J. Appl. Comput. Mech., 6(4) (2020) 848-861
DOI: 10.22055/JACM.2019.14803

ISSN: 2383-4536
jacm.scu.ac.ir

M&MoCS

Shahid Chamran University of Ahvaz

Journal of
Applied and Computational Mechanics

Research Paper

Numerical Solution of Caputo-Fabrizio Time Fractional Distributed Order Reaction-diffusion Equation via Quasi Wavelet based Numerical Method

Sachin Kumar[1], José Francisco Gómez-Aguilar[2]

[1] Department of Mathematical Sciences, Indian Institute of Technology (BHU), Varanasi, 221005, India, Email: sachinraghav522@gmail.com

[2] Departamento de Ingeniería Electrónica, CONACyT-Tecnológico Nacional de México/CENIDET, Interior Internado Palmira S/N Col. Palmira, C.P. 62490, Cuernavaca Morelos, México, Email: jgomez@cenidet.edu.mx

Received September 03 2019; Revised October 08 2019; Accepted for publication October 09 2019.
Corresponding author: J.F. Gómez-Aguilar (jgomez@cenidet.edu.mx)
© 2020 Published by Shahid Chamran University of Ahvaz
& International Research Center for Mathematics & Mechanics of Complex Systems (M&MoCS)

Abstract. In this paper, we derive a novel numerical method to find out the numerical solution of fractional partial differential equations (PDEs) involving Caputo-Fabrizio (C-F) fractional derivatives. We first find out the approximation formula of C-F derivative of function t^k. We approximate the C-F derivative in time with the help of the Legendre spectral method and approximation formula of t^k. The unknown function and their derivatives in spatial direction are approximated with the quasi wavelet-based numerical method. We apply this newly derived method to solve the nonlinear distributed order reaction-diffusion in which time-fractional derivative is of C-F type. The accuracy and validity of the proposed method is exhibited by giving a solution to some numerical examples. The obtained numerical results are compared with the analytical results and conclude that our proposed numerical method achieves accurate results. On the other hand, the method is easy to apply on higher-order fractional partial differential equations and variable-order fractional partial differential equations.

Keywords: Fractional Partial Differential Equations, Distributed order reaction-diffusion equation, Caputo-Fabrizio fractional derivative, Quasi wavelet method, Legendre polynomial method.

1. Introduction

In recent years fractional differential equations have received more attention from the researchers due to its exact description of the physical phenomenon. Many physical phenomenons have been described through fractional diffusion equation viz., transport in the porous medium, groundwater contamination problem through a porous medium, etc. Fractional calculus is an ancient topic of mathematics with history as ordinary or integer calculus [1]. It is developing progressively now. The theory of fractional calculus was developed by N.H. Abel and J. Liouville and the details can be found in [2]-[3]. Fractional calculus allows generalizing derivatives and integrals of integer order to real or variable order. It also can be considered as a branch of mathematical analysis that allows us to investigate with real differential operators and equations where types of integral are convolution or weakly singular. It has wide applications in control theory, stochastic process, and special functions. Fractional calculus was considered as an esoteric theory without applications but in the last few years, there has been a boost of research on its applications to economics, control system to finance [4]-[16].

Many different forms of fractional order differential operators were introduced as the Hadamard, Caputo, Grünwald-Letnikov, Riemann-Liouville, Riesz, and variable order operators. Due to its increasing applications, the researchers have paid their attention to find numerical and exact solutions of the fractional-order differential equations. As there are many difficulties to solve a fractional-order differential equation by the analytic method so there is a need for seeking numerical

Published online: October 14 2019

solutions. There are many numerical methods available in literature viz., eigenvector expansions, Adomain decomposition method [17], fractional differential transform method [18], homotopy perturbation method [19], predictor-corrector method [20] and generalized block pulse operational matrix method [21], etc. Some numerical methods based upon operational matrices of fractional order differentiation and integration with Legendre wavelets [22], Chebyshev wavelets [23], sine wavelets, Haar wavelets [24] have been developed to find the solutions of fractional order differential and integro-differential equations. The functions which are commonly used include Legendre polynomial [25], Laguerre polynomial [26], Chebyshev polynomial and semi-orthogonal polynomial as Genocchi polynomial [27].

