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Shahid Chamran
University of Ahvaz

Journal of Applied and Computational Mechanics



Research Paper

Numerical Simulation of Non-Newtonian Inelastic Flows in Channel based on Artificial Compressibility Method

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Received June 16 2019; Revised August 26 2019; Accepted for publication August 28 2019.

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& International Research Center for Mathematics & Mechanics of Complex Systems (M&MoCS)

Abstract. In this study, inelastic constitutive modelling is considered for the simulation of shear-thinning fluids through a circular channel. Numerical solutions are presented for power-law inelastic model, considering axisymmetric Poiseuille flow through a channel. The numerical simulation of such fluid is performed by using the Galerkin finite element approach based on artificial compression method (*AC*-method). Usually, the Navier-Stokes partial differential equations are used to describe fluid activity. These models consist of two partial differential equations; a continuity equation (mass conservation) and time-dependent conservation of momentum, which are maintained in the cylindrical coordinate system (axisymmetric) flow in current study. The effects of many factors such as Reynolds number (Re) and artificial compressibility parameter (β_{ac}) are discussed in this study. In particular, this study confirms the effect of these parameters on the convergence level. To meet the method analysis, Poiseuille flow along a circular channel under an isothermal state is used as a simple test problem. This test is conducted by taking a circular section of the pipe. The Findings reveal that, there is a significant effect from the inelastic parameters upon the the velocity temporal convergence-rates of velocity, while for pressure, the change in convergence is modest. In addition, the rate of convergence is increased as the values of artificial compressibility parameter (β_{ac}) are decreased.

Keywords: Finite element method, Galerkin method, Navier-Stokes, Non-Newtonian, Artificial compressibility method.

1. Introduction

There are many numerical studies of the Navier-Stokes equations by using finite element method that have been succeed and widely conducted based on artificial compressibility method (*AC*-method). Because of the *AC*-method can be used to deal with the difficulty of incompressibility condition by using the concept of the transformation of continuity equation, this method has been widely used for both of finite different and finite element method. The *AC*-method was originally introduced by Chorin (1967) with the objective of solving the steady state incompressible Navier-Stokes equations [1]. Moreover, it was also used to solve for unsteady case. For example, Peyret and Taylor [2] and Kao and Yang [3] were some of the first to extend the *AC*-method to the solution of the unsteady incompressible Navier-Stokes equations. The concept of this method is to transform the elliptic incompressible continuity equation to hyperbolic



compressible system by adding the artificial compressibility term in continuity equation. As a result, the new transformed equations can be solved directly by standard time-dependent approaches that is not complicated to apply in the solution. In this context, one can see various investigations of the *AC*-method that have been conducted for solving both of steady and unsteady cases (see for example [2], [4], [5], [6] and [7]).

Following the success of the finite difference scheme based on *AC*-method, several methods have been presented for example the characteristic-Based-Split algorithm (CBS method) based on *AC*-method, which features equal order of all variables was employed to solve steady and unsteady incompressible Navier-Stokes equations by Massarotti et al. [8]. There they used both of implicit and explicit scheme as the time integration approach to deal with the time dependent problem. The advantages of the proposed CBS scheme based on *AC*-method include easy parallelization and implementation procedure. However, this method make the large number of iterations to reach the steady state when using both of explicit and implicit scheme.

Furthermore, many methods are extended successfully to solve unsteady problems. At the first Peyret [9] and Peyret and Taylor [2] are extended the *AC*-method to find the solution of incompressible Navier Stokes equations in unsteady situation. The extension of *AC*-method for unsteady problems has had extensive coverage in the literature ([10]-[18]).

A general Newtonian fluid plays an important role in the new industrial processes. This process involves dealing with many materials; such as polymer melts and their solutions, soap solutions, foams, etc., which exhibit some rheological complexities like shear-thinning and shear-thickening. In this type of fluid the shear stress is defined as a non-linear function of shear rate at the particular time. So, various inelastic constitutive equations have been introduced to treat the non-linearity behaviour of shear stress. For that purpose, power law shear-thinning (pseudo-plastic), shear thickening model is implemented as a well presented model for simulating this type of fluid. This model describes shear viscous stress response of power-law form, which describes shear-thinning or shear-thickening behaviour depending upon the value of the power-law index. In addition, the power-law model is simple enough that it can be included in this study without great complexity, thereby extending the applicability of the results. More specifically, the behaviour of power-law fluids in a channel may prove useful to rheologists employing converging flows to characterize the response of viscoelastic fluids.

In addition to the industrial applications, recently there is a numerical and theoretical interest in investigating the role of non-Newtonian characteristics on the structure of the flow field in conduits (see for example [19]-[21]). Moreover, the literature on flows in conduits for various type of fluids is broad. In that field many studies of recent years have conducted (for more details see [22]-[25]). In contrast to this, through the recent years a few numerical studies related to axisymmetric power-law inelastic flows past a channel have been conducted. Thus this study is concerned with the investigation of this type of flows.

The present study aims to present a study on the incompressible power-law inelastic fluid with a viscosity dependent on simple shear-rate. The novelty here is to study the temporal convergence-rate of the system solution that is taken to be steady state, incompressible, axisymmetric, and laminar, which did not address by researchers previously. In this context, Poiseuille (Ps) flow along a two dimensional planar straight channel, under isothermal condition is studied. The main results of current study focused on the convergence rate of velocity and pressure solutions under the variation of power law parameters and Reynolds number presented. Furthermore, determination of the critical levels of Reynolds number (Re) is also represented the excited issue of this study. Numerical treatments are presented for governing system, where we are utilized the Galerkin finite element method based on *AC*-method, which also has not been addressed before. The unequal order primitive variable of velocity components and pressure will be employed as the main approach. For the numerical solution, the iterative method of Newton-Raphson will be used to solve the set of non-linear equations and the backward different scheme will be employed as the time-integration approach to deal with the time dependent term.

In the next section, the mathematical modelling of motion of the non-Newtonian flows is present. These equations are introduced in the cylindrical coordinates. The artificial compressibility technique is given in Section 3. Since these equations must be studied numerically, the numerical method is characterized in Section 4. The problem discretisation and the related numerical results are presented in Section 5 and 6, respectively.

