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Traveling Wave Solutions of 3D Fractionalized MHD Newtonian Fluid in Porous Medium with Heat Transfer

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Abstract. In the present paper, we get exact solutions of Magnetohydrodynamic (MHD) of the fractionalized three-dimensional flow of Newtonian fluid with porous and heat transfer through the traveling wave parameter. The governing equations are produced dependent on established Navier-stokes equations which can be diminished to ordinary differential equation by wave parameter $\xi = ax + by + nz + U t^\alpha / \Gamma(\alpha + 1)$. The new exact solutions are established for three various cases. In special cases the solution for Newtonian fluid with and without MHD and porous effects can also be found from the general solution by putting $M + \phi \rightarrow 0$ and solutions for simple Newtonian fluid can also be obtained by putting $\alpha \rightarrow 1$ in general solutions. Finally, the effect of the parameter of interest on the stream motion, as well as difference among the Newtonian fluids is examined by 2D and 3D graphical interpretations.

Keywords: 3D Newtonian fluid, MHD, Porous, Heat transfer, Exact solution, Traveling wave, 2D & 3D graphs.

1. Introduction

In the last forty years, several nonlinear partial differential equations (NPDEs) have been introduced to describe numerous nonlinear wave phenomena is split off from physics such as optical fibers, biology, solid-state physics, plasma physics, hydrodynamics and condensed matter physics [1 - 5]. These nonlinear phenomena must be understood better, then it is essential to determine their accurate solutions. They assist to discuss the solidity of these solutions and analyze the results by making 2D and 3D sketches of the exact solutions. Therefore, finding exact traveling wave solutions from many available other methods is a charming job from a theoretical and applied point of view [6 - 10].

It is a well-accepted fact that fractional calculus is a more powerful tool to explain technological applications than ordinary calculus. Therefore, the fractional differential equations are modeled in theoretical and applied science. The fractional differential equations have been used in the area of the science, fluid mechanics, chemistry, engineering, quantum mechanics and electromagnetic theory, etc. Many numerical and analytical methods have been suggested to determine accurate solutions of fractional partial differential equations and integral equations. The investigations of accurate solutions of non-linear fractional partial differential equations perform a dynamic character in non-linear science. A variety of reliable methods have been proposed to obtain the accurate solutions of non-linear fractional partial differential equations [11 - 12] such as variational iteration method [13 - 14], differential transformation method [15 - 16], Adomian decomposition method [17 - 18], exponent-function method [19 - 20], expansion method [21 - 22], reduced differential method [23] and finite difference method [24].

Magnetic fields have been effectively utilized to manage convection for electrically leading liquid, for example, incandescence melts and metals, composites, and semiconductors. Magnetohydrodynamic flows incited by the stretching

sheets have been a subject of extraordinary significance [25]. The correspondence between magnetic-flux and electrically conducting has been explored all the time in comparison to pump meters, atomic reactors, and industrial functions in cooling of the thermal reactor by the research worker. Some recent works appear with consideration of MHD, we cite here [26 - 34]. In chemical engineering, the stream of liquid through the porous media is significantly important in some specific areas like suction effects and agriculture engineering for the filtration process. The liquid crossing through the porous medium has various applications for examples aerogels, infusion of mud's, permeable rocks, compounds, slurries or bond gatherings to fortify clays, froths and frothed liquid and most utilization is boring liquids through infusion in rocks for the fortification of the wells and furthermore to upgrade oil recuperation [35 - 39].

Heat Transfer is characterized as the energy moved by the excellence of a temperature distinction. It runs from the area of higher temperature to locales of lower temperature and it is the product of temperature gradient and heat flux. It has several applications in engineering and industrial processes. Utilization of industrial procedures incorporates paper generation, production and drawing of plastics and elastic sheets, glass fiber, normalizing and tinning of copper wires, sustenance handling and heat-treated material traveling on conveyor belts, streamlined expulsion of plastic sheets, metal turning and miniaturized scale electromechanical framework and numerous others [40]. In constituent, the idea of heat generation is valuable in procedures like warmth expulsion from atomic fuel rubbish and underground transfer of radioactive waste material, stockpiling of foodstuffs and separating liquids. Some recent work appears in heat transfer are [41 - 44] and references cited therein.

In the present work, motivated by the above discussed and inspired from [45 - 46], we discuss here three-dimensional (3D) fractionalized MHD flow of Newtonian fluid in a porous medium with heat transfer using a traveling wave solution. The mathematical model that studied here is delicately settled in and judge with various standard cases. It is seen that the current model can cover such criterion cases and can give new material on the parameter progression of the dynamic system. To the best of our information, there is only one study which deals with the stream of the incompressible unsteady three-dimensional flow of Newtonian fluid that appeared in [45]. Thus, we have extended the work of [45] for greater class namely MHD three-dimensional flow of Newtonian fluid in a porous medium with heat transfer and fractional derivative approach in governing equations. The set of new exact traveling wave solutions are determined for three different cases. In particular cases, the solutions for with and without MHD and porous effects can easily be determined from the present general solutions by using $M + \phi \rightarrow 0$. The solutions for simple three-dimensional Newtonian fluid can also be obtained by putting $\alpha \rightarrow 1$ to the general solutions. In the end, the impact of material parameters on fluid motion are discussed through 2D and 3D graphical illustration and difference among the different Newtonian fluid are examined. This article is classified in the following manner: in section 2, we perform the formulations of Newtonian fluid with the 3D flow and also elaborate on the methodology and theoretical development of traveling wave methods. Section 3, we obtain the exact solution of the system of equation for three various cases. We have analyzed the results and conclusion of graphs in section 4. In section 5, appendixes are represented.

2. Basic Mathematical Model and Governing Equations

2.1 3D fractionalized MHD Newtonian equations in porous medium with heat transfer

In this paper, the unsteady three-dimensional fractionalized MHD Newtonian fluid in a porous medium with heat transfer is investigated. The governing equations are the combination of the equation of continuity, 3D Navier-Stokes equation with MHD and porous effects and lastly the conservation of energy equation, which are composed as pursues (disregarding radiation and inward heat source)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \quad (1)$$

$$\frac{\partial^\alpha u}{\partial t^\alpha} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = -\frac{1}{\rho} \left(\frac{\partial p}{\partial y} \right) + \nu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) - (M + \phi)u, \quad (2)$$

$$\frac{\partial^\alpha v}{\partial t^\alpha} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = -\frac{1}{\rho} \left(\frac{\partial p}{\partial y} \right) + \nu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) - (M + \phi)v, \quad (3)$$

$$\frac{\partial^\alpha w}{\partial t^\alpha} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = -\frac{1}{\rho} \left(\frac{\partial p}{\partial z} \right) + \nu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) - (M + \phi)w, \quad (4)$$

$$\begin{aligned} \frac{\partial^\alpha T}{\partial t^\alpha} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} = & \eta \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + \frac{\nu}{c_p} \left[2 \left(\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 + \left(\frac{\partial w}{\partial z} \right)^2 \right) \right. \\ & \left. + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)^2 + \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right)^2 \right], \end{aligned} \quad (5)$$

where $u(x, y, z, t)$, $v(x, y, z, t)$ and $w(x, y, z, t)$ are velocity components in x , y and z directions, respectively, the fluid pressure $p(x, y, z, t)$ and the temperature $T(x, y, z, t)$. The fluid density ρ , η the thermal diffusivity, the kinematic viscosity ν , ρ and the specific heat c_p are constant. Also, $\phi = \nu\psi / \kappa$, $M = \sigma B_0^2 / \rho$ are porosity constants and magnetic, where ψ is the porosity and κ is the permeability of the porous medium. σ is the electrically conductivity of fluid and B_0 is the magnetic of applied magnetic field. The fractional operator D_t^α named Caputo and α is fractional parameter is defined by:

$$D_t^\alpha = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(\tau)}{(t-\tau)^\alpha} d\tau, & 0 < \alpha < 1; \\ \frac{df(t)}{dt} & \alpha = 1, \end{cases}$$

It is clearly observed that to solve Eqs. (1 - 5), Eqs. (1 - 4) should be solved first to get u , v and w and afterward Eq. (5) is solved to acquire T . In the subsequent sections, the traveling wave solutions of Eqs. (1 - 5), will be acquired and the physical descriptions to Eqs. (1 - 5) are clarified.