The reaction-diffusion process has been investigated for a long time. In the process of reaction-diffusion, reacting molecules are used to move through space due to diffusion. This definition excludes other modes of transports as convection, drift that may arise due to the presence of externally imposed fields. When a reaction occurs within an element of space, molecules can be created or consumed. These events are added to the diffusion equation and lead to reaction-diffusion equation of the form

$$\frac{\partial c}{\partial t} = D\nabla^2 c + R(c,t), \tag{1}$$

where $R(c; t)$ denotes reaction term at time t. The extension of the reaction-diffusion equation in the fractional-order system can be found in the articles [28]. In nature, many of the beautiful systems in biology, physics, chemistry, and physiology can be described by reaction-diffusion equations. For example, the distribution and organization of vegetation-like bushes in arid ecosystems [29], the stripes and spots on fish [30], snakes [31] and the skin or fur of mammals [32] have been studied by the standing waves which are produced by reaction-diffusion equations.

In environmental fluid dynamics, groundwater contaminants through porous media is a hot topic like as seawater invasion in karstic, radionuclide through fractured rock, contaminant migrates at the waste landfill site. The study of these models is helpful to prevent and control contaminant diffusion and polluted water. We are aware of that diffusion of solute particles follow the Fickian diffusion law and Einstein relationship. But for particle diffusion in heterogeneous media does not follow these rules. Many mathematical and physical models have been derived to characterize the behavior of such non-Fickian diffusion in complex media. These models include a structural derivative diffusion model, Hausdorff derivative diffusion models, variable order models, and distributed order models.

The distributed order time-fractional diffusion equation is represented by

$$\int_0^1 p(\alpha)\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}d\alpha = D\frac{\partial^2 u(x,t)}{\partial x^2}, \tag{2}$$

where the weight function $p(\alpha)$ is a non-negative function satisfying the following inequality

$$\int_0^1 p(\alpha)d\alpha = \mu, \quad \mu > 0. \tag{3}$$

Here $p(\alpha)$ is the probability density of derivative order and $\mu = 1$.

Our article is outlined as follows. In section 2, we discussed Caputo, Riemann-Liouville and Caputo-Fabrizio fractional derivative. We have also discussed the quasi wavelet and quasi wavelet-based numerical method. In section 3, we derived the general formula of Caputo-Fabrizio derivative of the function t^k. Some properties of Legendre polynomial is also included in this section. In section 4, we developed the newly proposed method for solving the distributed order reaction-diffusion equation. In section 5, some numerical examples and results are presents. The last section includes the conclusion of all overwork.

2. Preliminaries

Here, some definitions and important properties of fractional calculus have been introduced. It is well known that the Riemann-Liouville definition has disadvantages when it comes to the modeling of real-world problems. But the definition of fractional differentiation given by Michele Caputo is more reliable for the application point of view. Nowadays new general types of fractional operators have been discovered. A brief description of the Caputo-Fabrizio derivative is discussed here.

2.1. Definition of Riemann-Liouville integration and derivative

Fractional integration of Riemann-Liouville type (RL) of a given order ϑ of a function $h(t)$ is given by [2, 3]

$$I^\vartheta h(t) = \frac{1}{\Gamma(\vartheta)}\int_0^t (t-\omega)^{\vartheta-1}h(\omega)d\omega, \quad t>0, \quad \vartheta \in R^+. \tag{4}$$

The fractional derivative of the Riemann-Liouville type of order $\vartheta>0$ can be defined as

$$D_t^{\vartheta} h(t) = \left(\frac{d}{dt}\right)^m (\mathrm{I}^{m-\vartheta}\mathrm{h})(\mathrm{t}), \quad (\vartheta > 0, \; m-1 < \vartheta < m). \tag{5}$$