2. Mathematical Modelling

For inelastic (non-Newtonian) constitutive modelling, the extra stress tensor may be reforested as

$$T = 2\mu(\dot{\gamma}, \dot{\epsilon})d, \quad (1)$$

where $\dot{\gamma}, \dot{\epsilon}$ represent shear-rate and strain-rate for simple shear flow and extensional flow, respectively, and is the deformation rate of the fluid, such that

$$\dot{\gamma} = 2\sqrt{II_d}, \quad (2)$$

$$\dot{\epsilon} = 3\frac{III_d}{II_d}, \quad (3)$$

$$d = \frac{1}{2}(\nabla u + \nabla u^T). \quad (4)$$

Here II_d and III_d represent the second and third invariants of the rate of strain tensor d which, in axisymmetric coordinate system can be defined as (see [26])

$$II_d = \frac{1}{2}tr(d^2) = \frac{1}{2}\left\{\left(\frac{\partial u_r}{\partial r}\right)^2 + \left(\frac{\partial u_z}{\partial z}\right)^2 + \left(\frac{u_r}{r}\right)^2 + \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right)^2\right\}, \quad (5)$$

and

$$III_d = det(d) = \frac{u_r}{r}\left\{\frac{\partial u_r}{\partial r}\frac{\partial u_z}{\partial z} - \frac{1}{4}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right)^2\right\}. \quad (6)$$

Hence, for incompressible inelastic fluid flows under an isothermal setting, the governing equations may be expressed as:

$$\nabla \cdot u = 0, \quad (7)$$

$$\rho \frac{\partial u}{\partial t} = \nabla \cdot (2\mu(\dot{\gamma}, \dot{\epsilon})d) - \rho(u \cdot \nabla u) - \nabla p, \quad (8)$$

where, p is the pressure and ρ is the density of the fluid. In contrast, the equation can be also defined by the non-dimensional groups of Reynolds number (Re) by using the scaling $Re = \rho UL / \mu$, such that (U), (L) and (ρ) are characteristic velocity, length and density, respectively (for more details see ([27]-[31])). Thus, in this case the non-dimensional momentum equation for general Newtonian can be written as:

$$Re \frac{\partial u}{\partial t} = \nabla \cdot (2\mu(\dot{\gamma}, \dot{\epsilon})d) - Re(u \cdot \nabla u) - \nabla p. \quad (9)$$

In this situation, we need to add a constitutive equation to treat the non-linearity behaviour of shear stress. For that purpose, power-law shear-thinning (pseudo-plastic), shear thickness model is implemented as a well-presented model for simulating this type of fluid. A power-law constitutive model is defined as:

$$\tau = (k |\dot{\gamma}|^{n-1})\dot{\gamma}, \quad (10)$$

where, k is a consistency parameter and n is the power-law index. In the cylindrical coordinates, the continuity equation for conservation of mass and time-dependent conservation of momentum equation are expressed as

$$\frac{\partial u_r}{\partial r} + \frac{1}{r}u_r + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0. \quad (11)$$

r -direction

$$\begin{aligned} \frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + u_\theta \left(\frac{1}{r} \frac{\partial u_r}{\partial \theta} - \frac{1}{r} u_\theta \right) + u_z \frac{\partial u_r}{\partial z} &= \frac{-1}{\rho} \frac{\partial p}{\partial r} + \frac{2\mu_s}{\rho} \frac{\partial^2 u_r}{\partial r^2} \\ + \frac{\mu_s}{\rho r^2} \frac{\partial^2 u_r}{\partial \theta^2} - \frac{\mu_s}{\rho r^2} \frac{\partial u_\theta}{\partial \theta} + \frac{2\mu_s}{\rho r} \frac{\partial u_r}{\partial r} + \frac{\mu_s}{\rho r} \frac{\partial^2 u_\theta}{\partial \theta \partial r} + \frac{\mu_s}{\rho} \frac{\partial^2 u_r}{\partial z^2} + \frac{\mu_s}{\rho} \frac{\partial^2 u_z}{\partial z \partial r}. \end{aligned} \quad (12)$$

θ -direction

$$\begin{aligned} \frac{\partial u_\theta}{\partial t} + u_r \frac{\partial u_\theta}{\partial r} + u_\theta \left(\frac{1}{r} u_r + \frac{\partial u_\theta}{\partial \theta} \right) + u_z \frac{\partial u_\theta}{\partial z} &= \frac{-1}{\rho r} \frac{\partial p}{\partial \theta} + \frac{2\mu_s}{\rho r^2} \frac{\partial u_r}{\partial \theta} \\ + \frac{2\mu_s}{\rho r^2} \frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{\mu_s}{\rho} \frac{\partial^2 u_\theta}{\partial r^2} + \frac{\mu_s}{\rho r} \frac{\partial^2 u_r}{\partial r \partial \theta} + \frac{\mu_s}{\rho} \frac{\partial^2 u_\theta}{\partial z^2} + \frac{\mu_s}{\rho r} \frac{\partial^2 u_z}{\partial z \partial \theta}. \end{aligned} \quad (13)$$

z -direction

$$\begin{aligned} \frac{\partial u_z}{\partial t} + u_r \frac{\partial u_z}{\partial r} + u_\theta \frac{1}{r} \frac{\partial u_z}{\partial \theta} + u_z \frac{\partial u_z}{\partial z} &= \frac{-1}{\rho} \frac{\partial p}{\partial z} + \frac{2\mu_s}{\rho} \frac{\partial^2 u_z}{\partial z^2} \\ + \frac{\mu_s}{\rho} \frac{\partial^2 u_z}{\partial r^2} + \frac{\mu_s}{\rho r} \frac{\partial u_r}{\partial z} + \frac{\mu_s}{\rho r} \frac{\partial u_z}{\partial r} + \frac{\mu_s}{\rho} \frac{\partial^2 u_r}{\partial r \partial z} + \frac{\mu_s}{\rho r^2} \frac{\partial^2 u_z}{\partial \theta^2} + \frac{\mu_s}{\rho r} \frac{\partial^2 u_\theta}{\partial \theta \partial z}. \end{aligned} \quad (14)$$

3. Artificial Compressibility Method (AC-Method)

One of the early techniques proposed for solving the incompressible Navier-Stokes equation in primitive variable form is the artificial compressibility method of Chorin (1967). The concept of this method is to transform the elliptic partial incompressible equation to hyperbolic compressible partial differential form by adding the artificial term into continuity equation. The addition of artificial compressibility term will be vanished when the steady state solution is reached [32]. With the addition of this term to continuity equation, the Navier-Stokes equation will be changed to a mixed type of hyperbolic-parabolic equations which can be solved by standard time dependent approach. In order to describe this method, let us apply the artificial term into continuity equation. Thus, the Eq. (11) is replaced by

$$\frac{\partial \rho}{\partial t} + \frac{\partial u_r}{\partial r} + \frac{1}{r} u_r + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0, \quad (15)$$

where ρ is the artificial density. The artificial density is related to the pressure by the artificial equation of state.

$$p = \rho \beta_{ac}, \quad (16)$$

where β_{ac} is the artificial compressibility parameter; $0 < 1/\beta_{ac} \ll 1$. By substituting Eq. (16) into Eq. (15), we get the new form of artificial compressibility continuity equation as

$$\frac{1}{\beta_{ac}} \frac{\partial p}{\partial t} + \frac{\partial u_r}{\partial r} + \frac{1}{r} u_r + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0. \quad (17)$$

