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On Approximate Stationary Radial Solutions for a Class of Boundary Value Problems Arising in Epitaxial Growth Theory

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Abstract. In this paper, we consider a non-self-adjoint, singular, nonlinear fourth order boundary value problem which arises in the theory of epitaxial growth. It is possible to reduce the fourth order equation to a singular boundary value problem of second order given by

$$w'' - \frac{1}{r} w' = \frac{w^2}{2r^2} + \frac{1}{2} \lambda r^2.$$

The problem depends on the parameter λ and admits multiple solutions. Therefore, it is difficult to pick multiple solutions together by any discrete method like finite difference method, finite element method etc. Here, we propose a new technique based on homotopy perturbation method and variational iteration method. We compare numerically the approximate solutions computed by Adomian decomposition method. We study the convergence analysis of both iterative schemes in $C^2([0, 1])$. For small values of λ , solutions exist whereas for large values of λ solutions do not exist.

Keywords: Singular boundary value problems, nonlinear boundary value problems, iterative method, convergence analysis, multiple solutions, non-self-adjoint operators, epitaxial growth.

1. Introduction

Nowadays semiconductor materials attain great attraction due to its elegant properties, e.g. Gallium arsenide (GaAs), Gallium Nitride (GaN), Silicon Carbide (SiC), Cadmium sulphide (CdS) etc. These are widely used in many structures like high performance RF devices, microwave ICs, photo resistors, solar cells and many electronic instruments like lasers, diodes, bipolar transistors etc. Such advanced structures can be achieved by choosing right epitaxial technique. More precisely, we can say epitaxial growth technique produces thin films ([1]) under high vacuum conditions from crystal in the semiconductor industry. There are several types of epitaxial growth technique like Liquid Phase Epitaxy (LPE), Vapour Phase Epitaxy (VPE), Metal Organic Vapour Phase Epitaxy (MOVPE), Molecular Beam Epitaxy (MBE) etc. Many models were produced depending on these epitaxial growth techniques. In general, these models have been introduced in two different ways, one is discrete probabilistic system and the other is differential equation ([1]). In this work, we mainly focus on differential equation which is described in [2]. The authors considered the following function as mathematical description of epitaxial growth

$$\sigma: \Omega \subset \mathbb{R}^N \times \mathbb{R}^+ \rightarrow \mathbb{R}, \quad (1)$$

which describes the height of the growing interface in the spatial point $x \in \Omega \subset \mathbb{R}^2$ at time $t \in \mathbb{R}^+$. For simplicity, they considered $N = 2$. The macroscopic description of the growing interface for σ is given by the stochastic partial differential equation (SPDE) ([1]). There are many such well-known SPDEs given by Kardar, Parisi and Zhang (KPZ) ([3]), Edwards and Wilkinson (EW), Mullins and Herring (MH) ([1]) and Villain, Lai, and Das Sharma (VLDS) ([4]). They considered height function which obeys a gradient flow equation with a forcing term

$$\frac{\partial \sigma}{\partial t} = (1 + (\nabla \sigma)^2)^{\frac{1}{2}} \left[-\frac{\partial J(\sigma)}{\partial \sigma} + \xi(x, t) \right]. \quad (2)$$

The functional $J(\sigma)$ denotes a potential that describes the microscopic properties of the interface at the macroscopic scale ([5]) and is given by

$$J(\sigma) = \int_{\Omega} Q(z)(1 + (\nabla \sigma)^2)^{\frac{1}{2}} dx, \quad (3)$$

where the square root term models growth along the normal to the surface, z denotes the mean curvature ([5]) and Q is an unknown function of z . The power series expansion of the function $Q(z)$ was introduced by them as follows

$$Q(z) = K_0 + K_1 z + \frac{K_2 z^2}{2} + \frac{K_3 z^3}{6} + \dots, \quad (4)$$

and apply the small gradient expansion, which is $|\nabla \sigma| \ll 1$. Therefore after simplification, final result reads

$$\frac{\partial \sigma}{\partial t} = K_0 \nabla^2 \sigma + 2K_1 \det(D^2 \sigma) - K_2 \nabla^4 \sigma - \frac{1}{2} K_3 \Delta(\Delta \sigma)^2 + \xi(x, t), \quad (5)$$

which is the same as conservative counterpart of the Kardar-Parisi-Zhang equation ([3, 4, 6]). In [7, 2], they explained each terms of equation (5) geometrically. In epitaxial growth theory, they considered $K_0 = 0$ and $K_3 = 0$ ([2]). Therefore, the equation (5) leads to the following equation

$$\frac{\partial \sigma}{\partial t} + K_2 \nabla^4 \sigma = 2K_1 \det(D^2 \sigma) + \xi(x, t). \quad (6)$$

The equation (6) is considered as a possible continuum hypothesis of epitaxial growth theory ([8, 9, 10]). Therefore the stationary version of equation (6) can be reduced in the following form

$$\Delta^2 \sigma = \det(D^2 \sigma) + \lambda P(x), \quad x \in \Omega \subset \mathbb{R}^2, \quad (7)$$

boundary conditions,

where $P(x)$ is a forcing term which is time independent, λ is the measure of the speed at which new particles are deposited.

Here, we will consider two types of boundary conditions. Corresponding to (7) homogeneous Dirichlet boundary condition is

$$\sigma = 0, \quad \frac{\partial \sigma}{\partial n} = 0 \text{ on } \partial \sigma, \quad (8)$$

where n is unit out drawn normal to $\partial \sigma$ and homogeneous Navier boundary condition is

$$\sigma = 0, \quad \Delta \sigma = 0 \text{ on } \partial \sigma. \quad (9)$$

The problem is set on a unit disk and we define

$$r = |x|, \quad \sigma = \phi(|x|). \quad (10)$$

Now, we have

$$\Delta^2 \sigma = \frac{\partial^4 \sigma}{\partial x_1^4} + \frac{\partial^4 \sigma}{\partial x_2^4} + 2 \frac{\partial^4 \sigma}{\partial x_1^2 \partial x_2^2} \quad \text{and} \quad \det(D^2 \sigma) = \frac{\partial^2 \sigma}{\partial x_1^2} \frac{\partial^2 \sigma}{\partial x_2^2} - \left(\frac{\partial^2 \sigma}{\partial x_1 \partial x_2} \right)^2. \quad (11)$$

By using the transformation (10) and equation (11), the above partial differential equation (7) is converted into an ordinary differential equation, which is given by

$$\frac{1}{r} \left(r \left[\frac{1}{r} (r\phi')' \right] \right)' = \frac{1}{r} \phi' \phi'' + \lambda P(r), \quad (12)$$

where $' = \frac{d}{dr}$. Corresponding to (12), our Dirichlet boundary conditions become

$$\phi'(0) = 0, \quad \phi(1) = 0, \quad \phi'(1) = 0, \quad \lim_{r \rightarrow 0} r \phi'''(r) = 0, \quad (13)$$

and Navier boundary conditions of type one are

$$\phi'(0) = 0, \quad \phi(1) = 0, \quad \phi'(1) + \phi''(1) = 0, \quad \lim_{r \rightarrow 0} r \phi'''(r) = 0, \quad (14)$$

and Navier boundary conditions of type two are

$$\phi'(0) = 0, \quad \phi(1) = 0, \quad \phi''(1) = 0, \quad \lim_{r \rightarrow 0} r \phi'''(r) = 0. \quad (15)$$

Again for simplicity, we have considered $P(r) = 1$. Integrating equation (12) and using $\lim_{r \rightarrow 0} r \phi'''(r) = 0$ gives

$$r \left[\frac{1}{r} (r\phi')' \right]' = \frac{1}{2} (\phi')^2 + \frac{1}{2} \lambda r^2. \quad (16)$$

Now, by using the transformation $w = r\phi'$, from equation (16) we get

$$r^2 w'' - rw' = \frac{1}{2} w^2 + \frac{1}{2} \lambda r^4. \quad (17)$$

Subject to the following boundary conditions.

- Now our Dirichlet boundary condition becomes

$$w'(0) = 0, \quad w(1) = 0, \quad (18)$$

- Navier boundary condition of type one is

$$w'(0) = 0, \quad w'(1) = 0, \quad (19)$$

- Navier boundary condition of type two is

$$w'(0) = 0, \quad w(1) = w'(1). \quad (20)$$

Equation (17) has been integrated numerically by using fourth order Runge-Kutta method in [2]. The aim of this work is to find the approximate solutions of fourth order differential equation (12) with $P(r) = 1$. To solve the equation (12), we first solve equation (17) by using homotopy perturbation method (HPM) and Adomian decomposition method (ADM). Recently, HPM and ADM have been studied by several researchers. In [11, 12, 13], J. H. He applied HPM to solve nonlinear equations. After that Momani and Odibat ([14]) applied HPM to solve Klein-Gordon equations. Attractively the idea found its way in research and has been used to found approximate numerical solution of nonlinear fractional IVPs ([15]), time dependent differential equations ([16, 17]), nonlinear Fredholm and Volterra integro-differential equations ([18]), Cauchy reaction diffusion problem ([19]), four point boundary value problems ([20]), system of second order BVP ([21]) and many other problems ([22, 23, 24, 25]).

In the beginning of the 1980's, Adomian ([26, 27, 28]) proposed a new and powerful method (called ADM) for solving linear and nonlinear singular differential equations. It has been used by many authors to find numerical or analytical solution for physical and mathematical problems involving differential equations ([29, 30]), fractional differential equations ([31]), the integral equation ([32]), integro-differential equation ([33]), system of such equations ([34]) etc. The

convergence of ADM was established by Cherruault, Adomian ([28]) and Hosseini, Nasabzadeh ([35]). To know about this method and its application, readers can read the reference and the references therein.