2.2. Definition of Caputo derivative

The fractional derivative of the Caputo type of order ϑ>0 can be defined as [2, 3]

$${}_0^C D_t^{\vartheta} h(\mathrm{t}) = \begin{cases} \dfrac{d^{\ell}}{dt^{\ell}} h(t), & \vartheta = \ell \in N, \\ \dfrac{1}{\Gamma(\vartheta)} \displaystyle\int_0^t (t-\eta)^{\ell-\vartheta-1} h^{\ell}(\eta) d\eta, & \iota\text{-}1 < \vartheta < \ell, \;\; \vartheta \in R^{+}. \end{cases} \tag{6}$$

Here, ℓ is an integer, *t>0*. Basic properties of Caputo fractional derivative are:

$$D_c^{\vartheta} C = 0, \tag{7}$$

where *C* is a constant.

$${}_0^C D_t^{\vartheta} t^{\sigma} = \begin{cases} 0, \;\; \sigma \in N \cup 0, \;\text{and}\; \sigma < \lfloor \vartheta \rfloor, \\ \dfrac{\Gamma(1+\sigma)}{\Gamma(1-\vartheta+\sigma)} t^{-\vartheta+\sigma}, \;\; \sigma \in N \cup 0, \;\text{and}\; \sigma < \lfloor \vartheta \rfloor \;\text{or}\; \notin N \;\text{and}\; \sigma > \lfloor \vartheta \rfloor, \end{cases} \tag{8}$$

where $\lfloor \vartheta \rfloor$ is floor function. The operator D_c^{ϑ} is linear, since

$${}_0^C D_t^{\vartheta} (c_1 h(t) + (c_2 g(t)) = c_1 \, {}_0^C D_t^{\vartheta} h(t) + c_2 \, {}_0^C D_t^{\vartheta} g(t), \tag{9}$$

where c_1 and c_2 are constants. Caputo and Riemann-Liouville operators have a relation given by

$$(\mathrm{I}^{\vartheta} \; {}_0^C D_t^{\vartheta} g)(\mathrm{t}) = \mathrm{g}(\mathrm{t}) - \sum_{k=0}^{\ell-1} g^k(0^+) \frac{t^k}{k!}, \quad \ell - 1 < \vartheta \le \ell. \tag{10}$$

2.3. Definition of Caputo-Fabrizio derivative

Let *g(t)* be a function which belongs to Sobolev space *H[1](a,b), b>a* then Caputo-Fabrizio derivative (CF) of order *n<θ<n+1* is defined by [33]

$${}_0^{CF} D_t^{\vartheta} g(\mathrm{t}) = \frac{B(\vartheta)}{\lceil \vartheta \rceil - \vartheta} \int_0^t \exp\left[-\frac{\vartheta(x,t)}{\lceil \vartheta \rceil - \vartheta} (t-s) \right] \times \frac{\partial^{n+1}}{\partial t^{n+1}} g(s) ds, \quad n < \vartheta \le n+1, \tag{11}$$

where B(ϑ) is a normalization function. In all our calculations we consider B(ϑ) = 1.

2.4. Definition of Caputo-Fabrizio Integral

The CF integral of order *n<θ<n+1* associated with the function *g(t)* is defined as follows

$${}_0^{CF} I_t^J g(t) = \sum_{i=0}^{n} \frac{t^i}{i!} g^{(i)}(0) + \frac{(1\text{-}\eta)}{B(\eta)(n\text{-}1)!} \int_0^t (t\text{-}s)^{n\text{-}1} g(s) ds + \frac{\eta}{M(\eta) n!} \int_0^t (t\text{-}s)^n g(s) ds, \quad n < \vartheta \le n+1, \tag{12}$$

where η denotes the fractional part of the order ϑ when η=0 then CF integral is given by

$${}_0^{CF} I_t^J g(t) = \frac{(1\text{-}\eta)}{B(\eta)} g(\mathrm{t}) + \frac{\eta}{M(\eta)} \int_0^t g(s) ds, \quad n < \vartheta \le n+1. \tag{13}$$