4. Numerical Method

The Galerkin finite element method is proposed to solve the system of Eqs. (12)-(14) and Eq. (17). The mine concept of this method is to find the weak form of the equation by using appropriate weight functions W and Q , such that the first function W for momentum equation and the second function Q for continuity equation. In addition, the quadratic shape functions of velocity components in cylindrical coordinates are utilized. These functions are given in natural coordinates as:

$$\begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \\ \psi_6 \end{bmatrix} = \begin{bmatrix} L_1^2 - L_1(L_2 + L_3) \\ L_2^2 - L_2(L_3 + L_1) \\ L_3^2 - L_3(L_1 + L_2) \\ 4L_1L_2 \\ 4L_2L_3 \\ 4L_3L_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} L_1^2 \\ L_2^2 \\ L_3^2 \\ L_1L_2 \\ L_2L_3 \\ L_3L_1 \end{bmatrix}. \quad (18)$$

In contrast, for the pressure, the following linear shape function are employed:

$$\begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} = \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix}, \quad (19)$$

where,

$$L_i = \frac{1}{2A_{area}}(a_i + b_i r + c_i z), \quad (\forall i = 1, 2, 3),$$

where A_{area} is the area of the triangular element and a_i , b_i and c_i are coefficients. Thus, from divergence theorem and rearranging the terms, we obtain the weak form of three dimensional Navier-Stokes equations as:

$$[M_p][\dot{p}] + [Q_1^r][u_r] + [q][u_r] + [Q_2^r][u_\theta] + [Q_3^r][u_z] = 0, \quad (20)$$

$$[M][\dot{u}_r] + [C(u_r, u_\theta, u_z)][u_r] + [c_r][u_\theta] - \frac{1}{Re}[Q_1][p] + [K_{rr}][u_r] + [K_{21}][u_\theta] + [K_{22}][u_r] + [k_r][u_r] + [k_\theta][u_\theta] + [K_{31}][u_z] + [K_{33}][u_r] = 0, \quad (21)$$

$$[M][\dot{u}_\theta] + [C(u_r, u_\theta, u_z)][u_\theta] + [c_r][u_r] - \frac{1}{Re}[Q_2][p] + [K_{11}][u_\theta] + [K_{12}][u_r] - [k_\theta][u_r] + [K_{\theta\theta}][u_\theta] + [K_{33}][u_\theta] + [K_{32}][u_z] = 0, \quad (22)$$

$$[M][\dot{u}_z] + [C(u_r, u_\theta, u_z)][u_z] - \frac{1}{Re}[Q_3][p] + [K_{11}][u_z] + [K_{13}][u_r] - [k_3][u_r] - [k_1][u_z] + [K_{22}][u_z] + [K_{23}][u_\theta] + [K_{zz}][u_z] = 0. \quad (23)$$

Consequently, by using the theory of area coordinates for triangular elements, the mass matrix can be expressed as

$$[M] = \int_{\Omega^e} \psi \psi^T d\Omega = \int_{A^e} \int_0^{2\pi} [N][H][H^T][N^T] r d\theta dA = 2\pi \int_{A^e} [N][H][H^T][N^T] r dA,$$

where,

$$r_m = \frac{r_1 + r_2 + r_3}{3}, \quad z_m = \frac{z_1 + z_2 + z_3}{3}.$$

Thus,

$$[M] = 2\pi r_m [N][\underline{H}][\underline{H}^T][N^T] \int_{A^e} A = 2\pi r_m A_{area} [N][\underline{H}][\underline{H}^T][N^T]. \quad (24)$$

In which the underline symbol refers to the evaluation of matrix at the centroid and the artificial compressibility matrix $[M_p]$ is given by

$$[M_p] = 2\pi r_m A_{area} \frac{1}{\beta_{ac}} [\underline{E}][\underline{E}^T]. \quad (25)$$

Also, the derivative form of shape functions can be defined as

$$\frac{\partial \psi}{\partial r} = [N] \frac{\partial [H]}{\partial r} = [N][B][E],$$

$$\frac{\partial \psi}{\partial \theta} = 0,$$

$$\frac{\partial \psi}{\partial z} = [N] \frac{\partial [H]}{\partial z} = [N][C][E],$$

where,

$$[B] = \frac{1}{2A_{area}} \begin{bmatrix} 2b_1 & 0 & 0 \\ 0 & 2b_2 & 0 \\ 0 & 0 & 2b_3 \\ b_2 & b_1 & 0 \\ 0 & b_3 & b_2 \\ b_3 & 0 & b_1 \end{bmatrix}, \quad [C] = \frac{1}{2A_{area}} \begin{bmatrix} 2c_1 & 0 & 0 \\ 0 & 2c_2 & 0 \\ 0 & 0 & 2c_3 \\ c_2 & c_1 & 0 \\ 0 & c_3 & c_2 \\ c_3 & 0 & c_1 \end{bmatrix}.$$

On the other hand, the final diffusion matrix formula can be written as

$$[K_{rr}] = 4\pi r_m A_{area} \frac{\beta}{Re} [N][B][\underline{E}][\underline{E}^T][B^T][N^T], \quad (26)$$

$$[K_{zz}] = 4\pi r_m A_{area} \frac{\beta}{Re} [N][C][\underline{E}][\underline{E}^T][C^T][N^T], \quad (27)$$

$$[K_{11}] = 2\pi r_m A_{area} \frac{\beta}{Re} [N][B][\underline{E}][\underline{E}^T][B^T][N^T], \quad (28)$$

$$[K_{33}] = 2\pi r_m A_{area} \frac{\beta}{Re} [N][C][\underline{E}][\underline{E}^T][C^T][N^T], \quad (29)$$

$$[K_{13}] = 2\pi r_m A_{area} \frac{\beta}{Re} [N][B][\underline{E}][\underline{E}^T][C^T][N^T], \quad (30)$$

$$[K_{31}] = 4\pi r_m A_{area} \frac{\beta}{Re} [N][C][\underline{E}][\underline{E}^T][B^T][N^T], \quad (31)$$

$$[k_r] = 4\pi A_{area} \frac{\beta}{Re} [N][\underline{H}][\underline{E}^T][B^T][N^T], \quad (32)$$

$$[k_1] = 2\pi A_{area} \frac{\beta}{Re} [N][H][E^T][B^T][N^T], \quad (33)$$

$$[k_3] = 2\pi A_{area} \frac{\beta}{Re} [N][H][E^T][C^T][N^T], \quad (34)$$

$$[K_{\theta\theta}] = 0, [K_{22}] = 0, [K_{12}] = 0, [K_{21}] = 0, [K_{23}] = 0, [K_{32}] = 0, [k_2] = 0, [k_\theta] = 0.$$