Motivation: Our problem (17) is non self-adjoint, singular, nonlinear and has no exact solution. Carlos et. al. ([9]) proved theoretically that, for $0 \leq \lambda \leq \lambda_c$, the problem (17) have two solutions, therefore it is not easy to capture all solutions by any discrete method like finite difference method as well as any iterative schemes. For every solution $w(r)$ of equation (17), they have $w(r) \leq 0, \forall r \in [0,1]$ and $\lim_{r \rightarrow 0} w(r) = 0$. In proposition (4.1) and proposition (4.2), they mention the estimates of λ_c . Corresponding to Dirichlet boundary condition λ_c admits the estimates $144 \leq \lambda_c \leq 307$ and corresponding to Navier boundary condition of type two λ_c admit the estimates $9 \leq \lambda_c \leq 11.63$. They did not find the estimates of λ_c corresponding to Navier boundary condition of type one. All these properties make this problem very very interesting and challenging for researchers. Till now there are only few papers which address certain issues related to the BVP considered in this paper and lot of investigations are still pending. Keeping this in view, we have put sincere effort to explore the given boundary value problem.

Our contribution: Since equation (17) is non self-adjoint, therefore it is not easy to get a suitable scheme which provides us better accuracy. In this paper, we have proposed two iterative schemes by using the boundary condition in a suitable way to get better accuracy. Also, we compare the approximate solutions which is produced by HPM based on variational iteration method (VIM) with ADM. Many authors ([36, 35]) proved convergence analysis of iterative schemes using Cauchy sequential criteria. But, to check the validity of our proposed techniques, we have proved the convergence analysis in a novel way. Furthermore, we verify the existing theoretical results ([9, 2]) by our numerical results.

Outline: This paper is organized in the following 8 sections. Sections 2 and 3 are devoted to the description of the iterative methods. In section 4, we have done the convergence analysis of these two methods. Numerical observations for all boundary conditions are presented in section 5. The residue error of approximate solutions and graphs are displayed in section 6 and 7 respectively. Finally, paper ends with conclusions in section 8.

2. ADM

To explain about this method, we rewrite equation (17) into the self-adjoint form given by

$$\left[\frac{w'}{r} \right]' = \frac{w^2}{2r^3} + \frac{1}{2} \lambda r. \quad (21)$$

Let L be the linear differential operator defined by

$$L(.) = \frac{d}{dr} \left[r^{-1} \frac{d}{dr} (.) \right]. \quad (22)$$

Therefore, equation (21) can be written as

$$L(w) = \frac{w^2}{2r^3} + \frac{1}{2} \lambda r. \quad (23)$$

We assume $w(r) \in C^2(0,1]$. Now we can write the equation (21) in normalized form and given by

$$w'' - \frac{w'}{r} = \frac{w^2}{2r^2} + \frac{1}{2} \lambda r^2. \quad (24)$$

We can easily see that zero is the singular point of equation (24). Taking limit $r \rightarrow 0$ from both sides of equation (24), we have

$$\lim_{r \rightarrow 0} w''(r) - \lim_{r \rightarrow 0} \frac{w'}{r} - \lim_{r \rightarrow 0} \frac{w^2}{2r^2} = 0. \quad (25)$$

Using the fact that $w'(0) = 0$, $w(0) = 0$ and equation (25), we can assume $w''(0)$ exists. Let us consider $w''(0) = a$, $a \in \mathbb{R}$. Integrating (21) from 0 to r twice, we compute the operator $L^{-1}(.)$, which is given by

$$L^{-1}(.) = \int_0^r x \int_0^x (.) dt dx. \quad (26)$$

Applying this operator to both sides of equation (21) yields

$$L^{-1} \left(\left[\frac{w'}{r} \right]' \right) = L^{-1} \left(\frac{w^2}{2r^3} + \frac{1}{2} \lambda r \right). \quad (27)$$

Now using the boundary condition $w'(0) = 0$ and integrating by parts, from equation (27), we get

$$w(r) = \frac{ar^2}{2} + \frac{\lambda r^4}{16} + \int_0^r x \int_0^x \frac{w^2}{2t^3} dt dx. \quad (28)$$

The inner integrand of the integral in equation (28) is nonlinear in w . To find the solution we approximate this nonlinearity by using Adomian's polynomial, which is given by

$$A_n = \frac{1}{n!} \frac{d^n}{d\beta^n} \left[N \left(\sum_{i=0}^n \beta^i w_i(r) \right) \right]_{\beta=0}, \quad n = 0, 1, \dots \quad (29)$$

Now from equation (28) and second fundamental theorem of calculus ([37], theorem 34.3, p-294), we can conclude that $w(r) \in C^2[0,1]$. Now, we write the polynomials expansion that corresponds to the nonlinear term as follows

$$A_0 = \frac{w_0^2}{2r^3}, \quad A_1 = \frac{w_0 w_1}{r^3}, \quad A_2 = \frac{w_1^2 + 2w_0 w_2}{2r^3}, \quad A_3 = \frac{w_1 w_2 + w_0 w_3}{r^3}, \dots \quad (30)$$

Therefore, from equation (28), we define

$$w_0(r) = \frac{ar^2}{2} + \frac{\lambda r^4}{16}, \quad (31)$$

$$w_1(r) = \int_0^r x \int_0^x A_0(w_0) dt dx, \quad (32)$$

$\vdots$

$$w_{n+1}(r) = \int_0^r x \int_0^x A_n(w_0, w_1, \dots, w_n) dt dx, \quad (33)$$

$\vdots$

After computing all the components of A_n , we can find the solution $w(r) = \sum_{n=0}^{\infty} w_n(r)$ of our original equation, provided the series $\sum_{n=0}^{\infty} w_n(r)$ is convergent. The convergence of the series $\sum_{n=0}^{\infty} w_n(r)$ will be discussed in section 4.1.

3. HPM Coupled with VIM (HPM-VIM)

To apply HPM, we can write equation (17) as

$$Lw + Nw = g(r), \quad (34)$$

where $L(w) = r^2 w'' - rw'$, $N(w) = -\frac{1}{2}w^2$ and $g(r) = \frac{1}{2}\lambda r^4$. Therefore after simplification as in [38], we have

$$q^0: L(v_0) - L(u_0) = 0, \quad (35)$$

$$q^1: L(v_1) + L(u_0) - \frac{1}{2}v_0^2 - \frac{1}{2}\lambda r^4 = 0, \quad (36)$$

$\vdots$

$$q^{n+1}: L(v_{n+1}) + H_n = 0, \quad (37)$$

$\vdots$

where H_n be the He's polynomial given by

$$H_n = \frac{1}{n!} \frac{d^n}{dq^n} \left[N \left(\sum_{i=0}^n q^i v_i(r) \right) \right]_{q=0}, \quad n = 0, 1, \dots \quad (38)$$

We see that equations (35-37) are all second order differential equations. Let us consider $u_0(r) = ar^2$ and $v_0(r) = ar^2$, where $a \in \mathbb{R}$. Obviously $u_0(r)$ and $v_0(r)$ easily satisfy the initial condition as well as equation (35). Put the values of $u_0(r)$ and $v_0(r)$ in equation (36), we have

$$L(v_1) = \frac{a^2 r^4}{2} + \frac{1}{2} \lambda r^4. \quad (39)$$

To solve the equation (39), we use VIM ([38]). We consider $v_1 = z$. Therefore, the correctional formula of VIM of equation (39) is as follows

$$z_{n+1}(r) = z_n(r) + \int_0^r \eta(t) \left(t^2 z_n''(t) - t z_n'(t) - \frac{1}{2} a^2 t^4 - \frac{1}{2} \lambda t^4 \right) dt, \quad n = 0, 1, \dots, \quad (40)$$

where $z_0(r)$ is the initial approximation satisfying the initial condition and $\eta(t)$ is the Lagrange multiplier. The Lagrange multiplier $\eta(t)$ can be identified optimally via variational principle. Using the analysis of [39], we find the stationary conditions which are given by the following three equations:

$$1 - \eta'(r)r^2 - 3r\eta(r) = 0, \quad (41)$$

$$\eta(r) = 0, \quad (42)$$

$$\eta''(t)t^2 + 5t\eta'(t) + 3\eta(t) = 0. \quad (43)$$

By solving (41-43), we get the optimal value of $\eta(t)$ and is given by

$$\eta(t) = \left(\frac{1}{2t} - \frac{r^2}{2t^3} \right). \quad (44)$$

Let $z_0(t) = 0$. By using (40) and (44), for $n = 0$, we get

$$z_1(r) = v_1(r) = \int_0^r \left(\frac{1}{2t} - \frac{r^2}{2t^3} \right) \left(-\frac{1}{2} a^2 t^4 - \frac{1}{2} \lambda t^4 \right) dt. \quad (45)$$

Hence by similar analysis, from (36-37) and (44), we have

$$q^2: v_2(r) = \int_0^r \left(\frac{1}{2t} - \frac{r^2}{2t^3} \right) H_1 dt, \quad (46)$$

$\vdots$

$$q^{n+1}: v_{n+1}(r) = \int_0^r \left(\frac{1}{2t} - \frac{r^2}{2t^3} \right) H_n dt, \quad (47)$$

$\vdots$

After solving the above system of equations, we can compute the values of v_i 's. Finally the approximate solutions of equation (17) is $w(r) = \sum_{i=0}^{\infty} v_i(r)$, provided the series is convergent. In section 4.2, we will discuss the convergence of $\sum_{i=0}^{\infty} v_i(r)$.