2.2 Basic procedure of traveling wave method

Assume that the nonlinear fractional partial differential equations in four variables and dependent variables are u , v and w in hereunder:

$$L_1(u, v, w) = 0, \quad (6)$$

$$L_2(u, v, w, p, u_t^\alpha, u_x, u_y, u_z, u_{xx}, \dots, v_t^\alpha, v_x, v_y, v_z, v_{xx}, \dots, w_t^\alpha, w_x, w_y, w_z, w_{xx}) = 0, \quad (7)$$

$$L_3(u, v, w, p, u_t^\alpha, u_x, u_y, u_z, u_{xx}, \dots, v_t^\alpha, v_x, v_y, v_z, v_{xx}, \dots, w_t^\alpha, w_x, w_y, w_z, w_{xx}) = 0, \quad (8)$$

$$L_4(u, v, w, p, u_t^\alpha, u_x, u_y, u_z, u_{xx}, \dots, v_t^\alpha, v_x, v_y, v_z, v_{xx}, \dots, w_t^\alpha, w_x, w_y, w_z, w_{xx}, \dots) = 0. \quad (9)$$

Here L_i is a polynomial function of u , v , w and p in which contain the nonlinear terms and higher order of derivative. $i = 1, 2, 3, 4$ We show traveling wave solution as:

$$u(x, y, z, t) = u(\xi), v(x, y, z, t) = v(\xi), w(x, y, z, t) = w(\xi) \text{ and } p(x, y, z, t) = p(\xi), \quad (10)$$

where $\xi = ax + by + nz + \frac{U t^\alpha}{\Gamma(\alpha+1)}$. Then the system Eqs. (6 - 9) can be transformed into the system of ordinary differential equations.

$$L_1(u', v', w') = 0, \quad (11)$$

$$L_2(u, v, w, p', u', u'', u''', \dots, v', v'', v''', \dots, w', w'', w''') = 0, \quad (12)$$

$$L_3(u, v, w, p', u', u'', u''', \dots, v', v'', v''', \dots, w', w'', w''') = 0, \quad (13)$$

$$L_4(u, v, w, p', u', u'', u''', \dots, w', w'', w''') = 0, \quad (14)$$

3. Exact Traveling Wave Solutions and Explanations

We can accept that the traveling wave type solution has the following representation, without loss of generality:

$$u = u(\xi), v = v(\xi), w = w(\xi), p = p(\xi), T = T(\xi) \text{ and } \xi = ax + by + nz + \frac{U t^\alpha}{\Gamma(\alpha+1)}$$

Here U is constant phase velocity. On putting the above traveling wave parameter into Eqs. (1 - 5), one obtains:

$$a u' + b v' + n w' = 0, \quad (15)$$

$$(U + au + bv + nw)u' = \frac{-ap'}{\rho} + \nu(a^2 + b^2 + n^2)u'' - (M + \phi)u, \quad (16)$$

$$(U + au + bv + nw)v' = \frac{-bp'}{\rho} + \nu(a^2 + b^2 + n^2)v'' - (M + \phi)v, \quad (17)$$

$$(U + au + bv + nw)w' = \frac{-np'}{\rho} + \nu(a^2 + b^2 + n^2)w'' - (M + \phi)w, \quad (18)$$

$$(U + au + bv + nw)T' = \eta(a^2 + b^2 + n^2)T'' + \frac{\nu}{c_p} \left[u'^2(2a^2 + b^2 + n^2) + v'^2(2b^2 + a^2 + n^2) + w'^2(2n^2 + b^2 + a^2) + 2ab u'v' + 2bn v'w' + 2an u'w' \right], \quad (19)$$

where prime denotes the differentiation with respect to ξ Integration of Eq. (15) w. r. t ξ yields:

$$a u + b v + n w = d_0. \quad (20)$$

Substituting Eq. (20) into Eqs. (16 - 19), provides:

$$(U + d_0)u' = \frac{-ap'}{\rho} + \nu(a^2 + b^2 + n^2)u'' - (M + \phi)u, \quad (21)$$

$$(U + d_0)v' = \frac{-bp'}{\rho} + \nu(a^2 + b^2 + n^2)v'' - (M + \phi)v, \quad (22)$$

$$(U + d_0)w' = \frac{-np'}{\rho} + \nu(a^2 + b^2 + n^2)w'' - (M + \phi)w, \quad (23)$$

$$(U + d_0)T' = \eta(a^2 + b^2 + n^2)T'' + \frac{\nu}{c_p} \left[u'^2(2a^2 + b^2 + n^2) + v'^2(2b^2 + a^2 + n^2) + w'^2(2n^2 + b^2 + a^2) + 2ab u'v' + 2bn v'w' + 2an u'w' \right]. \quad (24)$$

In order to find u , v , w , p and T from above four equations, we consider the following three cases independently.

3.1 Case-I: $U + d_0 \neq 0$

On eliminating p from Eqs. (21 - 23), we get:

$$(U + d_0)(bu' - av') = \nu(a^2 + b^2 + n^2)(bu'' - av'') - (M + \phi)(bu - av), \quad (25)$$

$$(U + d_0)(nu' - aw') = \nu(a^2 + b^2 + n^2)(nu'' - aw'') - (M + \phi)(nu - aw). \quad (26)$$

Solving Eqs. (25 - 26), we obtain:

$$u'' + \alpha_1 u' + \alpha_2 u = \alpha_3, \quad (27)$$

where α_1 , α_2 and α_3 are given in Appendix A. The solution of Eq. (27) is:

$$u(\xi) = \alpha_4 e^{\theta_{11}\xi} + \alpha_5 e^{\theta_{12}\xi} + \alpha_6, \quad (28)$$

where α_4 , α_5 are arbitrary constants and α_6 given in Appendix A and θ_{11} , θ_{12} are the roots of the auxiliary equations:

$$m^2 + \alpha_1 m + \alpha_2 = \alpha_3. \quad (29a)$$

Using Eq. (28) in Eq. (25), we find:



$$v'' + \alpha_7 v' + \alpha_8 v = \alpha_9 e^{\theta_{11} \xi} + \alpha_{10} e^{\theta_{12} \xi} + \alpha_{11}, \quad (29b)$$

where $\alpha_7 = \alpha_1$ and $\alpha_8 = \alpha_2$ and α_9, α_{10} and α_{11} are given in Appendix A. The solution of Eq. (29) is:

$$v(\xi) = \alpha_{17} e^{\theta_{11} \xi} + \alpha_{18} e^{\theta_{12} \xi} + \alpha_{16}, \quad (30)$$

Putting the Eqs. (28) and (30) in Eq. (20) yields:

$$w(\xi) = \alpha_{19} e^{\theta_{11} \xi} + \alpha_{20} e^{\theta_{12} \xi} + \alpha_{21}, \quad (31)$$

where $\alpha_{16}, \alpha_{17}, \alpha_{18}, \alpha_{19}, \alpha_{20}, \alpha_{21}$ are given in Appendix A. Using Eq. (28) in Eq. (21), we get the pressure as:

$$p(\xi) = \alpha_{22} e^{\theta_{11} \xi} + \alpha_{23} e^{\theta_{12} \xi} + \alpha_{24} \xi + \alpha_{25}, \quad (32)$$

where $\alpha_{22}, \alpha_{23}, \alpha_{24}$ are given in Appendix A and α_{25} is arbitrary constant. Put Eqs. (28), (30) and (31) in Eq. (24):

$$T'' - \alpha_{26} T' = \alpha_{27} e^{2\theta_{11} \xi} + \alpha_{28} e^{2\theta_{12} \xi} + \alpha_{29} e^{(\theta_{11} + \theta_{12}) \xi}. \quad (33)$$

Solution of Eq. (33) is

$$T(\xi) = \alpha_{30} + \alpha_{31} e^{\alpha_{26} \xi} + \alpha_{32} e^{2\theta_{11} \xi} + \alpha_{33} e^{2\theta_{12} \xi} + \alpha_{34} e^{(\theta_{11} + \theta_{12}) \xi}, \quad (34)$$

where $\alpha_{26}, \alpha_{27}, \alpha_{28}, \alpha_{29}, \alpha_{32}, \alpha_{33}, \alpha_{34}$ are given in Appendix A and α_{30}, α_{31} are arbitrary constants. The velocity components, pressure and temperature in the original variables are:

$$u(x, y, z, t) = \alpha_4 e^{\theta_{11} \left(ax + by + nz + \frac{U t^\alpha}{\Gamma(\alpha+1)} \right)} + \alpha_5 e^{\theta_{12} \left(ax + by + nz + \frac{U t^\alpha}{\Gamma(\alpha+1)} \right)} + \alpha_6, \quad (35)$$

$$v(x, y, z, t) = \alpha_{17} e^{\theta_{11} \left(ax + by + nz + \frac{U t^\alpha}{\Gamma(\alpha+1)} \right)} + \alpha_{18} e^{\theta_{12} \left(ax + by + nz + \frac{U t^\alpha}{\Gamma(\alpha+1)} \right)} + \alpha_{16}, \quad (36)$$

$$w(x, y, z, t) = \alpha_{19} e^{\theta_{11} \left(ax + by + nz + \frac{U t^\alpha}{\Gamma(\alpha+1)} \right)} + \alpha_{20} e^{\theta_{12} \left(ax + by + nz + \frac{U t^\alpha}{\Gamma(\alpha+1)} \right)} + \alpha_{21}, \quad (37)$$

$$p(x, y, z, t) = \alpha_{22} e^{\theta_{11} \left(ax + by + nz + \frac{U t^\alpha}{\Gamma(\alpha+1)} \right)} + \alpha_{23} e^{\theta_{12} \left(ax + by + nz + \frac{U t^\alpha}{\Gamma(\alpha+1)} \right)} + \alpha_{24} \left(ax + by + nz + \frac{U t^\alpha}{\Gamma(\alpha+1)} \right) + \alpha_{25}, \quad (38)$$

$$T(x, y, z, t) = \alpha_{30} + \alpha_{31} e^{\alpha_{26} \left(ax + by + nz + \frac{U t^\alpha}{\Gamma(\alpha+1)} \right)} + \alpha_{32} e^{2\theta_{11} \left(ax + by + nz + \frac{U t^\alpha}{\Gamma(\alpha+1)} \right)} + \alpha_{33} e^{2\theta_{12} \left(ax + by + nz + \frac{U t^\alpha}{\Gamma(\alpha+1)} \right)} + \alpha_{34} e^{(\theta_{11} + \theta_{12}) \left(ax + by + nz + \frac{U t^\alpha}{\Gamma(\alpha+1)} \right)}. \quad (39)$$

3.1.1 Verification of exact solutions

Partial derivative of Eqs. [35 – 37] with respect to x, y and z respectively then putting in Eq. (1), we get the LHS of Eq. (1):

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \theta_{11} (a \alpha_4 + b \alpha_{17} + n \alpha_{19}) e^{\theta_{11} \left(ax + by + nz + \frac{U t^\alpha}{\Gamma(\alpha+1)} \right)} + \theta_{12} (a \alpha_5 + b \alpha_{18} + n \alpha_{20}) e^{\theta_{12} \left(ax + by + nz + \frac{U t^\alpha}{\Gamma(\alpha+1)} \right)}, \quad (40a)$$