Moreover, the gradient matrix is defined as

$$[Q_1] = 2\pi r_m A_{area} [N][B][E][E^T], \quad (35)$$

$$[Q_3] = 2\pi r_m A_{area} [N][C][E][E^T], \quad (36)$$

$$[q] = 2\pi A_{area} [E][H^T][N^T], \quad (37)$$

$$[Q_2] = 0.$$

Finally, the convective matrix is given by

$$[C_r(u_r)] = 2\pi r_m A_{area} [N][H][H^T][N^T][u_r][E^T][B^T][N^T], \quad (38)$$

$$[C_z(u_z)] = 2\pi r_m A_{area} [N][H][H^T][N^T][u_z][E^T][C^T][N^T], \quad (39)$$

$$[c_\theta] = -2\pi A_{area} [N][H][H^T][N^T][u_\theta][H^T][N^T], \quad (40)$$

$$[c_r] = 2\pi A_{area} [N][H][H^T][N^T][u_\theta][H^T][N^T], \quad (41)$$

$$[C_\theta(u_\theta)] = 0.$$

It is known that the real challenge in the present problem is the non-linear term, which needs efficient treatment. So, to address this non-linear term of Eqs. (20)-(23), the Newton-Raphson method is implemented. As the result, the system of equation will be replaced by the following equation:

$$\begin{bmatrix} M & 0 & 0 & 0 \\ 0 & M & 0 & 0 \\ 0 & 0 & M & 0 \\ 0 & 0 & 0 & M_p \end{bmatrix} \begin{bmatrix} \dot{u}_r \\ \dot{u}_\theta \\ \dot{u}_z \\ \dot{p} \end{bmatrix} + \begin{bmatrix} \frac{\partial R_1}{\partial u_r} & \frac{\partial R_1}{\partial u_\theta} & \frac{\partial R_1}{\partial u_z} & \frac{\partial R_1}{\partial p} \\ \frac{\partial R_2}{\partial u_r} & \frac{\partial R_2}{\partial u_\theta} & \frac{\partial R_2}{\partial u_z} & \frac{\partial R_2}{\partial p} \\ \frac{\partial R_3}{\partial u_r} & \frac{\partial R_3}{\partial u_\theta} & \frac{\partial R_3}{\partial u_z} & \frac{\partial R_3}{\partial p} \\ \frac{\partial R_4}{\partial u_r} & \frac{\partial R_4}{\partial u_\theta} & \frac{\partial R_4}{\partial u_z} & \frac{\partial R_4}{\partial p} \end{bmatrix} \begin{bmatrix} u_r^{n+1} - u_r^n \\ u_\theta^{n+1} - u_\theta^n \\ u_z^{n+1} - u_z^n \\ p^{n+1} - p^n \end{bmatrix} = - \begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{bmatrix}. \quad (42)$$

or

$$[M]\dot{U} + [S(U)]\Delta U = -[R]. \quad (43)$$

5. Problem Specification

Poiseuille flow through a two dimensional axisymmetric straight channel is introduced in this study under the isothermal condition. For this context, two different triangular finite element meshes are implemented, 2×2 and 5×5 as shown in Fig. 1, with connectivity structure. In addition, the mesh characteristics are introduced in Table 1.

Table 1. Mesh characteristic parameters

Mesh	Total Element	Total Nodes	Boundary Nodes	Pressure Nodes
2×2	8	25	16	9
5×5	50	121	40	36

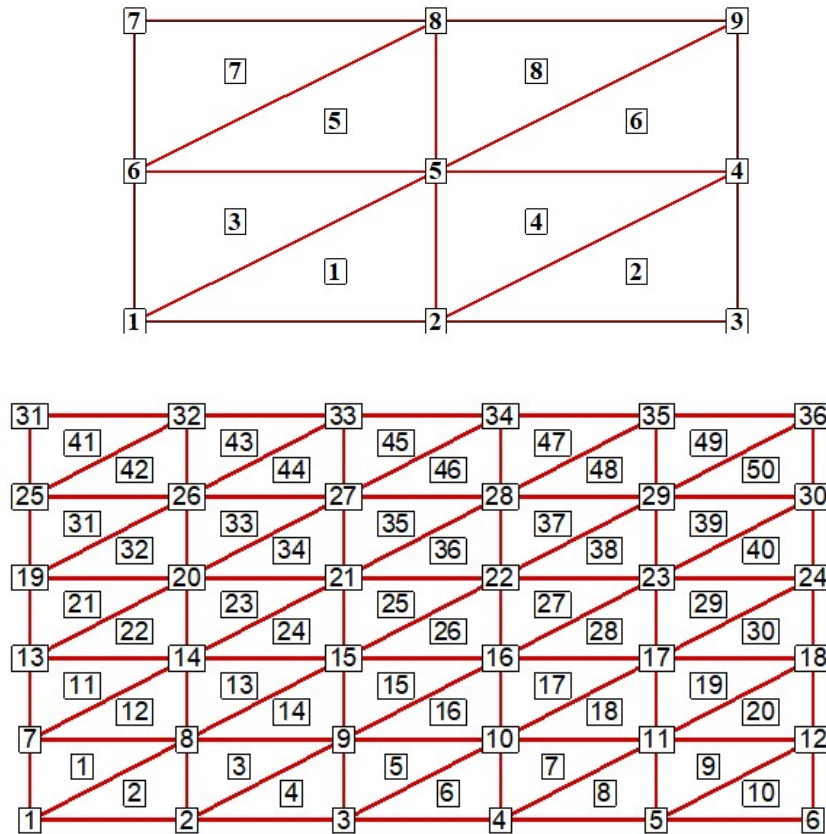


Fig. 1. Structured 2x2 and 5x5 finite element meshes.

Boundary conditions (BCs): The setting of BCs of the present channel problem are laid as follows (see Fig. 2):

- (a) Poiseuille (Ps) flow is specified at the inlet with zero radial velocity .
- (b) No-slip BCs is applied on the top and bottom walls of the channels.
- (c) Zero radial velocity and zero pressure are applied on the outlet of the channels.

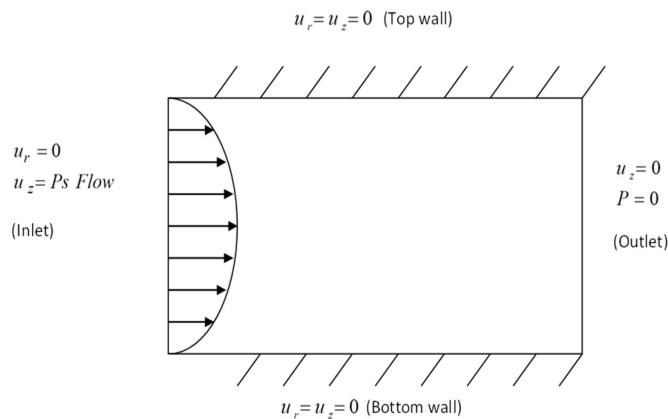


Fig. 2. Schematic for the flow problem and boundary conditions.