4. Convergence Analysis

Now, we prove the series $\sum_{i=0}^{\infty} v_i(r)$ defined by (45-47) and $\sum_{i=0}^{\infty} w_i(r)$ defined by (31-33) are convergent and converge to the exact solutions $w(r)$ of equation (17). We assume $\{S_n\}$ be the sequence of n^{th} partial sums of the series $\sum_{i=0}^{\infty} v_i(r)$ as well as $\sum_{i=0}^{\infty} w_i(r)$. We can write the series $\sum_{i=0}^{\infty} v_i(r)$ and $\sum_{i=0}^{\infty} w_i(r)$ in the following form

$$S_0 + \sum_{i=0}^{\infty} (S_{i+1} - S_i). \quad (48)$$

4.1 Convergence analysis of ADM

For every sequence $S_n = \sum_{i=0}^n w_i(r)$, we approximate $N(\sum_{i=0}^n w_i(r))$ by [40] and [41] is as follows

$$N\left(\sum_{i=0}^n w_i(r)\right) = \frac{1}{2r^3} \left(\sum_{i=0}^n w_i(r)\right)^2 = \sum_{i=0}^n A_i(w_0, w_1, \dots, w_i). \quad (49)$$

Therefore, we can write

$$N(S_n) = \frac{1}{2r^3} (S_n)^2 = \sum_{i=0}^n A_i(w_0, w_1, \dots, w_i). \quad (50)$$

Lemma 4.1.1. Let $w_0 = ar^2$, $a \in \mathbb{R}$ and $w_n \in C^2(0,1]$, $\forall n \in \mathbb{N}$. Then $w_n(0) = 0$, $w_n'(0) = 0$, $w_n''(r)$, $w_n'''(r)$, A_n are bounded on $[0,1]$ and $w_n \in C^2[0,1]$ for all $n \in \mathbb{N}$.

Proof. We use mathematical induction on n . Since $w_0(0) = 0$, $w_0'(0) = 0$, $w_0''(r)$ and $w_0'''(r)$ are bounded on $[0,1]$, then from the definition of Adomian's polynomials, we have $A_0 = N(w_0) = \frac{w_0^2}{2r^3}$. Now, $\lim_{r \rightarrow 0} \frac{w_0^2}{2r^3} = \lim_{r \rightarrow 0} \frac{2w_0 w_0'}{6r^2} = \lim_{r \rightarrow 0} \frac{2w_0 w_0'' + 2w_0' w_0'}{12r} = \lim_{r \rightarrow 0} \frac{2w_0 w_0'' + 6w_0' w_0'}{12} = 0$. Therefore, we get $A_0(w_0)$ is bounded on $[0,1]$. Now, from equation (32), we have $w_1(0) = 0$, $w_1'(r) = r \int_0^r A_0(w_0) dt$, $w_1''(r) = \int_0^r A_0(w_0) dt + r A_0(w_0)$ and $w_1'''(r) = 2A_0(w_0) + r A_0'(w_0) w_0'$. Therefore, we have, $w_1(0) = 0$, $w_1'(0) = 0$, $w_1''(r)$ and $w_1'''(r)$ are bounded on $[0,1]$. Now, from equation (29), we have

$$A_1(w_0, w_1) = \frac{1}{2} \sum_{i=0}^1 \frac{w_{1-i}(r) w_i(r)}{r^3}. \quad (51)$$

By similar analysis, we can easily show that, each $\frac{w_i(r)}{r^2}$, $i = 0,1$ and $\frac{w_i(r)}{2r}$, $i = 0,1$ are bounded on $[0,1]$. By second fundamental theorem of calculus ([37], theorem 34.3, p-294), we conclude that $w_1(0) = 0$, $w_1'(0) = 0$, $w_1''(r)$, $w_1'''(r)$, A_1 are bounded on $[0,1]$ and $w_1 \in C^2[0,1]$. Hence our assumptions are true for $n = 1$. Let our assumptions be true up to $n = m$. Therefore, $w_n(0) = 0$, $w_n'(0) = 0$, $w_n''(r)$, $w_n'''(r)$, A_n are bounded on $[0,1]$ and $w_n \in C^2[0,1]$, for all $n = 2, 3, \dots, m$. Therefore, for $n = m+1$, we have $w_{m+1}(0) = 0$, $w_{m+1}'(r) = r \int_0^r A_m(w_0, w_1, \dots, w_m) dt$, $w_{m+1}''(r) = \int_0^r A_m(w_0, w_1, \dots, w_m) dt + r A_m(w_0, w_1, \dots, w_m)$ and $w_{m+1}'''(r) = 2A_m(w_0, w_1, \dots, w_m) + r A_m'(w_0, w_1, \dots, w_m)$. So similarly $w_{m+1}(0) = 0$, $w_{m+1}'(0) = 0$, $w_{m+1}''(r)$, $w_{m+1}'''(r)$ are bounded on $[0,1]$ and $w_{m+1} \in C^2[0,1]$. Again, from the definition of Adomian's polynomials, we get

$$A_{m+1}(w_0, w_1, \dots, w_{m+1}) = \frac{1}{2} \sum_{i=0}^{m+1} \frac{w_{(m+1)-i}(r) w_i(r)}{r^3}. \quad (52)$$

In a similar manner, we can conclude that, $A_{m+1}(w_0, w_1, \dots, w_{m+1})$ is also bounded on $[0,1]$. Hence our assumptions are true for $n = m+1$. Hence, by mathematical induction, we get the results.

Lemma 4.1.2. Let $w_n(r) \in C^2[0,1]$, for all $n \in \mathbb{N} \cup \{0\}$. Let $p_i = \sup_{r \in [0,1]} |A_i|$, $i = 0, 1, \dots$. If there exists a natural number k ,

i.e., $p_{i+1} \leq p_i$ for all $i \geq k$, then there exists a real number $K_1 > 0$, such that $\left| \frac{S_n + S_{n-1}}{2r^2} \right| \leq K_1$, $\forall r \in [0,1]$, for all $n \in \mathbb{N}$.

Proof: Now, for $n = 1$ and using definition of S_n , there exists a real number $k_1 > 0$, i.e.,

$$\left| \frac{S_1(r) + S_0(r)}{2r^2} \right| = \left| \frac{w_1(r) + 2w_0(r)}{2r^2} \right| \leq k_1, \forall r \in [0,1]. \quad (53)$$

For $n = 2$, we have

$$\left| \frac{S_2(r) + S_1(r)}{2r^2} \right| = \left| \frac{2w_1(r) + 2w_0(r) + w_2(r)}{2r^2} \right| \leq \left| \frac{w_1(r) + 2w_0(r)}{2r^2} \right| + \left| \frac{w_1(r) + w_2(r)}{2r^2} \right|, \quad (54)$$

$$\leq k_1 + \left| \frac{w_1(r) + w_2(r)}{2r^2} \right|, \quad \forall r \in [0,1]. \quad (55)$$

By Lemma 4.1.1, we have $\left| \frac{w_1(r) + w_2(r)}{2r^2} \right|$ is bounded on $[0,1]$. Take $k_2 = k_1 + \sup_{r \in [0,1]} \left| \frac{w_1(r) + w_2(r)}{2r^2} \right|$. For $n = 3$, we have

$$\left| \frac{S_3(r) + S_2(r)}{2r^2} \right| = \left| \frac{2w_1(r) + 2w_0(r) + 2w_2(r) + w_3(r)}{2r^2} \right|, \quad (56)$$

$$\leq \left| \frac{2w_1(r) + 2w_0(r) + w_2(r)}{2r^2} \right| + \left| \frac{w_2(r) + w_3(r)}{2r^2} \right|, \quad (57)$$

$$\leq k_2 + \left| \frac{w_2(r) + w_3(r)}{2r^2} \right|, \forall r \in [0,1]. \quad (58)$$

Again, we take $k_3 = k_2 + \sup_{r \in [0,1]} \left| \frac{w_2(r) + w_3(r)}{2r^2} \right|$. In general, we have

$$\left| \frac{S_m(r) + S_{m-1}(r)}{2r^2} \right| \leq \left| \frac{2w_1(r) + 2w_0(r) + \dots + w_{m-1}(r)}{2r^2} \right| + \left| \frac{w_{m-1}(r) + w_m(r)}{2r^2} \right|, \quad (59)$$

$$\leq k_{m-1} + \left| \frac{w_{m-1}(r) + w_m(r)}{2r^2} \right|, \forall r \in [0,1]. \quad (60)$$

In similar argument, we have $k_m = k_{m-1} + \sup_{r \in [0,1]} \left| \frac{w_{m-1}(r) + w_m(r)}{2r^2} \right|$. For $n = m + 1$, we have

$$\left| \frac{S_{m+1}(r) + S_m(r)}{2r^2} \right| \leq \left| \frac{2w_1(r) + 2w_0(r) + \dots + w_m(r)}{2r^2} \right| + \left| \frac{w_m(r) + w_{m+1}(r)}{2r^2} \right|, \quad (61)$$

$$\leq k_m + \left| \frac{w_m(r) + w_{m+1}(r)}{2r^2} \right|, \forall r \in [0,1], \quad (62)$$

and we have, $k_{m+1} = k_m + \sup_{r \in [0,1]} \left| \frac{w_m(r) + w_{m+1}(r)}{2r^2} \right|$. Finally, we get,

$$\left| \frac{S_n(r) + S_{n-1}(r)}{2r^2} \right| \leq k_n, \quad \forall r \in [0,1], \quad \text{where } k_n = k_{n-1} + \sup_{r \in [0,1]} \left| \frac{w_{n-1}(r) + w_n(r)}{2r^2} \right|, \quad \forall n \in \mathbb{N}. \quad (63)$$

