with Variable Material Properties, *Acta Mech. Solida Sinica*, 26, 2013, 659–670.
[16] Prasad, R., Kumar, R., Mukhopadhyay, S., Propagation of Harmonic Plane Waves under Thermoelasticity with Dual-Phase-Lags, *Int. J. Eng. Sci.*, 48(12), 2010, 2028–2043.
[17] Borgmeyer, K., Quintanilla, R., Racke, R., Phase-Lag Heat Condition: Decay Rates for Limit Problems and Well-Posedness, *J. Evol. Equ.*, 14, 2014, 863-884.
[18] Liu, Z., Quintanilla, R., Time Decay in Dual-Phase-Lag Thermoelasticity: Critical Case, *Comm. Pure Appl. Analy.*, 17(1), 2018, 177-190.
[19] Guo, F.L., Wang, G.Q., Rogerson, G.A., Analysis of Thermoelastic Damping in Micro- and Nanomechanical Resonators based on Dual-Phase-Lag Generalized Thermoelasticity Theory, *Int. J. Eng. Sci.*, 60, 2012, 59-65.
[20] Abbas, I.A., A Dual Phase Lag Model on Thermoelastic Interaction in an Infinite Fiber-Reinforced Anisotropic Medium with a Circular Hole, *Mech. Based Des. Struct. Machines*, 43, 2015, 501–513.
[21] Green, A.E., Naghdi, P.M., A Re-examination of the Basic Postulates of Thermomechanics, *Proc. Roy. Soc. Lond. A*, 432, 1991, 171–194.
[22] Green, A.E., Naghdi, P.M., On Undamped Heat Waves in an Elastic Solid, *J. Therm. Stress.*, 15, 1992, 253–264.
[23] Green, A.E., Naghdi, P.M., Thermoelasticity without Energy Dissipation, *J. Elasticity*, 31, 1993, 189–208.
[24] Chandrasekharaiah, D.S., A Note on the Uniqueness of Solution in the Linear Theory of Thermoelasticity without Energy Dissipation, *J. Elasticity*, 43(3), 1996, 279–283.
[25] Chandrasekharaiah, D.S., A Uniqueness Theorem in the Theory of Thermoelasticity without Energy Dissipation, *J. Therm. Stresses*, 19(3), 1996, 267–272.
[26] Choudhuri, S.R., On a Thermoelastic Three-Phase-Lag Model, *J. Therm. Stresses*, 30(3), 2007, 231–238.
[27] El-Karamany, A.S., Ezzat, M.A., On the Phase-Lag Green-Naghdi Thermoelasticity Theories, *Appl. Math. Model.*, 40, 2016, 5643–5659.
[28] Ciarletta, M.A., Theory of Micropolar Thermoelasticity without Energy Dissipation, *J. Therm. Stresses*, 22, 2009, 581–594.
[29] Chiriţă, S., Ciarletta, M., Reciprocal and Variational Principles in Linear Thermoelasticity without Energy Dissipation, *Mech. Res. Commun.*, 37, 2010, 271–275.
[30] Ieşan, D., On a Theory of Thermoelasticity without Energy Dissipation for Solids with Microtemperatures, *Z. Angew. Math. Mech.*, 98(6), 2018, 870-885.
[31] Quintanilla, R., On Existence in Thermoelasticity without Energy Dissipation, *J. Therm. Stresses*, 25, 2002, 195-202.
[32] Allam, M.N., Elsibai K.A., Abouelregal, A.E., Electromagneto-Thermoelastic Problem in a Thick Plate using Green and Naghdi Theory, *Int. J. Eng. Sci.*, 47, 2009, 680-690.
[33] Allam, M.N., Elsibai K.A., Abouelregal, A.E., Electromagneto-Thermoelastic Plane Waves without Energy Dissipation for an Infinitely Long Annular Cylinder in a Harmonic Field, *J. Therm. Stresses*, 30, 2007,195–210.
[34] Marin, M., Baleanu, D., On Vibrations in Thermoelasticity without Energy Dissipation for Micropolar Bodies, *Bound. Val. Prob.*, 2016, 2016, 111.
[35] Khedr M.El., Khader, S.A., A Problem in Thermoelasticity with and without Energy Dissipation, *J. Phys. Math.*, 8(3), 2017, 1000243.
[36] Marin, M., Cesaro means in thermoelasticity of dipolar bodies, *Acta Mech.*, 122(1-4), 1997, 155-168.
[37] Hassan, M., Marin, M., Ellahi, R., Alamri, S.Z., Exploration of Convective Heat Transfer and Flow Characteristics Synthesis by Cu–Ag/Water Hybrid-Nanofluids, *Heat Transfer Research*, 49(18), 2018, 1837-1848.
[38] Chiriţă, S., Ciarletta, M., Tibullo, V., On the Wave Propagation in the Time Differential Dual-Phase-Lag Thermoelastic Model, *Proc. Royal Soc. A*, 471, 2015, 20150400.
[39] Chiriţă, S., High-Order Effects of Thermal Lagging in Deformable Conductors, *Int. J. Heat Mass Trans.*, 127, 2018, 965–974.
[40] Cattaneo, C., A Form of Heat Conduction Equation which Eliminates the Paradox of Instantaneous Propagation, *Comp. Rend.*, 247, 1958, 431-433.
[41] Vernotte, P., Les paradoxes de la Theorie Continue de l'Equation de la Chaleur, *Comp. Rend.*, 246, 1958, 3154-3155.
[42] Chiriţă, S., On the Time Differential Dual-Phase-Lag Thermoelastic Model, *Meccanica*, 52, 2017, 349–361.
[43] Zenkour, A.M., Abouelregal, A.E., Alnefaie, K.A., Abu-Hamdeh, N.H., Seebeck Effect on a Magneto-Thermoelastic Long Solid Cylinder with Temperature-Dependent Thermal Conductivity, European *J. Pure Appl. Math.*, 10(4), 2017, 786-808.
[44] Chandrasekharaiah, D.S., Srinath, K.S., Thermoelastic Interactions without Energy Dissipation due to a Point Heat Source, *J. Elasticity*, 50, 1998, 97–108.
[45] Morse, P., Feshbach, H., *Methods of Theoretical Physics*, 1st ed., McGraw-Hill, New York, 1953.
[46] Honig, G., Hirdes, U., A Method for the Numerical Inversion of the Laplace Transform, *J. Comput. Appl. Math.*, 10, 1984, 113–132.
[47] Mashat, D.S., Zenkour, A.M., Abouelregal, A.E., Fractional Order Thermoelasticity Theory for a Half-Space Subjected to an Axisymmetric Heat Distribution, *Mech. Adv. Mater. Struct.*, 22(11), 2015, 925–932.
[48] Zenkour, A.M. and Abouelregal, A.E., State-Space Approach for an Infinite Medium with a Spherical Cavity Based Upon Two-Temperature Generalized Thermoelasticity Theory and Fractional Heat Conduction, *Zeitsch. Angewandte Math.*