2.5. Introduction to the numerical method based on quasi-wavelets

The Quasi-wavelet based algorithm has been emerging as a new local spectral collocation method for finding numerical solutions of fractional PDEs and integro-differential equations. Firstly, we define the discrete singular convolution which is a mathematical transformation having significant importance in engineering and science. A singular convolution is defined in the distribution theory as

$$\psi(\upsilon) = (G * h)(\upsilon) = \int_{-\infty}^{\infty} G(\upsilon - x) h(x) dx, \tag{14}$$

where G is recognized as a singular kernel and h(x) is considered as a test function. A family of wavelets can be constructed from a single function called mother wavelet ϕ using operations of translation and dilation

$$\varphi_{\beta,\gamma}(x)=\beta^{-1/2}\varphi\left(\frac{x-\gamma}{\beta}\right). \tag{15}$$

Here β is used in dilation and γ in translation. An arbitrary wavelet subspace is generated by the help of orthonormal wavelet bases, which can be constructed by the corresponding orthonormal scaling functions. In this work, we shall use Shannon's delta sequence kernel which has the following form

$$\delta_\alpha(\mathrm{t})=\frac{1}{\pi}\int_0^\pi \cos(\mathrm{ty})\,\mathrm{dy}=\frac{\sin(\alpha\,\mathrm{t})}{\pi t}, \tag{16}$$

where $\lim_{\alpha\to\alpha_0}\delta_\alpha(\mathrm{t})=\delta(\mathrm{t}).$ Dirac first discussed δ in his text on quantum mechanics, so we called δ as Dirac delta function. Walter and Blum [34], gave the numerical use of delta sequences as probability density estimators. Especially when α=π, $\delta_\pi(\mathrm{t})$ is known as Shannon's wavelet scaling function. For a given α>0, Shannon's delta sequence kernel generates a basis for the Paley-Wiener reproducing kernel Hilbert space B_α^2 [35] that is a subspace of $L^2(\mathrm{R})$. A function $g(y)\in B_\alpha^2$ can be uniquely reproduced by

$$g(y)=\int_{-\infty}^{\infty} g(y)\delta_\alpha(y-t)dt=\int_{-\infty}^{\infty} g(y)\frac{\sin(\alpha(y-t))}{\pi(y-t)}dt,\ \ \forall g\in B_\alpha^2. \tag{17}$$

However, another form of this sampling scaling function in the Paley-Wiener reproducing kernel is given by

$$\delta_{\alpha,k}=\delta_\alpha(x-x_k)=\frac{\sin((x-x_k)\alpha)}{(x-x_k)\pi}, \tag{18}$$

where x_k is considered as a set of sampling points centered around x. Considering Eqs. (11) and (12) every function $\forall g\in B_\alpha^2$ can be represented in the following discrete form

$$g(y)=\sum_{k=-\infty}^{\infty} g(y_k)\delta_\alpha(y-y_k). \tag{19}$$

The Shannon sampling theorem states that the uniformly spatial discrete samples for a given bandlimited signal in B_α^2 a shown that sampling at the Nyquist frequency α. Here $\alpha=\pi/\Delta$ and Δ known as grid size in the spatial direction. Now, we have

$$g(y)=\sum_{k=-\infty}^{\infty} f(y_k)\delta_\alpha(y-y_k)=\sum_{k=-\infty}^{\infty} g(y_k)\frac{\sin\left(\frac{\pi(y-y_k)}{\Delta}\right)}{\frac{\pi(y-y_k)}{\Delta}}. \tag{20}$$

Wan [36] proposed a method for the improvement of the localized asymptotic behavior of Dirichlet's delta sequence kernel. By introducing a regularizer $R_\sigma(y)$ we increase its regularity such as

$$\delta_\alpha(y)\to\delta_{\alpha,\sigma}=\delta_\alpha(y)R_\sigma(y), \tag{21}$$

where regularizer R_σ satisfies

$$\lim_{\sigma\to\infty} R_\sigma(y)=1,$$

and

$$\int_{-\infty}^{\infty}\lim_{\sigma\to\infty} R_\sigma(y)\delta_\alpha(y)dy=R_\sigma(0)=1.$$