Hence, there exist a natural number m , such that

$$|k_n - k_{n-1}| = \sup_{r \in [0,1]} \left| \frac{w_{n-1}(r) + w_n(r)}{2r^2} \right|, \quad \forall n \geq m. \quad (64)$$

We need to show that the sequence $\{k_n\}$ defined by (64) is eventually constant sequence ([42], p-35). Now

$$w_0(r) = \frac{ar^2}{2} + \frac{\lambda r^4}{16}. \quad (65)$$

Therefore, from the definition of big oh, we can write $w_0(r) = O(r^2)$. Now, we have

$$|A_0(w_0)| = \left| \frac{1}{2} \frac{w_0(r)w_0(r)}{r^3} \right| \leq p_0 r, \quad p_0 \text{ is a fixed constant.} \quad (66)$$

Again, by using the properties of big oh, we have $A_0(w_0) = O(r)$. Now,

$$|w_1(r)| \leq \int_0^r x \int_0^x |A_0(w_0)| dt dx \leq \frac{p_0 r^4}{8}. \quad (67)$$

Therefore, $w_1(r) = O(r^4)$. Now,

$$|A_1(w_0, w_1)| = \left| \frac{w_0(r)w_1(r)}{r^3} \right| \leq p_1 r^3, \quad p_1 \text{ is a fixed constant.} \quad (68)$$

So, $A_1(w_0, w_1) = O(r^3)$. Let our assumptions be true up to $n = m$. Therefore, we have

$$w_0(r) = O(r^2), w_1(r) = O(r^4), \dots, w_m(r) = O(r^{2m+2}), \quad (69)$$

$$A_0(w_0) = O(r), A_1(w_0, w_1) = O(r^3), \dots, A_m(w_0, w_1, \dots, w_m) = O(r^{2m+1}). \quad (70)$$

Now,

$$|w_{m+1}(r)| \leq \int_0^r x \int_0^x |A_m(w_0, w_1, \dots, w_m)| dt dx \leq \int_0^r x \int_0^x p_m t^{2m+1} dt dx, \quad p_m \text{ is a fixed constant.} \quad (71)$$

Therefore $w_{m+1} = O(r^{2m+4})$. Similarly, from (52), we get $A_{m+1}(w_0, w_1, \dots, w_{m+1}) = O(r^{2m+3})$. Hence by mathematical induction, we get

$$w_0(r) = O(r^2), w_1(r) = O(r^4), \dots, w_n(r) = O(r^{2n+2}), \quad (72)$$

$$A_0(w_0) = O(r), A_1(w_0, w_1) = O(r^3), \dots, A_n(w_0, w_1, \dots, w_n) = O(r^{2n+1}), \quad (73)$$

for all $n \in \mathbb{N}$. Now

$$\left| \frac{w_{n+1}(r) + w_n(r)}{2r^2} \right| \leq \int_0^r x \int_0^x \frac{|A_n(w_0, w_1, \dots, w_n) + A_{n-1}(w_0, w_1, \dots, w_{n-1})|}{2r^2} dt dx, \quad (74)$$

$$\leq \frac{1}{r^2} \int_0^r \left(\frac{p_n}{2 \cdot (2n+2)} x^{2n+3} + \frac{p_{n-1}}{2 \cdot 2n} x^{2n+1} \right) dx, \quad (75)$$

$$\leq \frac{p_n}{2 \cdot (2n+2) \cdot (2n+4)} r^{2n+2} + \frac{p_{n-1}}{2 \cdot 2n \cdot (2n+2)} r^{2n}. \quad (76)$$

Hence, we have

$$\sup_{r \in [0,1]} \left| \frac{w_{n+1}(r) + w_n(r)}{2r^2} \right| \leq \frac{p_n}{2 \cdot (2n+2) \cdot (2n+4)} + \frac{p_{n-1}}{2 \cdot 2n \cdot (2n+2)}. \quad (77)$$

$$\text{Taking limit,} \quad \lim_{n \rightarrow \infty} \sup_{r \in [0,1]} \left| \frac{w_{n+1}(r) + w_n(r)}{2r^2} \right| \leq \lim_{n \rightarrow \infty} \left(\frac{p_n}{2 \cdot (2n+2) \cdot (2n+4)} + \frac{p_{n-1}}{2 \cdot 2n \cdot (2n+2)} \right). \quad (78)$$

Since $p_{n+1} \leq p_n$ for all $n \geq k$, therefore we have sequence $\{p_n\}$ is bounded. Hence by Squeeze theorem ([43], theorem 3.2.7, p-64), we have

$$\lim_{n \rightarrow \infty} \sup_{r \in [0,1]} \left| \frac{w_{n+1}(r) + w_n(r)}{2r^2} \right| = 0. \quad (79)$$

Therefore, from (64) and (79), there exists a natural number m , such that

$$k_n = k_{n+1}, \quad \forall n \geq m, \quad (80)$$

where each k_i , $i = 0, 1, \dots$ are fixed real constants. Hence, from (79), we conclude that the sequence $\{k_n\}$ is an eventually constant sequence. So, it is bounded. This completes the proof.

Lemma 4.1.3. Let $\{S_n\}$ be the sequence of partial sums of the series defined by (48). Then, we can write the sequence $\{S_n\}$ in the following form

$$S_n = C_1 + \int_0^r x \int_0^x N(S_{n-1}) dt dx, \quad n \geq 1, \quad (81)$$

where $C_1 = \frac{ar^2}{2} + \frac{\lambda r^4}{16}$ and N is the nonlinear operator defined by (49).

Proof. Now, we have

$$\begin{aligned} S_n &= w_0(r) + w_1(r) + \dots + w_n(r), \\ &= \frac{ar^2}{2} + \frac{\lambda r^4}{16} + \int_0^r x \int_0^x A_0(w_0) dt dx + \int_0^r x \int_0^x A_1(w_0, w_1) dt dx + \dots \\ &\quad + \int_0^r x \int_0^x A_{n-1}(w_0, w_1, \dots, w_{n-1}) dt dx, \\ &= C_1 + \int_0^r x \int_0^x \sum_{i=0}^{n-1} A_i dt dx, \\ &= C_1 + \int_0^r x \int_0^x N(S_{n-1}) dt dx, \end{aligned}$$

Theorem 4.1.1. Let $w_0 = \frac{ar^2}{2} + \frac{\lambda r^4}{16}$, $a \in \mathbb{R}$. Let $\{S_n\}$ be the sequence defined by (81). Then the series $S_0 + \sum_{i=0}^{\infty} (S_{i+1} - S_i)$ converges uniformly to the exact solutions of (17).

Proof. Now, from equation (81), for $n = 1$, we have

$$|S_1 - S_0| \leq \int_0^r \left| x \int_0^x N(S_0) dt \right| dx \leq \int_0^r \left| x \int_0^x \frac{S_0^2(t)}{2t^3} dt \right| dx. \quad (82)$$

We take $K_2 = \sup_{x \in [0,1]} \left| x \int_0^x \frac{S_0^2(t)}{2t^3} dt \right| = \frac{192a^2 + 24a\lambda + \lambda^2}{3072}$. Let $K = \max\{K_1, K_2\}$, where K_1 is defined by Lemma 4.1.2. Therefore, we have $|S_1 - S_0| \leq K_2 r \leq kr$. Now,

$$\begin{aligned} |S_2 - S_1| &\leq \int_0^r x \int_0^x |N(S_1) - N(S_0)| dt dx, \text{ by Lemma 4.1.3,} \\ &= \int_0^r x \int_0^x \left| \frac{S_1^2(t)}{2t^3} - \frac{S_0^2(t)}{2t^3} \right| dt dx, \\ &\leq \int_0^r x \int_0^x \frac{|S_1(t) - S_0(t)|}{t} \left| \frac{S_1(t) + S_0(t)}{2t^2} \right| dt dx, \\ &\leq \int_0^r x \int_0^x K \times K dt dx. \text{ by Lemma 4.1.2,} \\ &= \frac{K^2 r^3}{1^2 \cdot 3}. \end{aligned}$$

and,

$$\begin{aligned} |S_3 - S_2| &\leq \int_0^r x \int_0^x |N(S_2) - N(S_1)| dt dx, \\ &= \int_0^r x \int_0^x \left| \frac{S_2^2(t)}{2t^3} - \frac{S_1^2(t)}{2t^3} \right| dt dx, \\ &\leq \int_0^r x \int_0^x \frac{|S_2(t) - S_1(t)|}{t} \left| \frac{S_2(t) + S_1(t)}{2t^2} \right| dt dx, \\ &\leq \int_0^r x \int_0^x \frac{K^2 t^2}{1^2 \cdot 3} \times K dt dx = \frac{K^3 r^5}{1^2 \cdot 3^2 \cdot 5}. \end{aligned}$$

By similar analysis,

$$|S_4 - S_3| \leq \frac{K^4 r^7}{1^2 \cdot 3^2 \cdot 5^2 \cdot 7}. \quad (83)$$

In general, we have

$$|S_n - S_{n-1}| \leq \frac{K^n r^{2n-1}}{1^2 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot \dots \cdot (2n-3)^2 (2n-1)}. \quad (84)$$

Now,

$$\begin{aligned} |S_{n+1} - S_n| &\leq \int_0^r x \int_0^x |N(S_n) - N(S_{n-1})| dt dx, \\ &= \int_0^r x \int_0^x \left| \frac{S_n^2(t)}{2t^3} - \frac{S_{n-1}^2(t)}{2t^3} \right| dt dx, \\ &\leq \int_0^r x \int_0^x \frac{|S_n(t) - S_{n-1}(t)|}{t} \left| \frac{S_n(t) + S_{n-1}(t)}{2t^2} \right| dt dx, \\ &\leq \int_0^r x \int_0^x \frac{K^n t^{2n-2}}{1^2 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot \dots \cdot (2n-3)^2 (2n-1)} \times K dt dx, \\ &= \frac{K^{n+1} r^{2n+1}}{1^2 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot \dots \cdot (2n-3)^2 (2n-1)^2 (2n+1)}. \end{aligned}$$

Therefore, by using D'Alembert's test, we can show that the series

$$S_0 + \sum_{n=1}^{\infty} \frac{K^n r^{2n-1}}{1^2 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot \dots \cdot (2n-3)^2 (2n-1)} \quad (85)$$

is convergent. Hence, by Weierstrass's M test, we have uniformly convergent series (48). Therefore, from Lemma 4.1.1 and Lemma 4.1.3, we can conclude that the solutions converge to the exact solutions of equation (17).