Putting the value of α_{19} and α_{20} from Appendix A, we get:

$$\begin{aligned} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= \theta_{11} [a \alpha_4 + b \alpha_{17} - (a \alpha_4 + b \alpha_{17})] e^{\theta_{11} \left(ax + by + nz + \frac{U t^\alpha}{\Gamma(\alpha+1)} \right)} \\ &+ \theta_{12} [a \alpha_5 + b \alpha_{18} - (a \alpha_5 + b \alpha_{18})] e^{\theta_{12} \left(ax + by + nz + \frac{U t^\alpha}{\Gamma(\alpha+1)} \right)} = 0 \end{aligned} \quad (40b)$$

Hence Eq. (1) is verified. Taking partial derivative of Eqs. (35 – 37) with respect to x, y and z respectively and Eq. (38) with respect to x putting in Eq. (2), the LHS of Eq. (2) provides:

$$\begin{aligned}
\frac{\partial^\alpha u}{\partial t^\alpha} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} &= \alpha_4 \theta_{11} \left(U + au + bv + nw \right) e^{\theta_{11} \left(ax + by + nz + \frac{U t^\alpha}{\Gamma(\alpha+1)} \right)} \\
&+ \alpha_5 \theta_{12} \left(U + au + bv + nw \right) e^{\theta_{12} \left(ax + by + nz + \frac{U t^\alpha}{\Gamma(\alpha+1)} \right)} = \alpha_4 \theta_{11} \left(U + d_0 \right) e^{\theta_{11} \left(ax + by + nz + \frac{U t^\alpha}{\Gamma(\alpha+1)} \right)} \\
&+ \alpha_5 \theta_{12} \left(U + d_0 \right) e^{\theta_{12} \left(ax + by + nz + \frac{U t^\alpha}{\Gamma(\alpha+1)} \right)},
\end{aligned} \quad (40c)$$

The RHS of Eq. (2) gives:

$$\begin{aligned}
& -\frac{1}{\rho} \left(\frac{\partial p}{\partial x} \right) + \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) - (M + \phi) u \\
&= \left[\frac{-a \theta_{11} \alpha_{22}}{\rho} - \alpha_4 (M + \phi) \right] e^{\theta_{11} \left(ax + by + nz + \frac{U t^\alpha}{\Gamma(\alpha+1)} \right)} + \left[\frac{-a \theta_{12} \alpha_{23}}{\rho} - \alpha_5 (M + \phi) \right] e^{\theta_{12} \left(ax + by + nz + \frac{U t^\alpha}{\Gamma(\alpha+1)} \right)} - \alpha_6 (M + \phi) - \frac{a \alpha_{24}}{\rho} \\
&+ \nu \alpha_4 \theta_{11}^2 (a^2 + b^2 + n^2) e^{\theta_{11} \left(ax + by + nz + \frac{U t^\alpha}{\Gamma(\alpha+1)} \right)} + \nu \alpha_5 \theta_{12}^2 (a^2 + b^2 + n^2) e^{\theta_{12} \left(ax + by + nz + \frac{U t^\alpha}{\Gamma(\alpha+1)} \right)},
\end{aligned} \quad (41a)$$

Substituting the value of α_{22} , α_{23} and α_{24} we get:

$$\begin{aligned}
& -\frac{1}{\rho} \left(\frac{\partial p}{\partial x} \right) + \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) - (M + \phi) u \\
&= -\nu \alpha_4 \theta_{11}^2 (a^2 + b^2 + n^2) e^{\theta_{11} \left(ax + by + nz + \frac{U t^\alpha}{\Gamma(\alpha+1)} \right)} + \alpha_4 \theta_{11} (U + d_0) e^{\theta_{11} \left(ax + by + nz + \frac{U t^\alpha}{\Gamma(\alpha+1)} \right)} - \nu \alpha_5 \theta_{12}^2 (a^2 + b^2 + n^2) \\
&\times e^{\theta_{12} \left(ax + by + nz + \frac{U t^\alpha}{\Gamma(\alpha+1)} \right)} + \alpha_5 \theta_{12} (U + d_0) e^{\theta_{12} \left(ax + by + nz + \frac{U t^\alpha}{\Gamma(\alpha+1)} \right)} + \nu \alpha_4 \theta_{11}^2 (a^2 + b^2 + n^2) e^{\theta_{11} \left(ax + by + nz + \frac{U t^\alpha}{\Gamma(\alpha+1)} \right)} \\
&+ \nu \alpha_5 \theta_{12}^2 (a^2 + b^2 + n^2) e^{\theta_{12} \left(ax + by + nz + \frac{U t^\alpha}{\Gamma(\alpha+1)} \right)}, \\
&= \alpha_4 \theta_{11} (U + d_0) e^{\theta_{11} \left(ax + by + nz + \frac{U t^\alpha}{\Gamma(\alpha+1)} \right)} + \alpha_5 \theta_{12} (U + d_0) e^{\theta_{12} \left(ax + by + nz + \frac{U t^\alpha}{\Gamma(\alpha+1)} \right)}.
\end{aligned} \quad (41b)$$

Hence both side are equal and Eq. (2), is verified. Similarly, we can verify other equations.

3.2 Case-II: $U + d_0 = 0$

In this case, Eqs. (21 - 24) can be simplified as:

$$\frac{ap'}{\rho} = \nu (a^2 + b^2 + n^2) u'' - (M + \phi) u, \quad (42a)$$

$$\frac{bp'}{\rho} = \nu (a^2 + b^2 + n^2) v'' - (M + \phi) v, \quad (42b)$$

$$\frac{np'}{\rho} = \nu (a^2 + b^2 + n^2) w'' - (M + \phi) w, \quad (43a)$$

$$\begin{aligned}
\eta (a^2 + b^2 + n^2) T'' &= \frac{-\nu}{c_\rho} \left[u'^2 (2a^2 + b^2 + n^2) + v'^2 (2b^2 + a^2 + n^2) \right. \\
&\left. + w'^2 (2n^2 + b^2 + a^2) + 2ab u' v' + 2bn v' w' + 2an u' w' \right].
\end{aligned} \quad (43b)$$

Eliminating p from Eqs. (40) - (42) yields:



$$\nu(a^2 + b^2 + n^2)(bu'' - av'') - (M + \phi)(bu - av) = 0, \quad (44)$$

$$\nu(a^2 + b^2 + n^2)(nu'' - aw'') - (M + \phi)(nu - aw) = 0. \quad (45)$$

Solving Eqs. (44) – (45), we get:

$$u'' - \beta_1 u = \beta_2, \quad (46)$$

where β_1 and β_2 are given in Appendix B. The solution of Eq. (46) is:

$$u(\xi) = \beta_3 e^{\theta_{21}\xi} + \beta_4 e^{-\theta_{21}\xi} + \beta_5, \quad (47a)$$

where β_3, β_4 are arbitrary constants and β_5 given in Appendix B and $\theta_{21}, -\theta_{21}$ are the roots of the quadratic equation:

$$m^2 - \beta_1 = 0. \quad (47b)$$

Putting the Eq. (47) into Eq. (44):

$$v'' - \beta_6 v = \beta_7 e^{\theta_{21}\xi} + \beta_8 e^{-\theta_{21}\xi} + \beta_9, \quad (48)$$

where $\beta_6 = \beta_1$. The solution of Eq. (48):

$$v(\xi) = \beta_{15} e^{\theta_{21}\xi} + \beta_{16} e^{-\theta_{21}\xi} + \beta_{14}. \quad (49)$$

Using Eqs. (47) and (49) in Eq. (20):

$$w(\xi) = \beta_{17} e^{\theta_{21}\xi} + \beta_{18} e^{-\theta_{21}\xi} + \beta_{19}, \quad (50)$$

where $\beta_7, \beta_8, \beta_9, \beta_{14}, \beta_{15}, \beta_{16}, \beta_{17}, \beta_{18}$ and β_{19} are given in Appendix B. Substituting the Eq. (47) in Eq. (40), we find:

$$p(\xi) = \beta_{20} e^{\theta_{12}\xi} + \beta_{21} e^{-\theta_{21}\xi} + \beta_{22}\xi + \beta_{23}, \quad (51)$$

where $\beta_{20}, \beta_{21}, \beta_{22}$ are given in Appendix B and β_{23} is arbitrary constant. Putting the Eqs. (47), (49) and (50) in Eq. (43), we obtain:

$$T(\xi) = \beta_{24} e^{2\theta_{21}\xi} + \beta_{25} e^{-2\theta_{21}\xi} + \beta_{26}\xi^2 + \beta_{27}\xi + \beta_{28}, \quad (52)$$

where β_{24}, β_{25} and β_{26} are given in Appendix B and β_{27}, β_{28} are arbitrary constants. Finally, the velocity components, pressure and temperature in the original variables are:

$$u(x, y, z, t) = \beta_3 e^{\theta_{21}\left(ax+by+nz+\frac{U t^\alpha}{\Gamma(\alpha+1)}\right)} + \beta_4 e^{-\theta_{21}\left(ax+by+nz+\frac{U t^\alpha}{\Gamma(\alpha+1)}\right)} + \beta_5, \quad (53)$$

$$v(x, y, z, t) = \beta_{15} e^{\theta_{21}\left(ax+by+nz+\frac{U t^\alpha}{\Gamma(\alpha+1)}\right)} + \beta_{16} e^{-\theta_{21}\left(ax+by+nz+\frac{U t^\alpha}{\Gamma(\alpha+1)}\right)} + \beta_{14}, \quad (54)$$