However, there are many regularizers satisfying above two conditions but a commonly used regularizer is the Gaussian type

$$R_\sigma(y)=\exp\left(-\frac{y^2}{2\sigma^2}\right),\ \ \sigma>0, \tag{22}$$

where σ denotes the width parameter of the Gaussian envelope. The relation between Δ and σ is $\sigma = r\times\Delta$, where r is a parameter which will be chosen in computation. Using the Gaussian regularizer $R_\sigma(x)$, Gaussian regularized orthogonal sampling scaling function is defined as

$$\delta_{\Delta,\sigma}(\mathrm{x}) = \frac{\sin\left(\frac{\pi x}{\Delta}\right)}{\frac{\pi x}{\Delta}} \exp\left(-\frac{x^2}{2\sigma^2}\right). \tag{23}$$

Here

$$\lim_{\sigma\to\infty} \delta_{\Delta,\sigma}(\mathrm{x}) = \frac{\sin\left(\frac{\pi x}{\Delta}\right)}{\frac{\pi x}{\Delta}}.$$

Gaussian regularized sampling scaling function is a quasi scaling function as it does not follow the property of the exact orthonormal wavelet scaling function.

2.6. Description of the proposed method

An arbitrary function $g(y) \in B_\alpha^2$ can be written as by using quasi scaling function

$$g(y) = \sum_{k=-\infty}^{\infty} g(y_k)\delta_\alpha(y-y_k) = \sum_{k=-\infty}^{\infty} g(y_k)\delta_\alpha(y-y_k)R_\alpha(y-y_k). \tag{24}$$

We can clearly see that taking infinite sampling points is not possible for computation. Thus we have to restrict our computational domain to finite sampling points close to x perform numerical calculations. In practical numerical computation, we select the *(2W+1)* sampling points for this problem. Therefore Eq. (18) can be simplified as

$$g(y) = \sum_{k=-W}^{W} g(y_k)\delta_{\Delta,\sigma}(y-y_k), \tag{25}$$

for the approximation of n^{th} order derivatives of f(x)

$$g^n(y) = \sum_{k=-W}^{W} g(y_k)\delta_{\Delta,\sigma}^n(y-y_k), \quad n = 1, 2, \ldots, \tag{26}$$

where computational bandwidth is equal to *(2W+1)*, centered around x and superscript (n) denotes the nth-order derivative with respect to x. For the calculation purpose, we present the following detailed description of formulas of $\delta_{\Delta,\sigma}$, $\delta_{\Delta,\sigma}^1$ and $\delta_{\Delta,\sigma}^2$ [37]

$$\delta_{\Delta,\sigma}(\mathrm{x}) = \begin{cases} \dfrac{\exp\left(-\frac{x^2}{2\sigma^2}\right)\sin\left(\frac{\pi x}{\Delta}\right)}{\frac{\pi x}{\Delta}}, & x \neq 0, \\ 1, & x = 0, \end{cases} \tag{27}$$

$$\delta_{\Delta,\sigma}^1(\mathrm{x}) = \begin{cases} \left(-\dfrac{\sin\left(\frac{\pi x}{\Delta}\right)}{\frac{\pi x^2}{\Delta}} - \dfrac{\Delta\sin\left(\frac{\pi x}{\Delta}\right)}{\frac{\pi\sigma^2}{\Delta}} + \dfrac{\cos\left(\frac{\pi x}{\Delta}\right)}{x}\right)\exp\left(-\frac{x^2}{2\sigma^2}\right), & x \neq 0, \\ 0, & x = 0, \end{cases} \tag{28}$$