4.2 Convergence analysis of HPM-VIM

In a similar manner, we can approximate $N(\sum_{i=0}^n v_i(r))$ by each sequence $S_n = \sum_{i=0}^n v_i(r)$ ([40, 41]) in the following form

$$N\left(\sum_{i=0}^n v_i(r)\right) = -\frac{1}{2} \left(\sum_{i=0}^n v_i(r)\right)^2 = \sum_{i=0}^n H_i(v_0, v_1, \dots, v_i). \quad (86)$$

Therefore, we have

$$N(S_n) = -\frac{1}{2} (S_n)^2 = \sum_{i=0}^n H_i(v_0, v_1, \dots, v_i). \quad (87)$$

Lemma 4.2.1. Let $v_0 = ar^2$, $a \in \mathbb{R}$ and $v_n \in C^2[0,1]$, $\forall n \in \mathbb{N}$. Then $v_n(0) = 0$, $v_n'(0) = 0$, $v_n''(r)$, $v_n'''(r)$, $\frac{H_n(v_0, v_1, \dots, v_n)}{r^3}$ are bounded on $[0,1]$ and $v_n \in C^2[0,1]$ for all $n \in \mathbb{N}$.

Proof. The proof is the same as in Lemma 4.1.1.

Lemma 4.2.2. Let $\{S_n\}$ be the sequence of partial sums of the series defined by (48). Let $v_n \in C^2[0,1]$, $\forall n \in \mathbb{N}$. Let $q_i = \sup_{r \in [0,1]} |H_i|$, $i = 0, 1, \dots$. If there exists a natural number m such that $q_{i+1} \leq q_i$ for all $i \geq m$, then there exists a real number $K_3 > 0$, such that $\left| \frac{S_n + S_{n-1}}{r^2} \right| \leq K_3$ on $[0,1]$, for all $n \in \mathbb{N}$.

Proof. The proof is similar as in Lemma 4.1.2.



Lemma 4.2.3. Let $\{S_n\}$ be the sequence of partial sums of the series defined by (48). Then the sequence $\{S_n\}$ can be written as

$$S_n = C + \int_0^r \left(\frac{1}{2t} - \frac{r^2}{2t^3} \right) N(S_{n-1}) dt, \quad n \geq 1, \quad (88)$$

where $C = ar^2 + \frac{\lambda r^4}{16}$ and N is the nonlinear operator defined by (86).

Proof. Again, from the definition of S_n and with the help of the iterations (45-47), we have

$$\begin{aligned} S_n &= v_0(r) + v_1(r) + \dots + v_n(r), \\ &= ar^2 + \frac{\lambda r^4}{16} + \int_0^r \left(\frac{1}{2t} - \frac{r^2}{2t^3} \right) H_0 dt + \int_0^r \left(\frac{1}{2t} - \frac{r^2}{2t^3} \right) H_1 dt + \dots + \int_0^r \left(\frac{1}{2t} - \frac{r^2}{2t^3} \right) H_{n-1} dt, \\ &= C + \int_0^r \left(\frac{1}{2t} - \frac{r^2}{2t^3} \right) \sum_{i=0}^{n-1} H_i dt, \\ &= C + \int_0^r \left(\frac{1}{2t} - \frac{r^2}{2t^3} \right) N(S_{n-1}) dt. \end{aligned}$$

Theorem 4.2.1. Let $v_0 = ar^2$, $a \in \mathbb{R}$. Let $\{S_n\}$ be the sequence of partial sums defined by the recurrence relation (88). Then the series $S_0 + \sum_{n=1}^{\infty} (S_n - S_{n-1})$ converges uniformly to the exact solutions of (17).

Proof. From (88), for $n = 1$, we have

$$\begin{aligned} |S_1 - S_0| &= \left| C + \int_0^r \left(\frac{1}{2t} - \frac{r^2}{2t^3} \right) N(S_0) dt - S_0 \right|, \\ &\leq \left| \int_0^r \left(\frac{t^2 - r^2}{2} \right) \left(\frac{H_0}{t^3} - \frac{\lambda t}{2} \right) dt \right|. \end{aligned}$$

Now, we take $K_4 = \sup_{0 \leq t \leq r \leq 1} \left| \left(\frac{t^2 - r^2}{2} \right) \left(\frac{H_0}{t^3} - \frac{\lambda t}{2} \right) \right|$. Therefore

$$\begin{aligned} |S_2 - S_1| &\leq \int_0^r \left| \left(\frac{1}{2t} - \frac{r^2}{2t^3} \right) \right| |N(S_1) - N(S_0)| dt, \text{ by Lemma 4.2.3,} \\ &= \int_0^r \left| \left(\frac{1}{4t} - \frac{r^2}{4t^3} \right) \right| |(S_1)^2 - (S_0)^2| dt, \\ &\leq \int_0^r r \left| \left(\frac{t^2}{4r} - \frac{r}{4} \right) \right| \frac{|S_1 - S_0|}{t} \frac{|S_1 + S_0|}{t^2} dt, \end{aligned}$$

Since $0 \leq t \leq r \leq 1$, we have $K_5 = \sup_{0 \leq t \leq r \leq 1} \left| \left(\frac{t^2}{4r} - \frac{r}{4} \right) \right| = \frac{1}{2}$. Let $K = \max\{K_3, K_4, \frac{1}{2}\}$, where K_3 is defined by Lemma 4.2.2.

Therefore, we have $|S_1 - S_0| \leq K_4 r \leq Kr$ and $|S_2 - S_1| \leq \int_0^r rK \times K \times K dt \leq \frac{K^3 r^2}{1}$. Again, we have

$$\begin{aligned} |S_3 - S_2| &\leq \int_0^r \left| \left(\frac{1}{2t} - \frac{r^2}{2t^3} \right) \right| |N(S_2) - N(S_1)| dt, \text{ by Lemma 4.2.3,} \\ &= \int_0^r \left| \left(\frac{1}{4t} - \frac{r^2}{4t^3} \right) \right| |(S_2)^2 - (S_1)^2| dt, \\ &\leq \int_0^r r \left| \left(\frac{t^2}{4r} - \frac{r}{4} \right) \right| \frac{|S_2 - S_1|}{t} \frac{|S_2 + S_1|}{t^2} dt, \\ &\leq \int_0^r rK \times \frac{K^3 t}{1} \times K dt = \frac{K^5 r^3}{1.2}. \end{aligned}$$

Similarly,

$$|S_4 - S_3| \leq \frac{K^7 r^4}{1.2.3}, \quad (89)$$

$$|S_5 - S_4| \leq \frac{K^9 r^5}{1.2.3.4}. \quad (90)$$

In general, we have

$$|S_n - S_{n-1}| \leq \frac{K^{2n-1} r^n}{(n-1)!}. \quad (91)$$

Now, we have

$$\begin{aligned} |S_{n+1} - S_n| &\leq \int_0^r \left| \left(\frac{1}{2t} - \frac{r^2}{2t^3} \right) \right| |N(S_n) - N(S_{n-1})| dt, \text{ by Lemma 4.2.3,} \\ &= \int_0^r \left| \left(\frac{1}{4t} - \frac{r^2}{4t^3} \right) \right| |(S_n)^2 - (S_{n-1})^2| dt, \\ &\leq \int_0^r r \left| \left(\frac{t^2}{4r} - \frac{r}{4} \right) \right| \frac{|S_n - S_{n-1}|}{t} \frac{|S_n + S_{n-1}|}{t^2} dt, \\ &\leq \int_0^r r K \times \frac{K^{2n-1} t^{n-1}}{(n-1)!} \times K dt, \\ &= \frac{K^{2n+1} r^{n+1}}{n!}, \end{aligned}$$

and so on. Finally, we get the following series

$$S_0 + \sum_{n=1}^{\infty} \frac{K^{2n-1} r^n}{(n-1)!}. \quad (92)$$

Since, the series (92) is convergent for all $r \in [0,1]$, then by Weierstrass's M test we have the series (48) is uniformly convergent on $[0,1]$. Hence, from Lemma 4.2.1 and Lemma 4.2.3, we can conclude that the approximate solutions converge to the exact solutions of equation (17). Hence the proof is complete.

5. Approximate Solutions

In this section, we will discuss the solutions of the differential equation (12) corresponding to Dirichlet boundary condition and homogeneous Navier boundary condition. We apply both iterative methods to equation (17) and then integrate it to get the solution of (12). The problem at hand is not immediate, since equation (17) is nonlinear, present a non-self-adjoint operator, and a singularity at the origin. Here, we provide some numerical observations of the approximate solutions of (12) which is described in [2].