$$w(x, y, z, t) = \beta_{17} e^{\theta_{21}\left(ax+by+nz+\frac{U t^\alpha}{\Gamma(\alpha+1)}\right)} + \beta_{18} e^{-\theta_{21}\left(ax+by+nz+\frac{U t^\alpha}{\Gamma(\alpha+1)}\right)} + \beta_{19}, \quad (55)$$

$$p(x, y, z, t) = \beta_{20} e^{\theta_{21}\left(ax+by+nz+\frac{U t^\alpha}{\Gamma(\alpha+1)}\right)} + \beta_{21} e^{-\theta_{21}\left(ax+by+nz+\frac{U t^\alpha}{\Gamma(\alpha+1)}\right)} + \beta_{22}\left(ax+by+nz+\frac{U t^\alpha}{\Gamma(\alpha+1)}\right) + \beta_{23}, \quad (56)$$

$$T(x, y, z, t) = \beta_{24} e^{2\theta_{21}\left(ax+by+nz+\frac{U t^\alpha}{\Gamma(\alpha+1)}\right)} + \beta_{25} e^{-2\theta_{21}\left(ax+by+nz+\frac{U t^\alpha}{\Gamma(\alpha+1)}\right)} + \beta_{19} \\ + \beta_{26}\left(ax+by+nz+\frac{U t^\alpha}{\Gamma(\alpha+1)}\right)^2 + \beta_{27}\left(ax+by+nz+\frac{U t^\alpha}{\Gamma(\alpha+1)}\right) + \beta_{28}. \quad (57)$$

3.3 Case-III: $U = 0$

In this case, Eqs. (21 - 24) become:

$$d_0 u' = \frac{-ap'}{\rho} + \nu(a^2 + b^2 + n^2)u'' - (M + \phi)u, \quad (58)$$

$$d_0 v' = \frac{-bp'}{\rho} + \nu(a^2 + b^2 + n^2)v'' - (M + \phi)v, \quad (59)$$

$$d_0 w' = \frac{-np'}{\rho} + \nu(a^2 + b^2 + n^2)w'' - (M + \phi)w, \quad (60)$$

$$d_0 T' = \eta(a^2 + b^2 + n^2)T'' + \frac{\nu}{c_p} \left[u'^2(2a^2 + b^2 + n^2) + v'^2(2b^2 + a^2 + n^2) + w'^2(2n^2 + b^2 + a^2) + 2ab u'v' + 2bn v'w' + 2an u'w' \right]. \quad (61)$$

Eliminating p from Eqs. (58) - (60):

$$d_0 (bu' - av') = \nu(a^2 + b^2 + n^2)(bu'' - av'') - (M + \phi)(bu - av), \quad (62)$$

$$d_0 (nu' - aw') = \nu(a^2 + b^2 + n^2)(nu'' - aw'') - (M + \phi)(nu - aw). \quad (63)$$

Solving Eqs. (62 - 63) yields:

$$u'' + \gamma_1 u' + \gamma_2 u = \gamma_3, \quad (64)$$

where γ_1, γ_2 and γ_3 are given in Appendix A. Solution of Eq. (64) is:

$$u(\xi) = \gamma_4 e^{\theta_{13}\xi} + \gamma_5 e^{\theta_{32}\xi} + \gamma_6, \quad (65)$$

where γ_4, γ_5 are arbitrary constants and γ_6 given in Appendix C and θ_{31}, θ_{32} are the roots of the auxiliary equations:

$$m^2 + \gamma_1 m + \gamma_2 = 0. \quad (66a)$$

Putting the Eq. (65) into Eq. (62), we find:

$$v'' + \gamma_7 v' + \gamma_8 v = \gamma_9 e^{\theta_{31}\xi} + \gamma_{10} e^{\theta_{32}\xi} + \gamma_{11}, \quad (66b)$$

where $\gamma_7 = \gamma_1$ and $\gamma_8 = \gamma_2$ and γ_9, γ_{10} and γ_{11} are given in Appendix C. The solution of Eq. (66) is:

$$v(\xi) = \gamma_{17} e^{\theta_{31}\xi} + \gamma_{18} e^{\theta_{32}\xi} + \gamma_{16}. \quad (67)$$

Using the Eqs. (65) and (67) in Eq. (20) yields:

$$w(\xi) = \gamma_{19} e^{\theta_{31}\xi} + \gamma_{20} e^{\theta_{32}\xi} + \gamma_{21}, \quad (68)$$

where $\gamma_{16}, \gamma_{17}, \gamma_{18}, \gamma_{19}, \gamma_{20}, \gamma_{21}$ are given in Appendix C. Put Eq. (65) in Eq. (58):

$$p(\xi) = \gamma_{22} e^{\theta_{31}\xi} + \gamma_{23} e^{\theta_{32}\xi} + \gamma_{24} \xi + \gamma_{25}, \quad (69)$$

where $\gamma_{22}, \gamma_{23}, \gamma_{24}$ are given in Appendix A and γ_{25} is arbitrary constant. Put Eqs. (28), (30) and (31) in Eq. (24):

$$T'' - \gamma_{26} T' = \gamma_{27} e^{2\theta_{31}\xi} + \gamma_{28} e^{2\theta_{32}\xi} + \gamma_{29} e^{(\theta_{31} + \theta_{32})\xi}. \quad (70)$$

The Solution of Eq. (70) is:

$$T(\xi) = \gamma_{30} + \gamma_{31} e^{\gamma_{26}\xi} + \gamma_{32} e^{2\theta_{31}\xi} + \gamma_{33} e^{2\theta_{32}\xi} + \gamma_{34} e^{(\theta_{31} + \theta_{32})\xi}, \quad (71)$$

where $\gamma_{26}, \gamma_{27}, \gamma_{28}, \gamma_{29}, \gamma_{32}, \gamma_{33}, \gamma_{34}$ are given in Appendix C and γ_{30}, γ_{31} are arbitrary constants. The velocity components, pressure and temperature in the original variables are:

$$u(x, y, z, t) = \gamma_4 e^{\theta_{31} \left(ax + by + nz + \frac{U t^\alpha}{\Gamma(\alpha+1)} \right)} + \gamma_5 e^{\theta_{32} \left(ax + by + nz + \frac{U t^\alpha}{\Gamma(\alpha+1)} \right)} + \gamma_6, \quad (72)$$

$$v(x, y, z, t) = \gamma_{17} e^{\theta_{31} \left(ax + by + nz + \frac{U t^\alpha}{\Gamma(\alpha+1)} \right)} + \gamma_{18} e^{\theta_{32} \left(ax + by + nz + \frac{U t^\alpha}{\Gamma(\alpha+1)} \right)} + \gamma_{16}, \quad (73)$$

$$w(x, y, z, t) = \gamma_{19} e^{\theta_{31} \left(ax + by + nz + \frac{U t^\alpha}{\Gamma(\alpha+1)} \right)} + \gamma_{20} e^{\theta_{32} \left(ax + by + nz + \frac{U t^\alpha}{\Gamma(\alpha+1)} \right)} + \gamma_{21}, \quad (74)$$

$$p(x, y, z, t) = \gamma_{22} e^{\theta_{31} \left(ax + by + nz + \frac{U t^\alpha}{\Gamma(\alpha+1)} \right)} + \gamma_{23} e^{\theta_{32} \left(ax + by + nz + \frac{U t^\alpha}{\Gamma(\alpha+1)} \right)} + \gamma_{24} \left(ax + by + nz + \frac{U t^\alpha}{\Gamma(\alpha+1)} \right) + \gamma_{25}, \quad (75)$$

$$T(x, y, z, t) = \gamma_{30} + \gamma_{31} e^{\gamma_{26} \left(ax + by + nz + \frac{U t^\alpha}{\Gamma(\alpha+1)} \right)} + \gamma_{32} e^{2\theta_{31} \left(ax + by + nz + \frac{U t^\alpha}{\Gamma(\alpha+1)} \right)} + \gamma_{33} e^{2\theta_{32} \left(ax + by + nz + \frac{U t^\alpha}{\Gamma(\alpha+1)} \right)} + \gamma_{34} e^{(\theta_{31} + \theta_{32}) \left(ax + by + nz + \frac{U t^\alpha}{\Gamma(\alpha+1)} \right)}. \quad (76)$$

4. Numerical Results and Discussions

In this study, it is attempted to find exact solutions of three-dimensional fractionalized MHD Newtonian fluid in a porous medium with heat transfer. The methodology in the latest work is easily diminished the nonlinear fractional partial differential equations to linearly ordinary differential equations. The method has been used without any restrictive assumption and laborious calculation. Defining fractional traveling wave parameter ξ , we can solve the nonlinear partial differential equations of 3D fractionalized MHD Newtonian fluid in a porous medium with heat transfer into linear ordinary differential equations and then solve for three interesting cases. Further, we write all cases and general solutions in the exponential form. By imposing some conditions on the roots of auxiliary equations of all cases, we can write a periodic form of general form. In special cases, we can determine the solutions for the case without MHD and porous effects from the general solutions by taking $M + \phi \rightarrow 0$.