$$\delta_{\Delta,\sigma}^2(\mathrm{x}) = \begin{cases} \left[\left[\dfrac{2\Delta\sin\left(\frac{\pi x}{\Delta}\right)}{\pi x^3} - \dfrac{2\cos\left(\frac{\pi x}{\Delta}\right)}{x^2} + \dfrac{\Delta\,\mathrm{x}\sin\left(\frac{\pi x}{\Delta}\right)}{\pi\sigma^4}\right.\right. \\ \left. + \dfrac{\Delta\sin\left(\frac{\pi x}{\Delta}\right)}{\pi\sigma^2 x} - \dfrac{2\cos\left(\frac{\pi x}{\Delta}\right)}{\sigma^2} - \dfrac{\pi\sin\left(\frac{\pi x}{\Delta}\right)}{x\Delta}\right]\exp\left(-\frac{x^2}{2\sigma^2}\right), & x \neq 0, \\ 0, & x = 0. \end{cases} \tag{29}$$

3. Approximation of Caputo-Fabrizio Derivative

In the following theorem, we will find out an approximation formula of the Caputo-Fabrizio derivative of function $f(t)=t^k$.

Theorem 1: The CF derivative of order n<α<n+1 of function $f(t) = t^k$ with $k \geq \lceil \alpha \rceil$ is given by

$$ {}_0^{CF}D_t^{\alpha} t^k = \frac{B(\alpha)\Gamma(1+k)}{\lceil \alpha \rceil - \alpha} \left(\sum_{r=0}^{k-n-1} \frac{(-1)^r t^{k-n-1-r}}{\Gamma(k-n-r)\left(-\frac{\alpha}{\lceil \alpha \rceil - \alpha}\right)^{r+1}} + \frac{(-1)^{k-n}}{\left(-\frac{\alpha}{\lceil \alpha \rceil - \alpha}\right)^{k-n}} \exp\left(-\frac{\alpha}{\lceil \alpha \rceil - \alpha} t\right) \right). \tag{30} $$

Proof: By the definition of CF derivative $D^n t^k = 0,\ k = 0,1,\ldots,\lceil \alpha \rceil - 1$. Now for $k \geq \lceil \alpha \rceil$, we have

$$ {}_0^{CF}D_t^{\alpha} t^k = \frac{B(\alpha)}{\lceil \alpha \rceil - \alpha} \int_0^t D^{n+1} s^k \exp\left[-\frac{\alpha}{\lceil \alpha \rceil - \alpha}(t-s)\right] ds, $$

$$ = \frac{B(\alpha)}{\lceil \alpha \rceil - \alpha} \int_0^t \frac{\Gamma(k+1)}{\Gamma(k-n)} s^{k-n-1} \exp\left[-\frac{\alpha}{\lceil \alpha \rceil - \alpha}(t-s)\right] ds, $$

$$ = \frac{B(\alpha)}{\lceil \alpha \rceil - \alpha} \frac{\Gamma(k+1)}{\Gamma(k-n)} \exp\left[-\frac{\alpha}{\lceil \alpha \rceil - \alpha} t\right] \int_0^t s^{k-n-1} \exp\left[-\frac{\alpha}{\lceil \alpha \rceil - \alpha} s\right] ds, $$

$$ = \frac{B(\alpha)}{\lceil \alpha \rceil - \alpha} \times \frac{\Gamma(k+1)}{\Gamma(k-n)} \exp\left[-\frac{\alpha}{\lceil \alpha \rceil - \alpha} t\right] \times \left\{ \exp\left[-\frac{\alpha}{\lceil \alpha \rceil - \alpha} t\right] \sum_{r=0}^{k-n-1} (-1)^r \cdot \right. $$

$$ \left. \cdot \frac{\Gamma(k-n) t^{k-n-1-r}}{\Gamma(k-n-r)\left(\frac{\alpha}{1-\alpha}\right)^{r+1}} - \frac{(-1) k^{-n-1} \Gamma(k-n)}{\left(\frac{\alpha}{1-\alpha}\right)^{k-n}} \right\}, $$

$$ = \frac{B(\alpha)\Gamma(k+1)}{\lceil \alpha \rceil - \alpha} \left[\sum_{r=0}^{k-n-1} \frac{(-1)^r t^{k-n-1-r}}{\Gamma(k-n-r)\left(\frac{\alpha}{1-\alpha}\right)^{r+1}} - \frac{(-1)^{k-n-1}}{\left(\frac{\alpha}{1-\alpha}\right)^{k-n}} \exp\left[-\frac{\alpha}{\lceil \alpha \rceil - \alpha} t\right] \right]. $$