Case (a): $\lambda \geq 0$,

It is observed that, for $\lambda = 0$ there are two solutions: the trivial one but also a nontrivial positive solution. For $0 < \lambda < \lambda_{critical}$, there are two non-trivial solutions. Both solutions are always positive and ordered. $\lambda_{critical}$ is the critical value of λ . For Dirichlet boundary condition $\lambda_{critical}$ is numerically estimated to be 169 and for Navier boundary condition of type two $\lambda_{critical}$ is estimated to be 11.34. It is observed that both solutions are moving to each other for increasing the value of λ . For Navier boundary condition of type one, no estimate of λ is given. For $\lambda > \lambda_{critical}$ no longer numerical solutions exist.

Case (b): $\lambda < 0$, No conclusion for negative λ is given.

5.1 Example 1

Let us consider the equation (17) with homogeneous boundary condition



$$w'(0) = 0, \quad w'(1) = 0. \quad (93)$$

Now, by means of equations (31-33), we find

$$w_1(r) = \frac{a^2 r^4}{64} + \frac{a \lambda r^6}{768} + \frac{\lambda^2 r^8}{24576}, \quad (94)$$

$$w_2(r) = \frac{a^3 r^6}{3072} + \frac{5a^2 \lambda r^8}{147456} + \frac{a \lambda^2 r^{10}}{786432} + \frac{\lambda^3 r^{12}}{47185920}, \quad (95)$$

$$w_3(r) = \frac{7a^4 r^8}{1179648} + \frac{17a^3 \lambda r^{10}}{23592960} + \frac{a^2 \lambda^2 r^{12}}{28311552} + \frac{a \lambda^3 r^{14}}{1174405120} + \frac{13 \lambda^4 r^{16}}{1352914698240}, \quad (96)$$

;

and so on. Again, from equations (45-47), we have

$$v_1(r) = \frac{a^2 r^4}{16} + \frac{\lambda r^4}{16}, \quad (97)$$

$$v_2(r) = \frac{a^3 r^6}{384} + \frac{a \lambda r^6}{384}, \quad (98)$$

$$v_3(r) = \frac{7a^4 r^8}{73728} + \frac{5a^2 \lambda r^8}{36864} + \frac{\lambda^2 r^8}{24576}, \quad (99)$$

;

and so on. The approximate solution of equation (17) can be computed as $w(r) = \sum_{i=0}^{N_1} w_i$ and $w(r) = \sum_{i=0}^{N_2} v_i$, where N_1 and N_2 are chosen to get better accuracy. Using the boundary condition $w'(1) = 0$, we calculate the values of a that correspond to each λ . Finally, employing the transformation $w = r\phi'$ and the boundary condition $\phi(1) = 0$, we can build the approximate solution $\phi(r)$ of equation (12). We divide our results in two cases depending on the sign of λ .

Case (c): $\lambda \geq 0$. Our observations are similar as in case (a). The critical value of $\lambda_{critical}$ is 31.94. In Table 1, we tabulate maximum absolute residual errors. The graphs of $\phi(r)$ versus r for varying λ are plotted in the next section [see: Figure 7.1].

Case (d): $\lambda < 0$. For any negative value λ , we always get two non-trivial numerical solutions. One of them is positive and the other is negative. No negative critical λ has been numerically detected. We have listed maximum absolute residual error in Table 2. The plots of the graph that correspond to the negative λ will appear in the next section [see: Figure 7.2]. We observe that two nontrivial solutions moving away from each other as the value of λ decreases.

5.2 Example 2

Here, we consider the equation (17) corresponding to the boundary condition

$$w'(0) = 0, \quad w'(1) = w(1). \quad (100)$$

By similar analysis, we get the same iterations w_i 's defined by (94-96) and v_i 's (97-99). Similarly to subsection 5.1, herein two solutions are computed for different values of λ . Same remarks are made to cases (c) and cases (d). The critical value of λ , i.e., $\lambda_{critical}$, is numerically estimated to be 11.34. In Table 3 and Table 4, we have displayed maximum absolute residual errors. We have plotted the graphs of the approximate solutions $\phi(r)$ versus r in Figure 7.3 and Figure 7.4.

5.3 Example 3

In this subsection, we consider

$$w'(0) = 0, \quad w(1) = 0 \quad (101)$$

corresponding to equation (17).

In a similar manner, we find the same iterations w_i 's and v_i 's. Rest of the observation are the same as in subsection 5.1 except the critical value of the λ , which is $\lambda_{critical} = 168.76$. In Table 5 and Table 6, we have displayed maximum absolute residual errors of approximate solutions. We have also plotted the graphs of the approximate solutions $\phi(r)$ versus r that correspond to $\lambda \geq 0$ and $\lambda < 0$ in Figure 7.5 and Figure 7.6 respectively.

6. Tables

We define the maximum absolute residue error in the following

$$||E||_{\infty} = \max_{i=0,1,\dots,9} |R(r_i)|, \quad (102)$$

where $r_i = r_0 + ih$, $r_0 = 0$ and $h = 0.1$. The residue of the equation (12) is given by

$$R(r) = \frac{1}{r} \left(r \left[\frac{1}{r} (r\phi')' \right]' \right)' - \frac{1}{r} \phi' \phi'' - \lambda. \quad (103)$$

In the following, we have placed some numerical data of the approximate solutions corresponding to different types of boundary condition.

Table 1. Lower and upper solution maximum absolute residue errors corresponding to some λ for example 1:

$ E _{\infty}$			$ E _{\infty}$		
λ	ADM	HPM-VIM	λ	ADM	HPM-VIM
0	0	0	0	5.10×10^{-11}	1.05×10^{-12}
31.94	1.08×10^{-14}	2.48×10^{-14}	31.94	1.33×10^{-14}	1.95×10^{-14}

Table 2. Negative and positive solution maximum absolute residue errors corresponding to few λ for example 1:

$ E _{\infty}$			$ E _{\infty}$		
λ	ADM	HPM-VIM	λ	ADM	HPM-VIM
-30	4.82×10^{-15}	4.88×10^{-15}	-30	1.17×10^{-07}	2.04×10^{-10}
-100	1.33×10^{-14}	1.95×10^{-14}	-100	3.64×10^{-03}	7.24×10^{-04}

Table 3. Lower and upper solution maximum absolute residue errors corresponding to some λ for example 2:

$ E _{\infty}$			$ E _{\infty}$		
λ	ADM	HPM-VIM	λ	ADM	HPM-VIM
0	0	0	0	6.97×10^{-14}	4.56×10^{-14}
11.34	4.44×10^{-14}	1.29×10^{-14}	11.34	1.33×10^{-14}	1.78×10^{-15}

Table 4. Negative and positive solution maximum absolute residue errors corresponding to few λ for example 2:

$ E _{\infty}$			$ E _{\infty}$		
λ	ADM	HPM-VIM	λ	ADM	HPM-VIM
-30	1.78×10^{-15}	1.11×10^{-16}	-30	2.39×10^{-13}	2.40×10^{-13}
-110	3.43×10^{-14}	1.99×10^{-15}	-110	2.08×10^{-01}	1.34×10^{-06}

Table 5. Lower and upper solution maximum absolute residue errors corresponding to few λ for example 3:

$ E _{\infty}$			$ E _{\infty}$		
λ	ADM	HPM-VIM	λ	ADM	HPM-VIM
0	0	0	0	8.25×10^{-01}	1.41×10^{-01}
168.76	6.66×10^{-12}	2.80×10^{-10}	168.76	2.09×10^{-12}	1.25×10^{-09}

Table 6. Negative and positive solution maximum absolute residue errors corresponding to few λ for example 3:

$ E _{\infty}$			$ E _{\infty}$		
λ	ADM	HPM-VIM	λ	ADM	HPM-VIM
-200	1.07×10^{-14}	2.46×10^{-14}	-200	8.11×10^{-01}	2.89×10^{-01}
-700	2.56×10^{-13}	8.30×10^{-12}	-700	1.90×10^{-01}	5.76×10^{-01}

7. Figures

In this section, we present several examples of the graphs of the approximate solutions $\phi(r)$ corresponding to different boundary conditions. The agreement between the results given by different methods is evident. We observe that, we do not find a unique solution numerically for the critical value of λ .