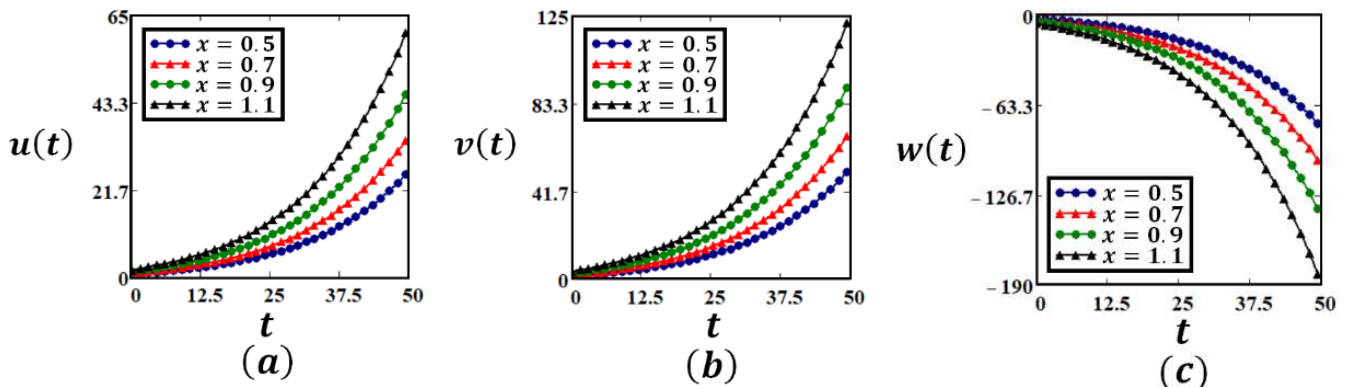


Fig. 1. 2D graphs of $u(x, t)$, $v(x, t)$ and $w(x, t)$ for MHD Newtonian fluid in porous medium given by Eqs. (35), (36) & (37) for $U = 0.1$, $\nu = 0.055$, $M = 0.1$, $\phi = 0.1$, $\alpha = 0.8$, $a = b = 1$, $y = z = 1$, $d_0 = 0.001$ and diverse values of x .

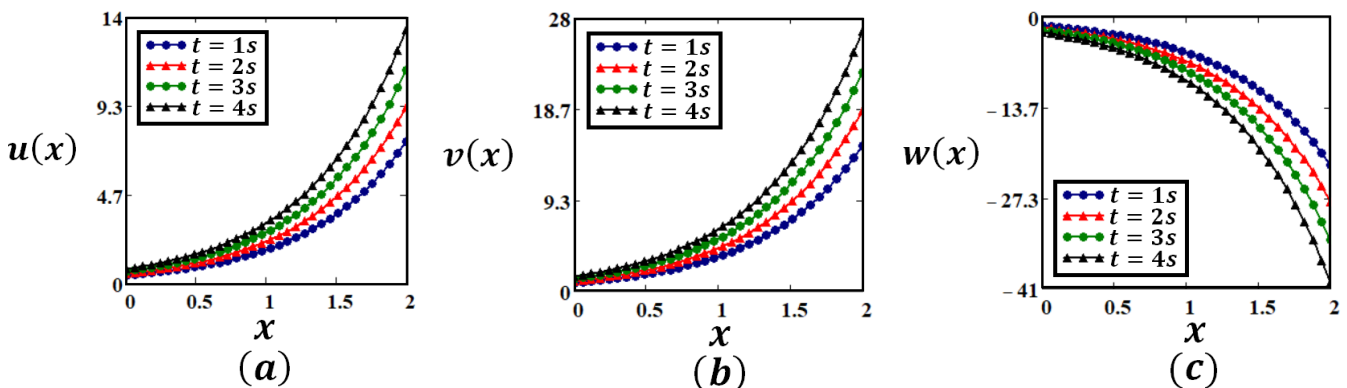


Fig. 2. 2D graphs of $u(x, t)$, $v(x, t)$ and $w(x, t)$ for MHD Newtonian fluid in porous medium given by Eqs. (35), (36) & (37) for $U = 0.1$, $\nu = 0.055$, $M = 0.1$, $\phi = 0.1$, $\alpha = 0.8$, $a = b = 1$, $y = z = 1$, $d_0 = 0.001$ and diverse values of t .

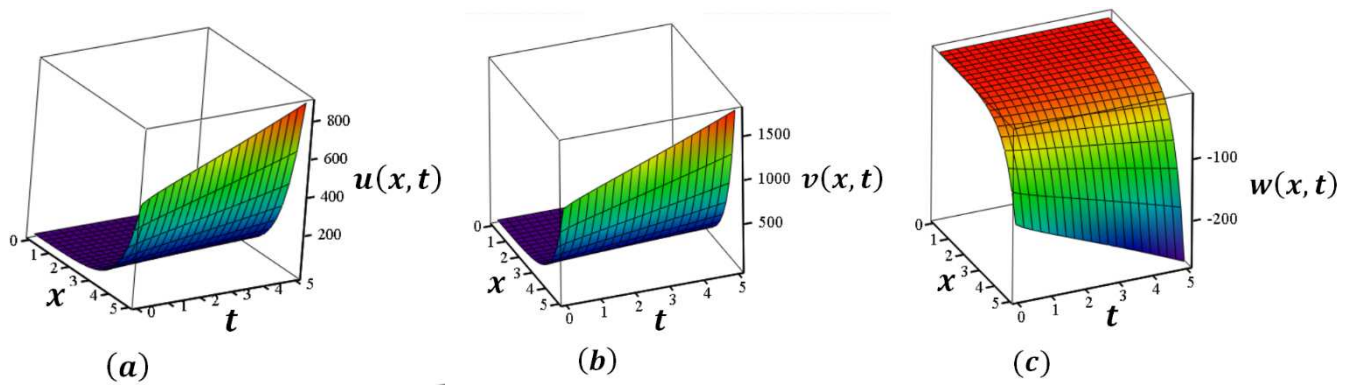


Fig. 3. 3D graphs of $u(x,t)$, $v(x,t)$ and $w(x,t)$ for MHD Newtonian fluid in porous medium given by Eqs. (35), (36) & (37) for $U = 0.1$, $\nu = 0.055$, $M = 0.1$, $\phi = 0.1$, $\alpha = 0.8$, $a = b = 1$, $y = z = 1$, $d_0 = 0.001$.

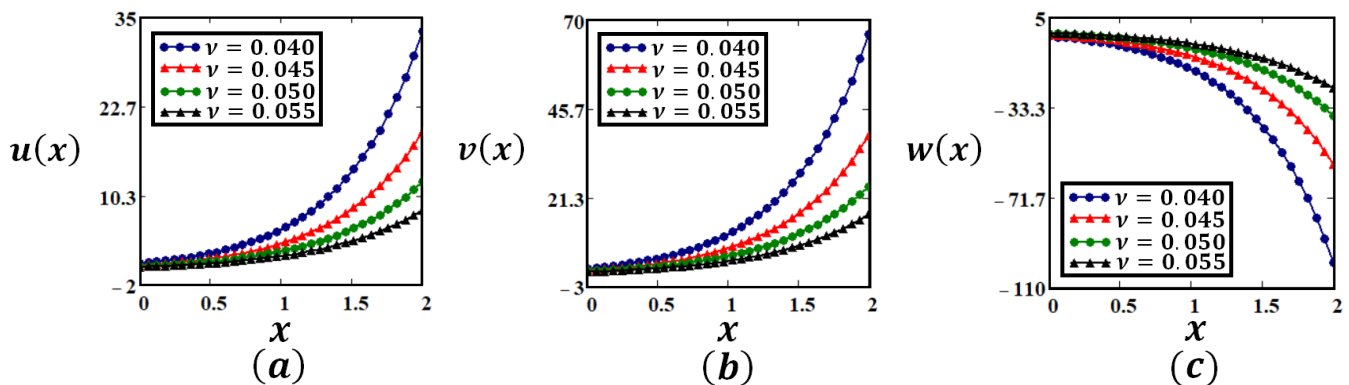


Fig. 4. 2D graphs of $u(x,t)$, $v(x,t)$ and $w(x,t)$ for MHD Newtonian fluid in porous medium given by Eqs. (35), (36) & (37) for $U = 0.1$, $\nu = 0.055$, $M = 0.1$, $\phi = 0.1$, $\alpha = 0.8$, $a = b = 1$, $y = z = 1$, $d_0 = 0.001$, $t = 2s$ and diverse values of ν .

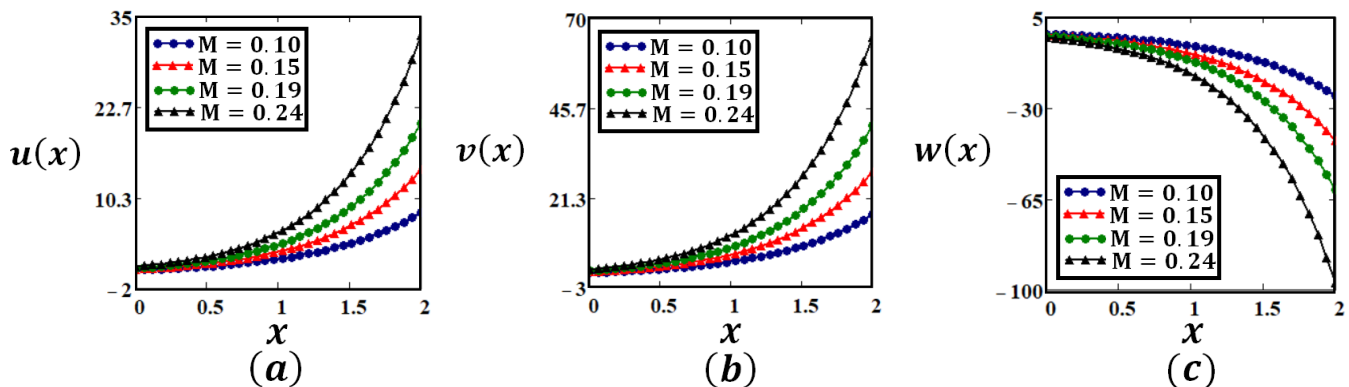


Fig. 5. 2D graphs of $u(x,t)$, $v(x,t)$ and $w(x,t)$ for MHD Newtonian fluid in porous medium given by Eqs. (35), (36) & (37) for $U = 0.1$, $\nu = 0.055$, $M = 0.1$, $\phi = 0.1$, $\alpha = 0.8$, $a = b = 1$, $y = z = 1$, $d_0 = 0.001$, $t = 2s$ and diverse values of M .