Phys., 65(1), 2014, 149–164.
[49] Quintanilla, R., Exponential Stability in the Dual-Phase-Lag Heat Conduction Theory, *J. Non-Equil. Thermod.*, 27, 2002, 217–227.
[50] El-Karamany, A.S., Ezzat, M.A., On the Phase-Lag Green-Naghdi Thermoelasticity Theories, *Appl Math. Model.*, 40, 2016, 5643–5659.
[51] Ezzat, M.A., El-Karamany, A.S., Fractional Order Heat conduction Law in Magneto-Thermoelasticity Involving Two Temperatures, *Zeitsch. Angewandte Math. Phys.*, 62, 2011, 937–952.
[52] Ezzat, M.A., El-Karamany, A.S., Theory of Fractional Order in Electro-thermoelasticity, *Eur. J. Mech. A/Solid*, 30, 2011, 491–500.
[53] Ezzat, M.A., Fayik, M., Fractional Order Theory of Thermoelastic Diffusion, *J. Therm. Stresses*, 34, 2013, 851–872.
[54] Xu, H.Y., Jiang, X.Y., Time Fractional Dual-Phase-Lag Heat Conduction Equation, *Chin. Phys. B*, 24(3), 2015, 034401.

ORCID iD

Ahmed Abouelregal https://orcid.org/0000-0003-3363-7924