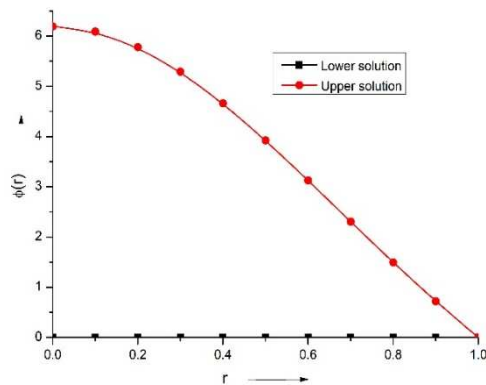
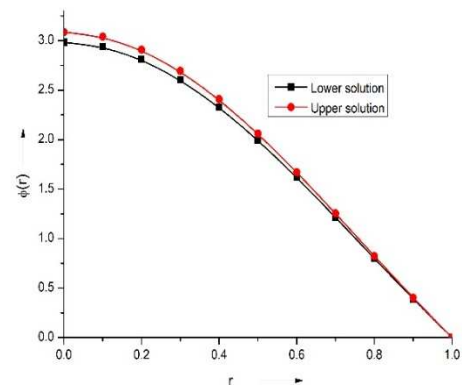
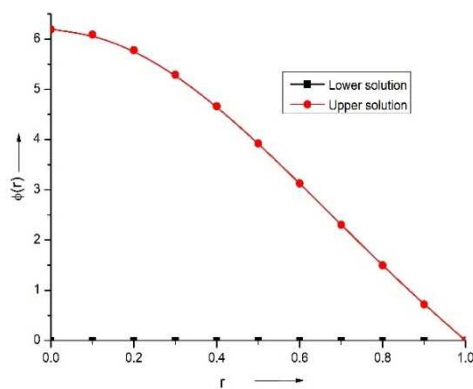
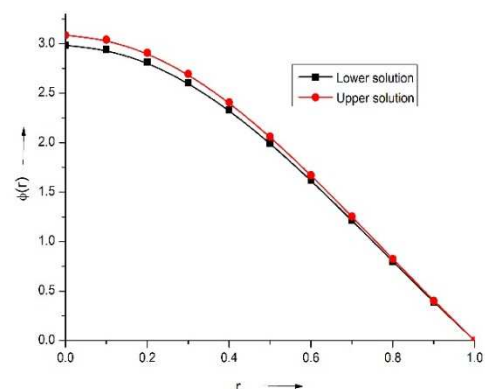
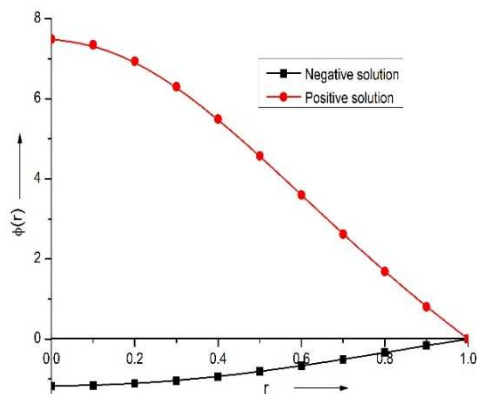
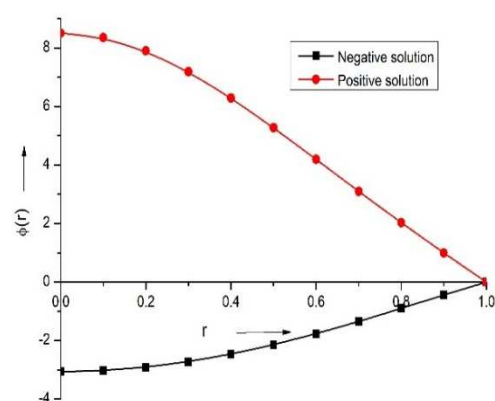
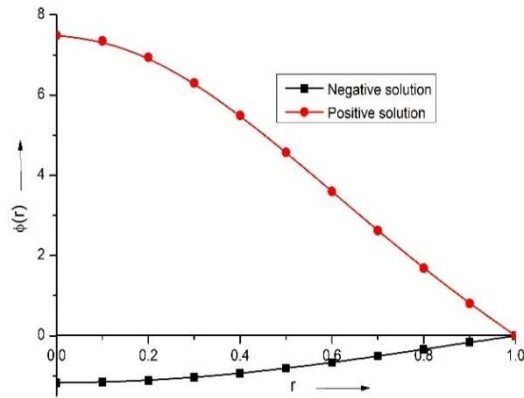
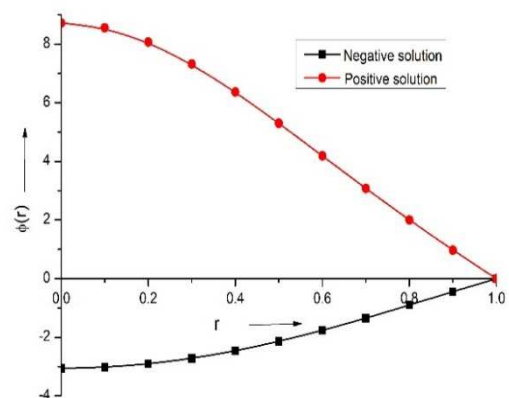
(a) $\lambda = 0$ (b) $\lambda = 31.94$ (c) $\lambda = 0$ (d) $\lambda = 31.94$

Figure 7.1. (ADM) approximate solutions corresponding to (a) $\lambda = 0$, (b) $\lambda = 31.94$, and (HPM-VIM) approximate solutions corresponding to (c) $\lambda = 0$, (d) $\lambda = 31.94$.

(a) $\lambda = -30$ (b) $\lambda = -100$

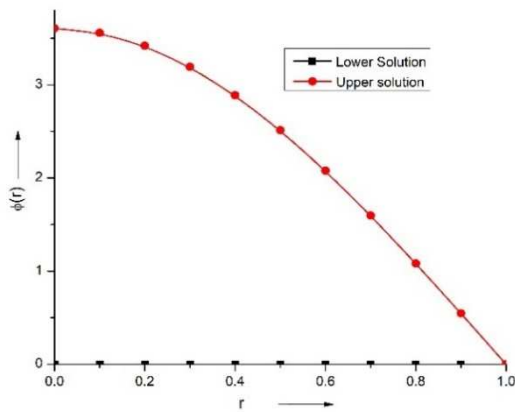


(c) $\lambda = -30$

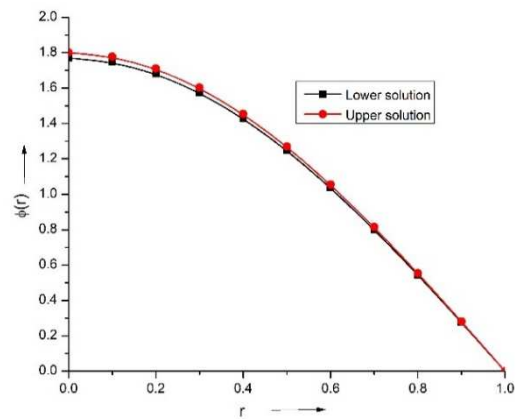


(d) $\lambda = -100$

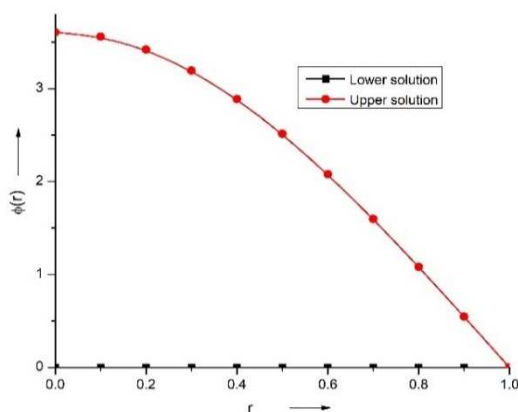
Figure 7.2. (ADM) approximate solutions corresponding to (a) $\lambda = -30$, (b) $\lambda = -100$, and (HPM-VIM) approximate solutions corresponding to (c) $\lambda = -30$, (d) $\lambda = -100$.



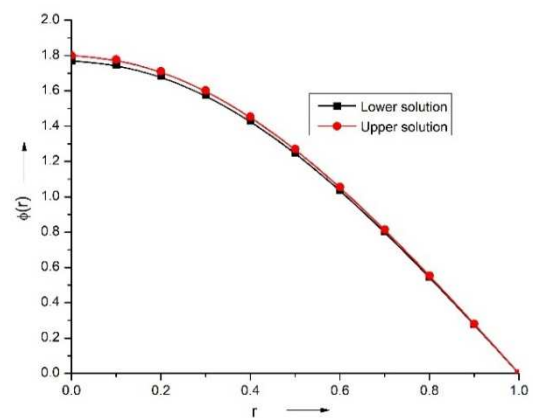
(a) $\lambda = 0$



(b) $\lambda = 11.34$



(c) $\lambda = 0$



(d) $\lambda = 11.34$

Figure 7.3. (ADM) approximate solutions corresponding to (a) $\lambda = 0$, (b) $\lambda = 11.34$, and (HPM-VIM) approximate solutions corresponding to (c) $\lambda = 0$, (d) $\lambda = 11.34$.