Now, in order to reveal some relevant physical aspect of the determine results the diagram of the fluid velocity components $u(x,t)$, $v(x,t)$ and $w(x,t)$ are present against x for different values of t and of the pertinent parameters of the viscous fluid. For convenience we make graph only for case-I when $U + d_0 \neq 0$ the similar illustration can be made for remaining cases. Figs. 1, 2 and 3 discussed the effect of time, space variable x and 3D sketches of u, v and w with respect to t and x . It shows clearly that the first two velocity components u and v are increasing functions of space variable x and time t . The third velocity component w is also increasing function (but in the absolute sense) of these variables. These phenomena are also represented by 3D graphs of u, v and w as shown in Figs. 3.

The effect of kinematic viscosity ν , on the fluid motion is shown in Figs. 4. The outcome of these figures as we except is that all velocity components are decreasing the function of ν . Figs. 5 and 6 show the effects of magnetic parameter M and porosity parameter ϕ , it is observed from these figures that the velocity component u, v and w (in an absolute sense) are increasing functions of these parameters. The last important parameter for us is the fractional parameter α , Figs. 7 clearly shows that the increasing values of fractional parameter α , increase the motion of fluid which is common experience that

when $\alpha \rightarrow 1$ the fractionalized Newtonian fluid reduces to an ordinary Newtonian fluid. Therefore, we conclude that the Newtonian fluid flows faster than the fractionalized Newtonian fluid. In Figs. 8, we have presented the influence of the parameter a appear in the traveling wave parameter $\xi = ax + by + nz + U t^a / \Gamma(\alpha + 1)$. It is noted that for the giving time t in every position of the domain of flow the velocity components u, v and w varies proportionally to the values of $a > 0$.

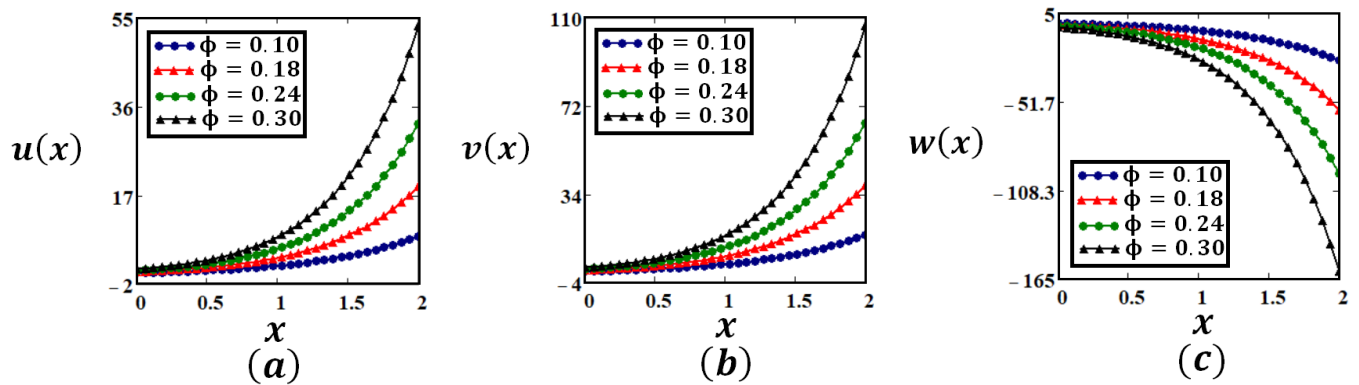


Fig. 6. 2D graphs of $u(x,t)$, $v(x,t)$ and $w(x,t)$ for MHD Newtonian fluid in porous medium given by Eqs. (35), (36) & (37) for $U = 0.1$, $\nu = 0.055$, $M = 0.1$, $\phi = 0.1$, $\alpha = 0.8$, $a = b = 1$, $y = z = 1$, $d_0 = 0.001$, $t = 2s$ and diverse values of ϕ .

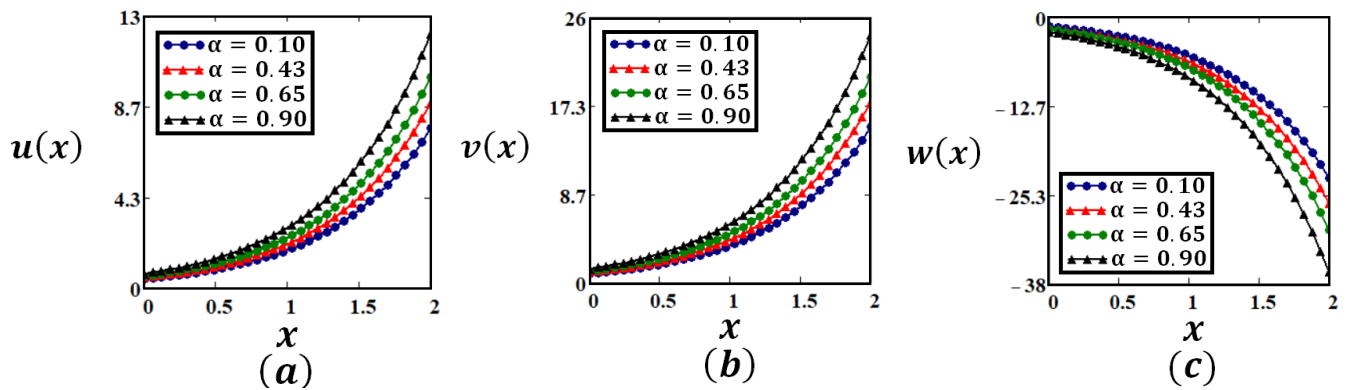


Fig. 7. 2D graphs of $u(x,t)$, $v(x,t)$ and $w(x,t)$ for MHD Newtonian fluid in porous medium given by Eqs. (35), (36) & (37) for $U = 0.1$, $\nu = 0.055$, $M = 0.1$, $\phi = 0.1$, $\alpha = 0.8$, $a = b = 1$, $y = z = 1$, $d_0 = 0.001$, $t = 5s$ and diverse values of α .

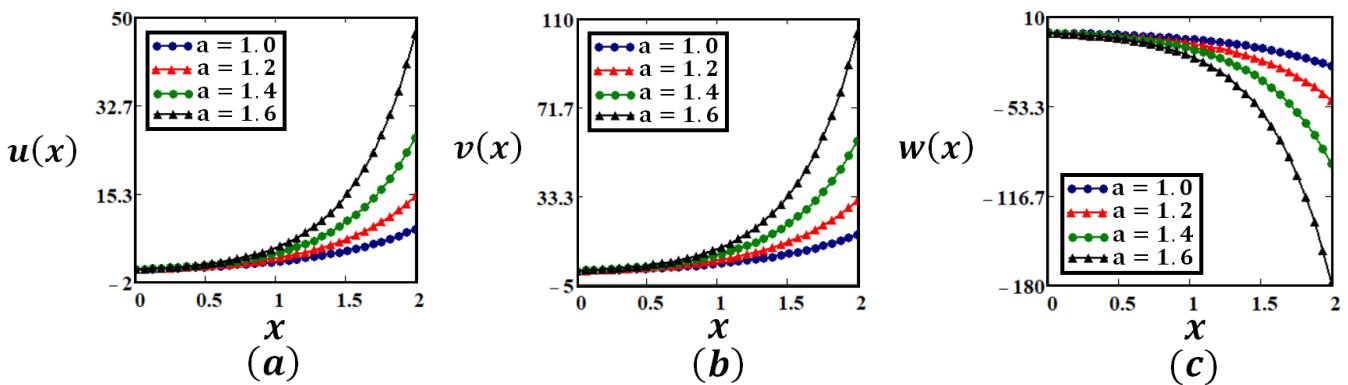


Fig. 8. 2D graphs of $u(x,t)$, $v(x,t)$ and $w(x,t)$ for MHD Newtonian fluid in porous medium given by Eqs. (35), (36) & (37) for $U = 0.1$, $\nu = 0.055$, $M = 0.1$, $\phi = 0.1$, $\alpha = 0.8$, $a = b = 1$, $y = z = 1$, $d_0 = 0.001$, $t = 2s$ and diverse values of a .

Finally, for differentiation, the profiles of the velocity components $u(x,t)$, $v(x,t)$ and $w(x,t)$ corresponding to the flow of the three types of fluids (MHD Newtonian fluid in porous medium for $\alpha = 0.5$, MHD Newtonian fluid in porous medium for $\alpha = 0.8$ and simple Newtonian fluids for $\alpha = 1$) are considered and presented in Figs. 9 for similar values of material constants. As expected, and justified from Figs. 7, that simple Newtonian fluid flows much faster than fractionalized MHD Newtonian fluid in a porous medium. Furthermore, it is noted in all figures that in the absolute sense the velocity components $u(x,t)$, $v(x,t)$ and $w(x,t)$ satisfy the inequality $u(x,t) < v(x,t) < w(x,t)$.