6. Numerical Results

The numerical results is concerned with the rate of error convergence of the problem under consideration by using Galerkin finite element method. Here the effect of power-law index (n), consistency parameter (k) and Reynolds number (Re) on the numerical convergence is investigated.

β_{ac} -variation: The rate of convergence for axial velocity components and pressure are illustrated in Fig. 3 for different values of the artificial compressibility parameter ($\beta_{ac}=10, 40, 100, 150$) and fixed parameters $\{Re = 0.001, n = 0.8, k = 1\}$. Generally, the level of convergence for velocity component is high compared to pressure because of the influence of nonlinearity behaviour. In addition and as anticipated, the findings show that the rate of convergence is increased as the values of β_{ac} are decreased due to the compressibility effects.

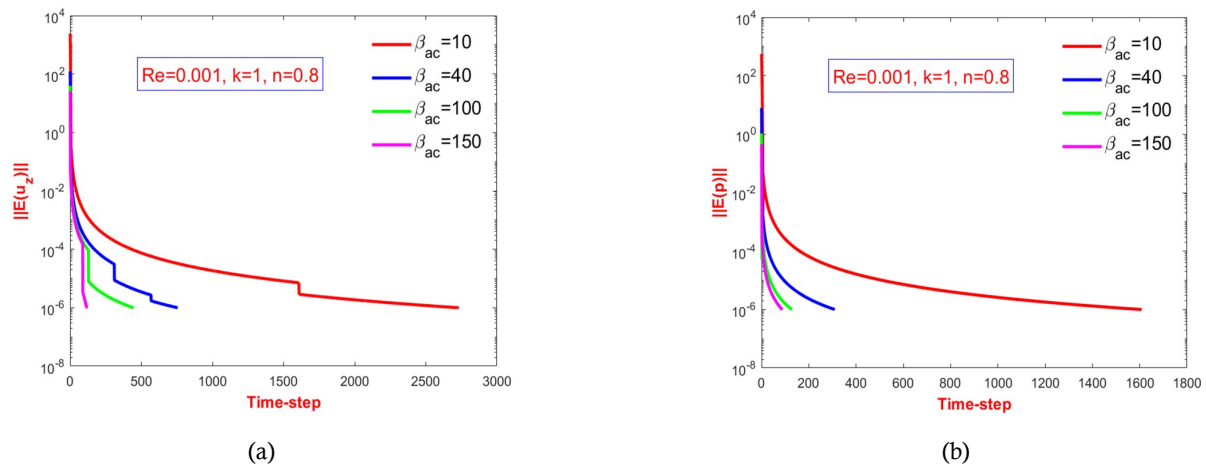


Fig. 3. Convergence of velocity and pressure; β_{ac} -variation, $n = 0.8$, $k = 1$, $Re = 0.001$.

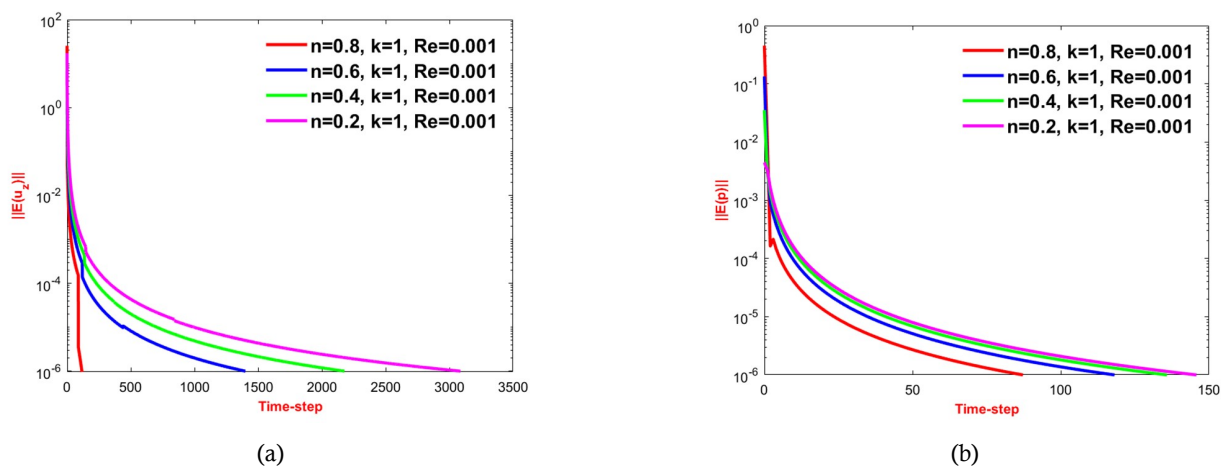


Fig. 4. Convergence of velocity and pressure; n -variation, $\beta_{ac} = 150$, $k = 1$, $Re = 0.001$.

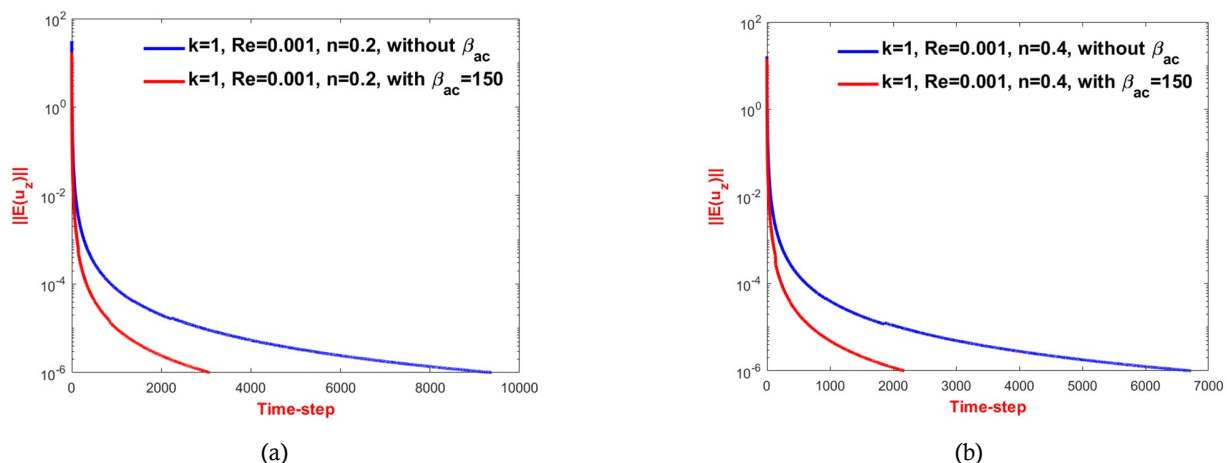


Fig. 5. Comparison of velocity convergence without AC -method against AC -method $\beta_{ac} = 150$, $Re=0.001$, $k=1$, n -variation.