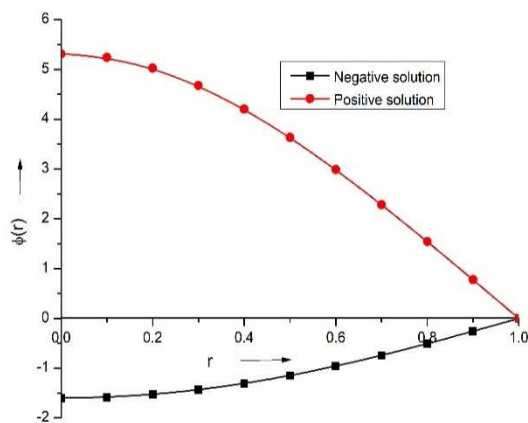
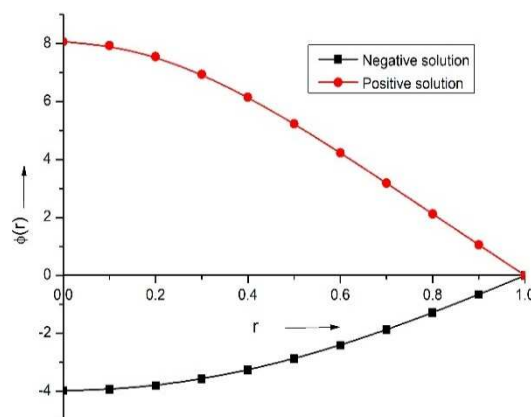
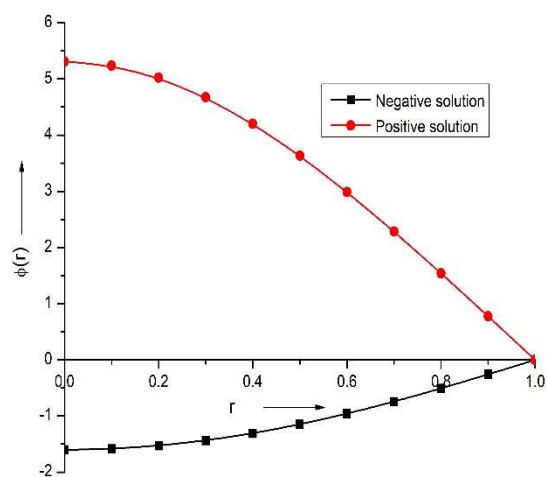
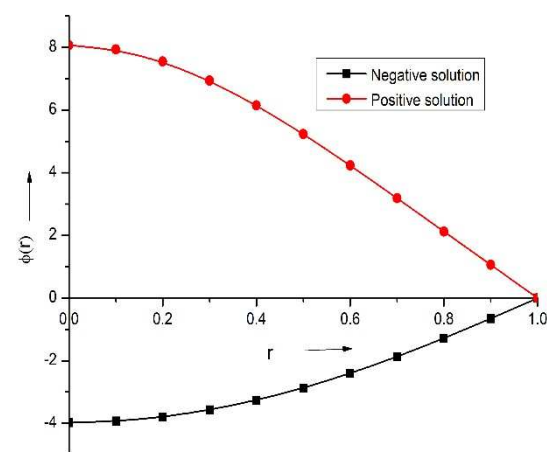
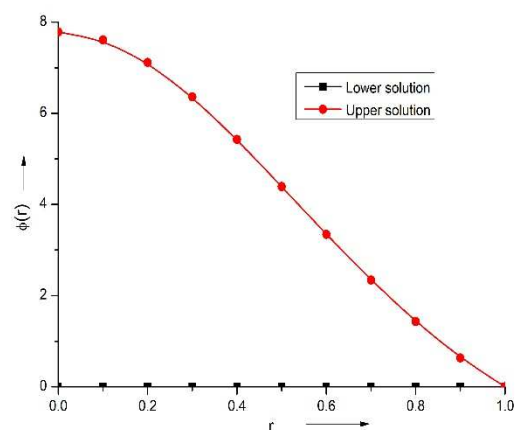
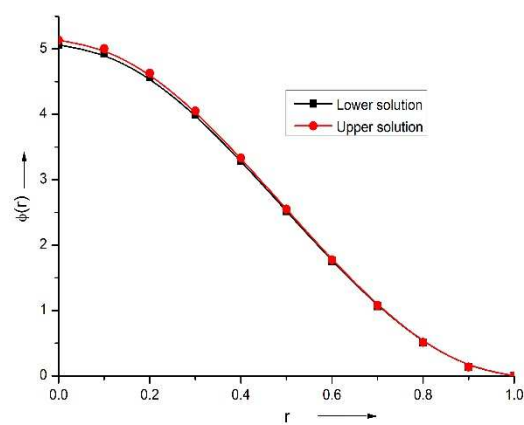
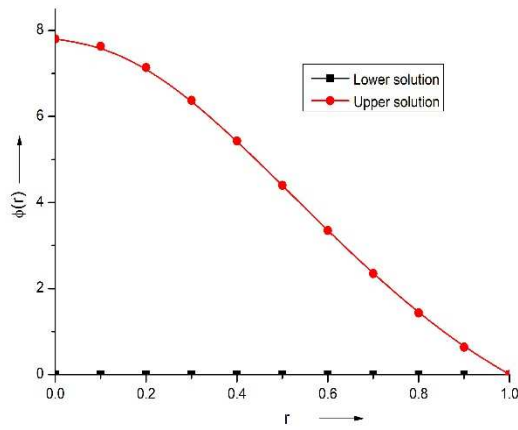
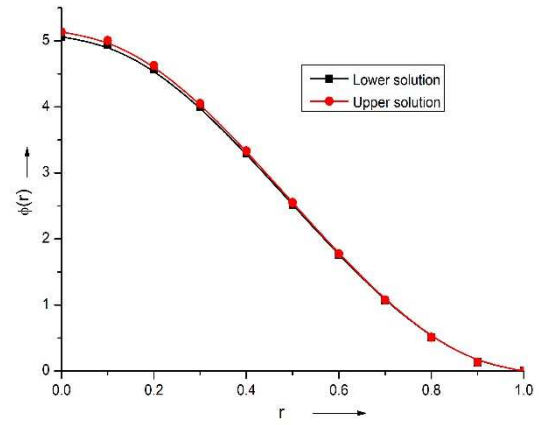
(a) $\lambda = -30$ (b) $\lambda = -110$ (c) $\lambda = -30$ (d) $\lambda = -110$

Figure 7.4. (ADM) approximate solutions corresponding to (a) $\lambda = -30$, (b) $\lambda = -110$, and (HPM-VIM) approximate solutions corresponding to (c) $\lambda = -30$, (d) $\lambda = -110$.

(a) $\lambda = 0$ (b) $\lambda = 168.76$

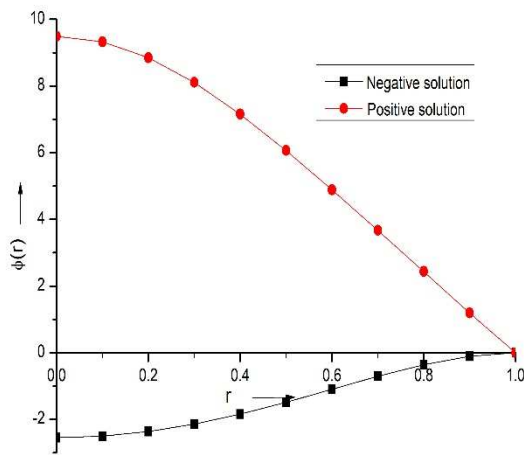


(c) $\lambda = 0$

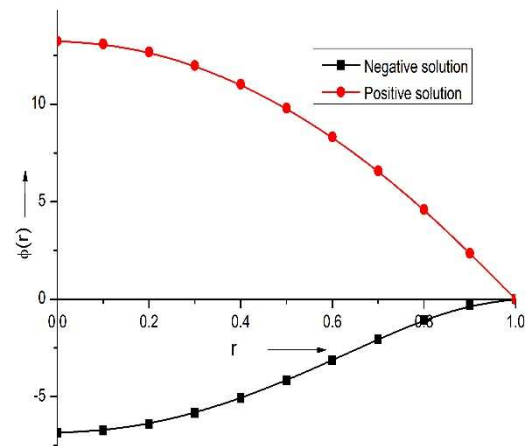


(d) $\lambda = 168.76$

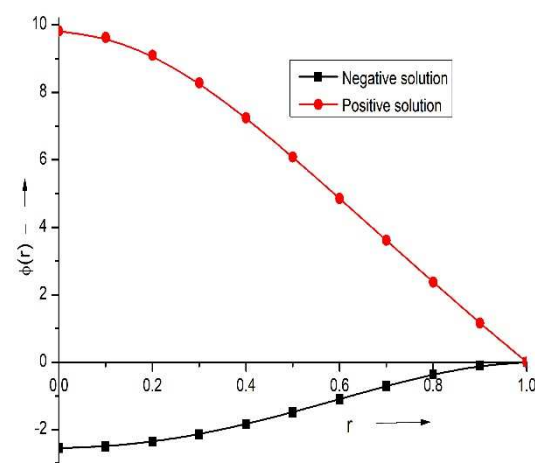
Figure 7.5. (ADM) approximate solutions corresponding to (a) $\lambda = 0$, (b) $\lambda = 168.76$, and (HPM-VIM) approximate solutions corresponding to (c) $\lambda = 0$, (d) $\lambda = 168.76$.



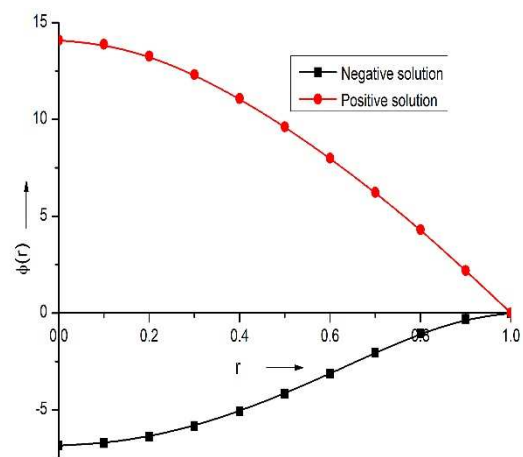
(a) $\lambda = -200$



(b) $\lambda = -700$



(c) $\lambda = -200$



(d) $\lambda = -700$

Figure 7.6. (ADM) approximate solutions corresponding to (a) $\lambda = -200$, (b) $\lambda = -700$, and (HPM-VIM) approximate solutions corresponding to (c) $\lambda = -200$, (d) $\lambda = -700$.

8. Conclusions

In this paper, we applied two iterative schemes to solve a nonlinear second order boundary value problem provided with a non-self-adjoint differential operator. We used these approximate solutions to compute the solutions of a fourth order boundary value problem which arises in the theory of epitaxial growth. The proposed techniques generates numerical solutions with high accuracy which can be observed from Table 1 to Table 6 and Figure 7.1 to Figure 7.6. For both Navier and Dirichlet boundary conditions, we obtained solutions, which are in agreement with the previous analytical and numerical results ([9, 2]). Altogether, these results offered a set of numerical techniques that may be used to approach the present as well as related problems which does not have a unique solution. Furthermore, our numerical techniques will be an effective tool to engineers as well as mathematicians.

Author Contributions

Basic idea of the paper was conceived by A.K. Verma. B. Pandit has implemented these ideas. R.P. Agarwal has identified major issues with correctness of the paper and A.K. Verma and B. Pandit have rectified these in consultation with R.P. Agarwal. The manuscript was written through the contribution of all authors. All authors discussed the results, reviewed and approved the final version of the manuscript.

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Conflict of Interest

The authors declared no potential conflicts of interest with respect to the research, authorship and publication of this article.

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