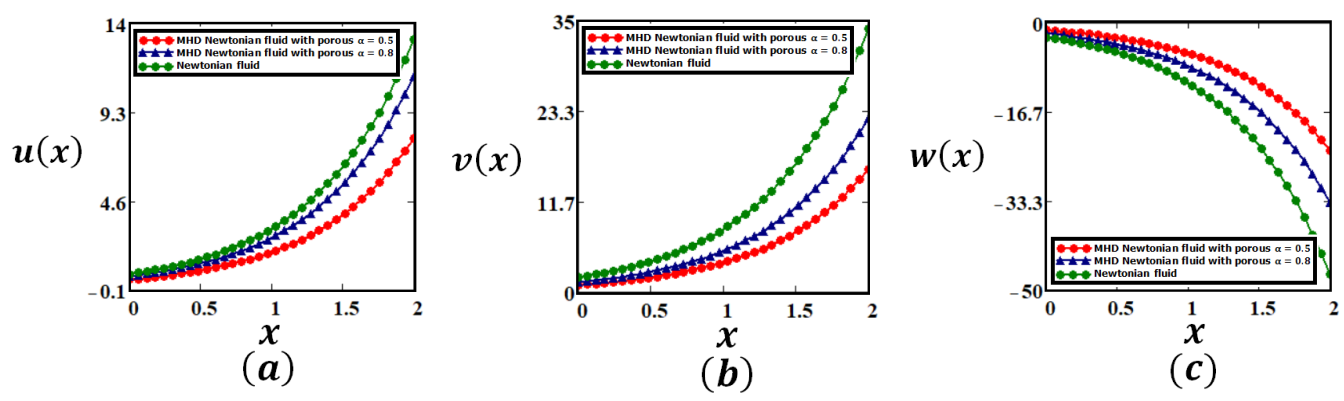


Fig. 9. 2D graphs of $u(x,t)$, $v(x,t)$ and $w(x,t)$ for MHD Newtonian fluid in porous medium for $\alpha = 0.5$ MHD Newtonian fluid in porous medium for $\alpha = 0.8$ and Newtonian fluid given by Eqs.(35),(36)&(37) for $U = 0.1$, $\nu = 0.055$, $M = 0.1$, $\phi = 0.1$, $\alpha = 0.8$, $a = b = 1$, $y = z = 1$, $d_0 = 0.001$, $t = 5s$.

5. Conclusions

In this paper exact traveling wave solutions obtained for three-dimensional fractionalized MHD Newtonian fluid in porous medium with heat transfer. Traveling wave methods easily convert the nonlinear partial differential equations into simple linear ordinary differential equations and then we solved for three different cases. At the end, in order to give some physical insight of the obtained solutions, we presented 2D and 3D graphical interpretation of the results for important parameters of interest.

Author Contributions

In this paper, the main idea and graphical part and their interpretation completed by Muhammad Jamil and Arsalan Ahmed performed the mathematical calculations and wrote the manuscript. The manuscript was written through the contribution of all authors. All authors discussed the results, reviewed and approved the final version of the manuscript.

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Conflict of Interest

The authors declared no potential conflicts of interest with respect to the research, authorship and publication of this article.

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Appendix A

$$\alpha_1 = \frac{-(U+d_0)}{\nu(a^2+b^2+n^2)}, \quad \alpha_2 = \frac{-(M+\phi)}{\nu(a^2+b^2+n^2)}, \quad \alpha_3 = \frac{-a(M+\phi)d_0}{\nu(a^2+b^2+n^2)^2}, \quad \alpha_6 = \frac{\alpha_3}{\alpha_2}, \quad \alpha_7 = \frac{-(U+d_0)}{\nu(a^2+b^2+n^2)} \quad (\text{A.1})$$

$$\alpha_8 = \frac{-(M+\phi)}{\nu(a^2+b^2+n^2)}, \quad \alpha_9 = \frac{1}{\nu a(a^2+b^2+n^2)} \left[\nu b \alpha_4 \theta_{11}^2 (a^2+b^2+n^2) - b \alpha_4 \theta_{11} (U+d_0) - b \alpha_4 (M+\phi) \right] \quad (\text{A.2})$$

$$\alpha_{10} = \frac{1}{\nu a(a^2+b^2+n^2)} \left[\nu b \alpha_5 \theta_{12}^2 (a^2+b^2+n^2) - b \alpha_5 \theta_{12} (U+d_0) - b \alpha_5 (M+\phi) \right] \quad (\text{A.3})$$

$$\alpha_{11} = \frac{-\alpha_6 (M+\phi)}{\nu a(a^2+b^2+n^2)}, \quad \alpha_{14} = \frac{\alpha_9}{(\theta_{11}^2 + \alpha_7 \theta_{11} + \alpha_8)}, \quad \alpha_{15} = \frac{\alpha_{10}}{(\theta_{12}^2 + \alpha_7 \theta_{12} + \alpha_8)}, \quad \alpha_{16} = \frac{\alpha_{11}}{\alpha_8} \quad (\text{A.4})$$

$$\alpha_{17} = \alpha_{12} + \alpha_{14}, \quad \alpha_{18} = \alpha_{13} + \alpha_{15} \quad (\text{A.5})$$

where α_{12} and α_{13} are arbitrary constants.

$$\alpha_{19} = \frac{-1}{n} (a \alpha_4 + b \alpha_{17}), \quad \alpha_{20} = \frac{-1}{n} (a \alpha_5 + b \alpha_{18}), \quad \alpha_{19} = \frac{-1}{n} (a \alpha_6 + b \alpha_{16} - d_0), \quad (\text{A.6})$$

$$\alpha_{22} = \frac{1}{a \theta_{11}} \left[\nu \rho \alpha_4 \theta_{11}^2 (a^2+b^2+n^2) - \rho \alpha_4 \theta_{11} (U+d_0) - \rho \alpha_4 (M+\phi) \right], \quad (\text{A.7})$$

$$\alpha_{23} = \frac{1}{a \theta_{12}} \left[\nu \rho \alpha_5 \theta_{12}^2 (a^2+b^2+n^2) - \rho \alpha_5 \theta_{12} (U+d_0) - \rho \alpha_5 (M+\phi) \right], \quad (\text{A.8})$$

$$\alpha_{24} = \frac{-1}{n} [\rho \alpha_6 (M+\phi)], \quad \alpha_{26} = \frac{(U+d_0)}{\eta(a^2+b^2+n^2)}, \quad (\text{A.9})$$

$$\alpha_{27} = \frac{-\nu}{\eta c_\rho (a^2 + b^2 + n^2)} \left[(\theta_{11} \alpha_4)^2 (2a^2 + b^2 + n^2) + (\theta_{11} \alpha_{17})^2 (2b^2 + a^2 + n^2) + (\theta_{11} \alpha_{19})^2 (2n^2 + a^2 + b^2) \right. \\ \left. + 2ab \alpha_4 \alpha_{17} \theta_{11}^2 + 2bn \alpha_{17} \alpha_{19} \theta_{11}^2 + 2an \alpha_4 \alpha_{19} \theta_{11}^2 \right] \quad (\text{A.10})$$

$$\alpha_{28} = \frac{-\nu}{\eta c_\rho (a^2 + b^2 + n^2)} \left[(\theta_{12} \alpha_5)^2 (2a^2 + b^2 + n^2) + (\theta_{12} \alpha_{18})^2 (2b^2 + a^2 + n^2) + (\theta_{12} \alpha_{20})^2 (2n^2 + a^2 + b^2) \right. \\ \left. + 2ab \alpha_5 \alpha_{18} \theta_{12}^2 + 2bn \alpha_{18} \alpha_{20} \theta_{12}^2 + 2an \alpha_5 \alpha_{20} \theta_{12}^2 \right] \quad (\text{A.11})$$

$$\alpha_{29} = \frac{-\nu}{\eta c_\rho (a^2 + b^2 + n^2)} \left[2 \theta_{11} \theta_{12} \alpha_4 \alpha_5 (2a^2 + b^2 + n^2) + 2 \theta_{11} \theta_{12} \alpha_{17} \alpha_{18} (2b^2 + a^2 + n^2) + 2 \theta_{11} \theta_{12} \alpha_{19} \alpha_{20} \right. \\ \left. \times (2n^2 + a^2 + b^2) + 2ab \theta_{11} \theta_{12} (\alpha_4 \alpha_{18} + \alpha_5 \alpha_{17}) + 2bn \theta_{11} \theta_{12} (\alpha_{17} \alpha_{20} + \alpha_{19} \alpha_{18}) + 2an \theta_{11} \theta_{12} (\alpha_4 \alpha_{20} + \alpha_5 \alpha_{19}) \right] \quad (\text{A.12})$$

$$\alpha_{32} = \frac{\alpha_{27}}{(2 \theta_{11})^2 - 2 \alpha_{26} \theta_{11}}, \quad \alpha_{33} = \frac{\alpha_{28}}{(2 \theta_{12})^2 - 2 \alpha_{26} \theta_{12}}, \quad \alpha_{34} = \frac{\alpha_{31}}{(\theta_{11} + \theta_{12})^2 - 2 \alpha_{26} (\theta_{11} + \theta_{12})} \quad (\text{A.13})$$

Appendix B

$$\beta_1 = \frac{-(M + \phi)}{\nu(a^2 + b^2 + n^2)}, \quad \beta_2 = \frac{-a(M + \phi)d_0}{\nu(a^2 + b^2 + n^2)^2}, \quad \beta_3 = \frac{\beta_2}{\beta_1}, \quad \beta_4 = \frac{-(M + \phi)}{\nu(a^2 + b^2 + n^2)} \quad (\text{B.1})$$

$$\beta_7 = \frac{b \beta_3}{\nu a(a^2 + b^2 + n^2)} \left[\nu \theta_{21} (a^2 + b^2 + n^2) - (M + \phi) \right] \quad (\text{B.2})$$

$$\beta_8 = \frac{b \beta_4}{\nu a(a^2 + b^2 + n^2)} \left[\nu \theta_{12} (a^2 + b^2 + n^2) - (M + \phi) \right], \quad \beta_9 = \frac{-\beta_5 (M + \phi)}{\nu a(a^2 + b^2 + n^2)} \quad (\text{B.3})$$

$$\beta_{12} = \frac{\beta_7}{(\theta_{21})^2 - \beta_6}, \quad \beta_{13} = \frac{\beta_8}{(-\theta_{21})^2 - \beta_6}, \quad \beta_{14} = \frac{-\beta_9}{\beta_6} \quad (\text{B.4})$$

$$\beta_{15} = \beta_{10} + \beta_{12}, \quad \beta_{16} = \beta_{11} + \beta_{13} \quad (\text{B.5})$$

where β_{10} and β_{11} are arbitrary constants.