n -variation: To study the effect of shear thinning properties of power-law model on the rate of convergence, the level of convergence for axial velocity and pressure components with different values of the power-law index (n) ($n = 0.8, 0.6, 0.4, 0.2$) are presented in Fig. 4, with $Re = 0.001$, $k = 1$ and $\beta_{ac} = 150$. The results reveal that the level of velocity convergence have been increased as the value of power-law index (n) decreases (see Fig. 4a) due to the shear thinning behaviours. In content and from Fig. 4b, one can observe that there is no significant change in the level of convergence of pressure at the same setting of the power-law index (n). Fig. 5 provides the comparison for the temporal convergence rates for velocity with and without β_{ac} under same setting of Re , k and n . Generally, one can observe that the rates of convergence for velocity are higher for the case of without β_{ac} when compared to that with β_{ac} .

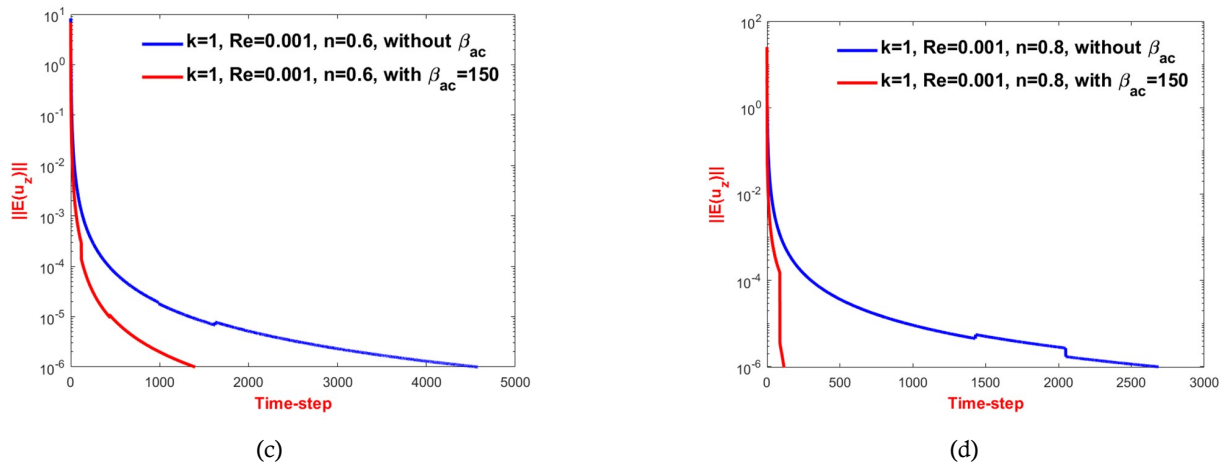


Fig. 5. Comparison of velocity convergence without AC -method against AC -method $\beta_{ac} = 150$.

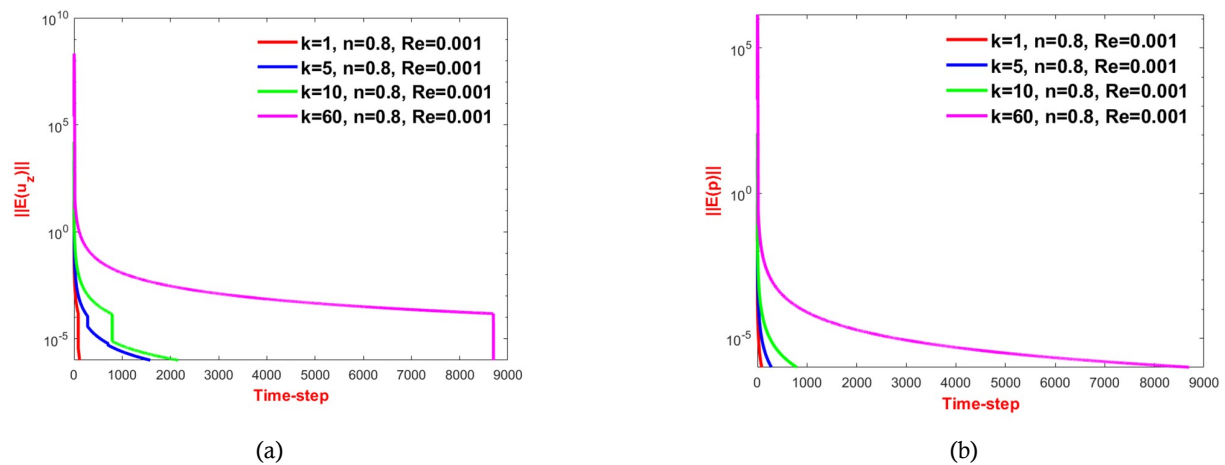


Fig. 6. Convergence of velocity and pressure; k -variation, $\beta_{ac} = 150$, $n = 0.8$, $Re = 0.001$.

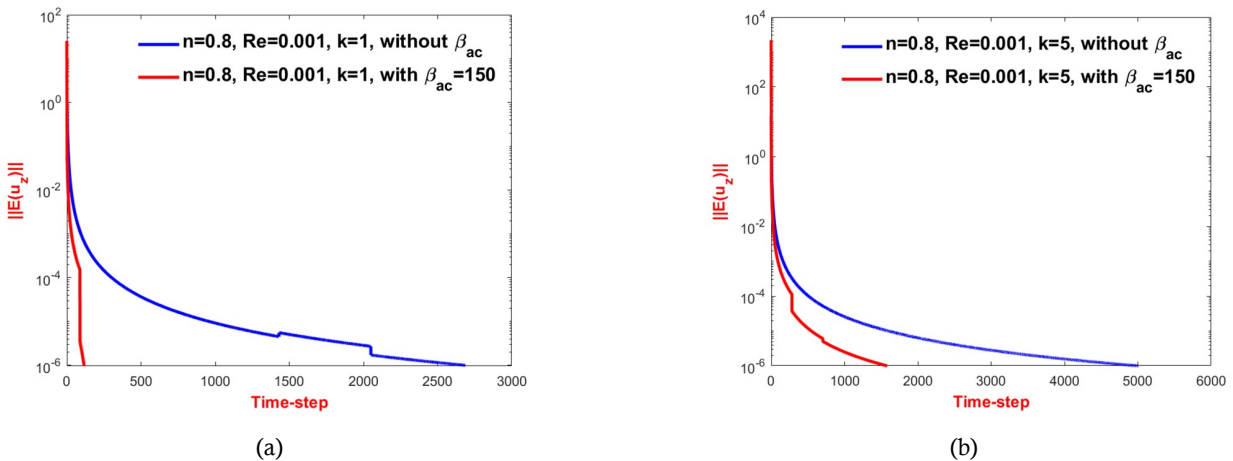


Fig. 7. Comparison of velocity convergence without AC -method against AC -method at $\beta_{ac} = 150$, $Re = 0.001$, $n = 0.8$, k -variation.

k -variation: Same feature have been observed in the study of k -variation under fixed $n = 0.8$ and $Re = 0.001$. Here, the level of convergence for velocity and pressure almost closed for all values of k . The interesting outcome in the results is that the level of convergence in the k -variation is almost double compared to the case of n -variation (compare Fig. 4 and Fig. 6), which reflects a significant influence of viscous fluid characteristics on the level of convergence. Numerical comparison between the results with and without β_{ac} is illustrated in Fig. 7 to assess the development of the solution with k -variation. Clearly one can detect the improvement in the convergence under β_{ac} when compared to the case of without β_{ac} .

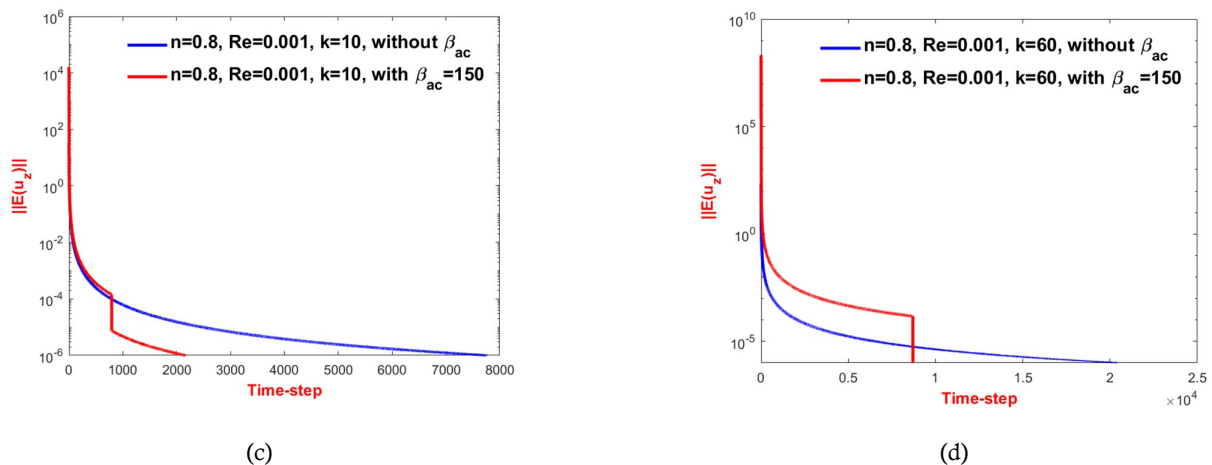


Fig. 7. Comparison of velocity convergence without AC -method against AC -method at $\beta_{ac} = 150$, $Re = 0.001$, $n = 0.8$, k -variation.

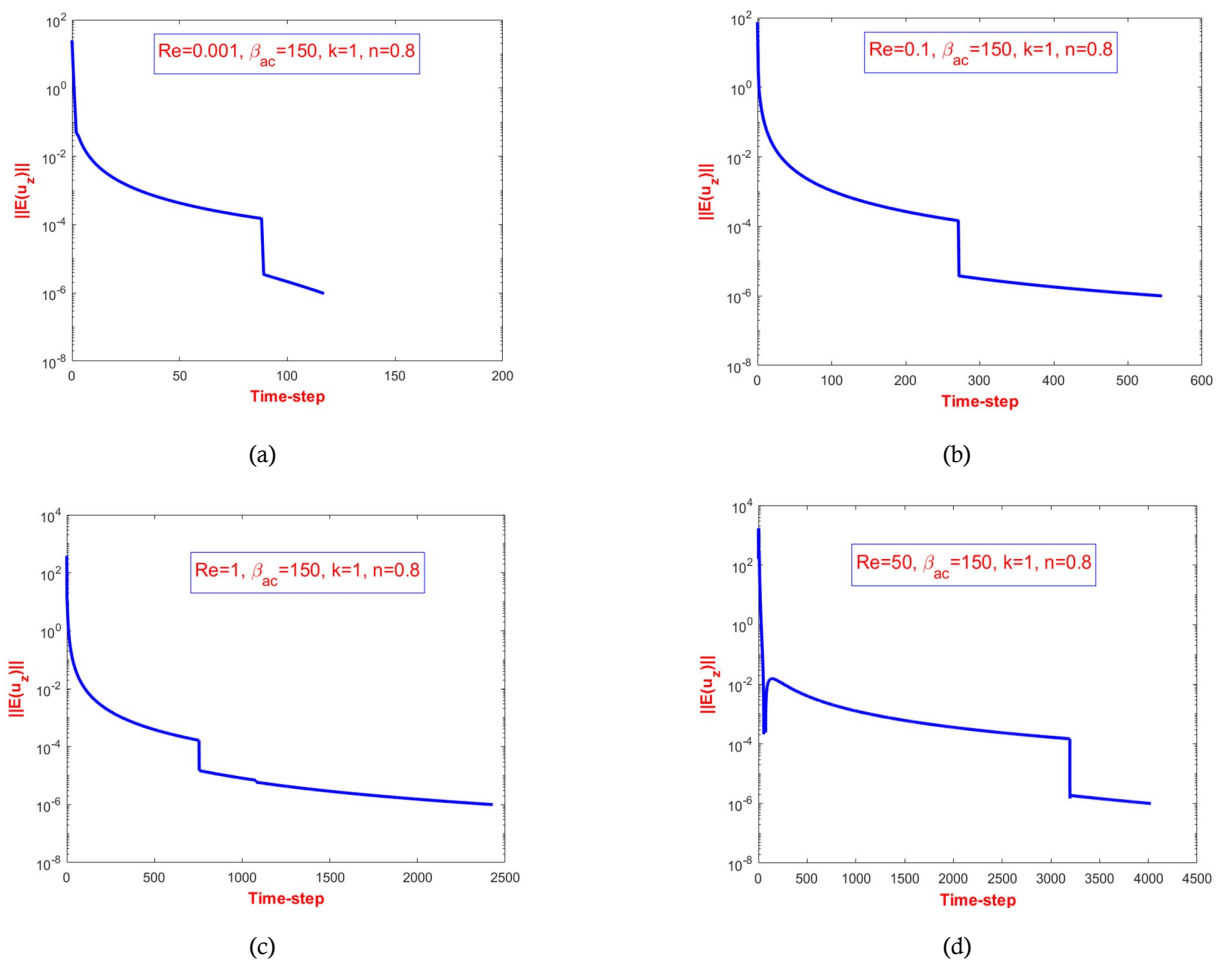


Fig. 8. Velocity convergence, $\beta_{ac} = 150$, $n = 0.8$, $k = 1$, Re -variation.