$$\beta_{17} = \frac{-1}{n}(a \beta_3 + b \beta_{15}), \quad \beta_{18} = \frac{-1}{n}(a \beta_4 + b \beta_{16}), \quad \beta_{19} = \frac{-1}{n}(a \beta_5 + b \beta_{14} - d_0) \quad (\text{B.6})$$

$$\beta_{20} = \frac{\rho \beta_3}{a \theta_{21}} \left[\nu \theta_{21} (a^2 + b^2 + n^2) - (M + \phi) \right] \quad (\text{B.7})$$

$$\beta_{21} = \frac{\rho \beta_4}{a \theta_{21}} \left[\nu \theta_{21} (a^2 + b^2 + n^2) - (M + \phi) \right], \quad \beta_{22} = \frac{-1}{a} [\rho \beta_5 (M + \phi)] \quad (\text{B.8})$$

$$\beta_{24} = \frac{-\nu}{4 \theta_{21}^2 \eta c_\rho (a^2 + b^2 + n^2)} \left[(\theta_{21} \beta_3)^2 (2a^2 + b^2 + n^2) + (\theta_{21} \beta_{15})^2 (2b^2 + a^2 + n^2) + (\theta_{21} \beta_{17})^2 (2n^2 + a^2 + b^2) \right. \\ \left. + 2ab \beta_3 \beta_{15} \theta_{21}^2 + 2bn \beta_{15} \beta_{17} \theta_{21}^2 + 2an \beta_3 \beta_{17} \theta_{21}^2 \right] \quad (\text{B.9})$$

$$\beta_{25} = \frac{-\nu}{4\theta_{21}^2 \eta c_\rho (a^2 + b^2 + n^2)} \left[(\theta_{21} \beta_4)^2 (2a^2 + b^2 + n^2) + (\theta_{21} \beta_{16})^2 (2b^2 + a^2 + n^2) + (\theta_{21} \beta_{18})^2 (2n^2 + a^2 + b^2) \right. \\ \left. + 2ab \beta_4 \beta_{16} \theta_{21}^2 + 2bn \beta_{13} \beta_{18} \theta_{21}^2 + 2an \beta_4 \beta_{18} \theta_{21}^2 \right] \quad (\text{B.10})$$

$$\beta_{26} = \frac{-\nu \theta_{21}^2}{2\eta c_\rho (a^2 + b^2 + n^2)} \left[-2\beta_3 \beta_4 (2a^2 + b^2 + n^2) - 2\beta_{15} \beta_{16} (2b^2 + a^2 + n^2) - 2\beta_{17} \beta_{18} (2n^2 + a^2 + b^2) \right. \\ \left. - 2ab(\beta_3 \beta_{16} + \beta_4 \beta_{15}) - 2bn(\beta_{15} \beta_{18} + \beta_{16} \beta_{17}) - 2an \theta_{11} \theta_{12} (\beta_3 \beta_{18} + \beta_4 \beta_{17}) \right] \quad (\text{B.11})$$

Appendix C

$$\gamma_1 = \frac{-d_0}{\nu(a^2 + b^2 + n^2)}, \quad \gamma_2 = \frac{-(M + \phi)}{\nu(a^2 + b^2 + n^2)}, \quad \gamma_3 = \frac{-a(M + \phi)d_0}{\nu(a^2 + b^2 + n^2)^2}, \quad \gamma_6 = \frac{\gamma_3}{\gamma_2}, \quad \gamma_7 = \frac{-d_0}{\nu(a^2 + b^2 + n^2)} \quad (\text{C.1})$$

$$\gamma_8 = \frac{-(M + \phi)}{\nu(a^2 + b^2 + n^2)}, \quad \gamma_9 = \frac{1}{\nu a(a^2 + b^2 + n^2)} \left[\nu b \gamma_4 \theta_{31}^2 (a^2 + b^2 + n^2) - b \gamma_4 \theta_{31} d_0 - b \gamma_4 (M + \phi) \right] \quad (\text{C.2})$$

$$\gamma_{10} = \frac{1}{\nu a(a^2 + b^2 + n^2)} \left[\nu b \gamma_5 \theta_{32}^2 (a^2 + b^2 + n^2) - b \gamma_5 \theta_{32} d_0 - b \gamma_5 (M + \phi) \right] \quad (\text{C.3})$$

$$\gamma_{11} = \frac{-\gamma_6 (M + \phi)}{\nu a(a^2 + b^2 + n^2)}, \quad \gamma_{14} = \frac{\gamma_9}{(\theta_{31}^2 + \gamma_7 \theta_{31} + \gamma_8)}, \quad \gamma_{15} = \frac{\gamma_{10}}{(\theta_{32}^2 + \gamma_7 \theta_{32} + \gamma_8)}, \quad \gamma_{16} = \frac{\gamma_{11}}{\gamma_8} \quad (\text{C.4})$$

$$\gamma_{17} = \gamma_{12} + \gamma_{14}, \quad \gamma_{18} = \gamma_{13} + \gamma_{15} \quad (\text{C.5})$$

where γ_{12} and γ_{13} are arbitrary constants.

$$\gamma_{19} = \frac{-1}{n}(a\gamma_4 + b\gamma_{17}), \quad \gamma_{20} = \frac{-1}{n}(a\gamma_5 + b\gamma_{18}), \quad \gamma_{19} = \frac{-1}{n}(a\gamma_6 + b\gamma_{16} - d_0) \quad (\text{C.6})$$

$$\gamma_{22} = \frac{1}{a\theta_{31}} \left[\nu \rho \gamma_4 \theta_{31}^2 (a^2 + b^2 + n^2) - \rho \gamma_4 \theta_{31} d_0 - \rho \gamma_4 (M + \phi) \right] \quad (\text{C.7})$$

$$\gamma_{23} = \frac{1}{a\theta_{32}} \left[\nu \rho \gamma_5 \theta_{32}^2 (a^2 + b^2 + n^2) - \rho \gamma_5 \theta_{32} d_0 - \rho \gamma_5 (M + \phi) \right] \quad (\text{C.8})$$

$$\gamma_{24} = \frac{-1}{n}[\rho \gamma_6 (M + \phi)], \quad \gamma_{26} = \frac{d_0}{\eta(a^2 + b^2 + n^2)} \quad (\text{C.9})$$

$$\gamma_{27} = \frac{-\nu}{\eta c_\rho (a^2 + b^2 + n^2)} \left[(\theta_{31} \gamma_4)^2 (2a^2 + b^2 + n^2) + (\theta_{31} \gamma_{17})^2 (2b^2 + a^2 + n^2) + (\theta_{31} \gamma_{19})^2 (2n^2 + a^2 + b^2) \right. \\ \left. + 2ab \gamma_4 \gamma_{17} \theta_{31}^2 + 2bn \gamma_{17} \gamma_{19} \theta_{31}^2 + 2an \gamma_4 \gamma_{19} \theta_{31}^2 \right] \quad (\text{C.10})$$

$$\gamma_{28} = \frac{-\nu}{\eta c_\rho (a^2 + b^2 + n^2)} \left[(\theta_{32} \gamma_5)^2 (2a^2 + b^2 + n^2) + (\theta_{32} \gamma_{18})^2 (2b^2 + a^2 + n^2) + (\theta_{32} \gamma_{20})^2 (2n^2 + a^2 + b^2) \right. \\ \left. + 2ab \gamma_5 \gamma_{18} \theta_{32}^2 + 2bn \gamma_{18} \gamma_{20} \theta_{32}^2 + 2an \gamma_5 \gamma_{20} \theta_{32}^2 \right] \quad (\text{C.11})$$

$$\gamma_{29} = \frac{-\nu}{\eta c_p (a^2 + b^2 + n^2)} \left[2\theta_{31}\theta_{32}\gamma_4\gamma_5(2a^2 + b^2 + n^2) + 2\theta_{31}\theta_{32}\gamma_{17}\gamma_{18}(2b^2 + a^2 + n^2) + 2\theta_{31}\theta_{32}\gamma_{19}\gamma_{20} \right. \\ \left. \times (2n^2 + a^2 + b^2) + 2ab\theta_{31}\theta_{32}(\gamma_4\gamma_{18} + \gamma_5\gamma_{17}) + 2bn\theta_{31}\theta_{32}(\gamma_{17}\gamma_{20} + \gamma_{19}\gamma_{18}) + 2an\theta_{31}\theta_{32}(\gamma_4\gamma_{20} + \gamma_5\gamma_{19}) \right] \quad (C.12)$$

$$\gamma_{32} = \frac{\gamma_{27}}{(2\theta_{31})^2 - 2\gamma_{26}\theta_{31}}, \quad \gamma_{33} = \frac{\gamma_{28}}{(2\theta_{32})^2 - 2\gamma_{26}\theta_{32}}, \quad \gamma_{34} = \frac{\gamma_{31}}{(\theta_{31} + \theta_{32})^2 - 2\gamma_{26}(\theta_{31} + \theta_{32})} \quad (C.13)$$

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