Re-variation: when the Reynolds number is increased, the velocity gradients will be developed which reflects the difficulties of convergence for large Re number. Thus, we have directed our interest in the discussing and studying the effects and levels of Reynolds number. Fig. 8 shows that the convergence of axial velocity for various levels of Re and fixed $\{\beta_{ac} = 150, n = 0.8 \text{ and } k = 1\}$. From this figure, one can see that the level of time increments increases whenever you get increased in Re . For example, when $Re = 0.001$, the level of time steps is much less than that of $Re = 100$ (see Fig. 8a and 8e), so the level of convergence of velocity is faster when Re is small. We note also that for the critical value of $Re = 139$, as shown in the Fig. 8f, we found that it is so difficult to reach the convergence criteria (around 198050 time-step).

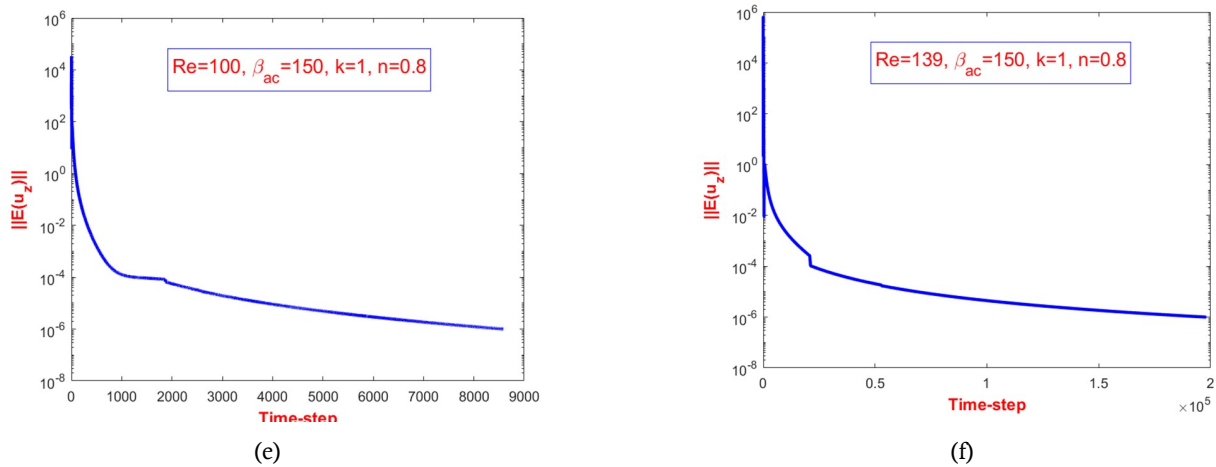


Fig. 8. Velocity convergence, $\beta_{ac} = 150$, $n = 0.8$, $k = 1$, Re -variation.

For clear feature, Fig. 9 demonstrates critical Re profile as a function of n at $k = 1$ and $\beta_{ac} = 150$. The results on Re level show the effect of shear thinning behaviour on the critical level of Re . Here, Re level is observed to reduce as the power-law index (n) raises, so that, the maximum Re corresponds to the largest power-law index ($n = 0.8$), which is consistent with the results reported by Coelho and Pinho [33-34]; Sivakumar et al. [35].

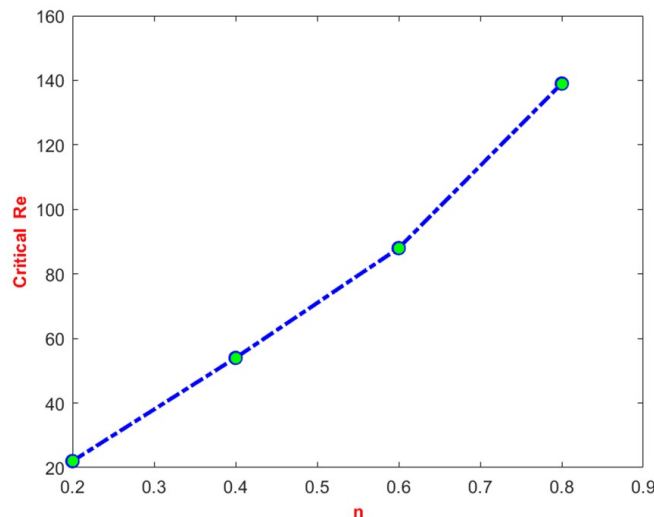


Fig. 9. Critical Reynolds number (Re) against power-law index (n); $\beta_{ac} = 150$, $k = 1$.

7. Summary and Conclusion

In this paper, the numerical simulation for incompressible power-law inelastic fluid was conducted based on the Galerkin finite element method in Cylindrical coordinate system. The artificial compressibility method was employed to transform the elliptic incompressible continuity equation to hyperbolic compressible system by adding artificial compressibility term to continuity equation in order to deal with the difficulty of incompressibility constraint conditions and singular matrix. To treat the non-linear equations, the Newton-Raphson iterative method based on backward difference scheme was employed. The convergence analysis of velocity and pressure was done to identify the effect of a consistency parameter (k), power-law index (n), Reynolds number (Re) and artificial compressible parameter β_{ac} on the acceleration of convergence. In this matter, a strong influence of shear-thinning causes a rising in the level of convergence. Same trend was observed for effect of viscosity. In this context, the results revealed that the rate of convergence is reduced as Re reduces, while the rate of convergence is reduced as β_{ac} increases. In addition, the critical value of Re was around 139, which represents a good point in inelastic fluid. The influence of shear thinning behaviours on the rate of convergence and critical level of Reynolds number (Re) was shown clearly as well. With decrease in shear thinning tendency, the rate of convergence increases and the level of (Re) decreases, such this is in agreement with experimental results and findings of others. In contrast, an opposite feature was appeared with consistency parameter (k), where the level of convergence becomes higher with higher k -value.

Conflict of Interest

The authors declared no potential conflicts of interest with respect to the research, authorship and publication of this article.

Funding

The authors received no financial support for the research, authorship and publication of this article.

Nomenclature

Re	Reynolds number	$\dot{\gamma}, \dot{\epsilon}$	Shear-rate, Strain-rate
β_{ac}	The artificial compressibility parameter	μ_s	The solvent viscosity
ρ	Fluid density	∇	Derivative operator (divergence)

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