□

3.1. Legendre polynomials

Now we discussed here Legendre polynomials and their properties. We shifted Legendre polynomials on the [0,1] from the interval [-1,1] by the transformation $z = 2x-1$. The analytical form of these polynomials of degree i are given as follows

$$ \psi_i = \sum_{k=0}^{i} \frac{(-1)^{i+k}(i+k)!}{(k!)^2(\ell-k)!} x^k, \tag{31} $$

where i=0,1,…

These polynomials are orthogonal on the interval [-1,1] with respect to the weight function 1 and the orthogonality condition can be described as

$$ \int_0^1 \psi_j(x)\psi_i(x) = \begin{cases} \frac{1}{2i+1}, & i = j, \\ 0, & i \neq j. \end{cases} \tag{32} $$

A function $u(x)$ which belongs to the $L^2[0,1]$ can be approximated by a linear combination of shifted Legendre polynomials as

$$ u(x) = u_m(x) = \sum_{j=0}^{m} a_j \psi_i(x), \tag{33} $$

where the linear coefficients are given by

$$a_j = (2j+1)\int_0^1 u(x)\psi_j(x). \tag{34}$$

Similarly, a function *u(x,t)* of two variables can be approximated as

$$u(x,t) = \sum_{i=0}^{m-1}\sum_{\ell=0}^{m-1} a_{i\ell}\psi_i(x)\psi_\ell(t), \tag{35}$$

where $a_{i\ell}$ are unknown coefficients.

4. Proposed New Algorithm

In this section, we develop a new algorithm with the combination of the Legendre spectral method and the quasi wavelet method and then apply it to find out the solution of C-F time-fractional non-linear distributed order reaction-diffusion equations. We approximate the C-F time-fractional derivative by using Legendre spectral method. On the other hand, spatial derivatives and unknown functions are approximated with the help of the quasi wavelet-based numerical method.

Distributed order reaction-diffusion equation involving C-F fractional derivative is given by

$$\int_0^1 s(\alpha)\,{}_0^{CF}D_t^\alpha u(t,x)d\alpha = \frac{\partial^2 u(x,t)}{\partial x^2} + g(u(x,t),x,t) + f(x,t). \tag{36}$$

The initial and boundary conditions for this model are taken as follows

$$\begin{aligned} u(0,x) &= f_1(x),\\ u(t,0) &= f_2(\mathrm{t}),\\ u(\mathrm{t},1) &= f_3(\mathrm{t}), \end{aligned} \tag{37}$$

where $0<\alpha\le 1,\ 0\le x\le 1$ and $0\le t\le 1$. Now we develop the method with the help of the Legendre spectral and quasi wavelet method to investigate the models given by Eq. (36). Approximating the unknown function in terms of shifted Legendre polynomial, we have

$$u(x,t) = \sum_{i=0}^{m-1}\sum_{\ell=0}^{m-1} c_{i\ell}\psi_i(x)\psi_\ell(t), \tag{38}$$

where $c_{i\ell}$ are unknown coefficients for i=0,2,…., and ℓ =0,1,2,… Now considering the C-F time-fractional operator and using Eq. (38), we get

$$\begin{aligned} {}_0^{CF}D_t^\alpha u(t,x) &= \sum_{i=0}^{m-1}\sum_{\ell=0}^{m-1} c_{i\ell}\psi_i(x)\left({}_0^{CF}D_t^\alpha \psi_\ell(t)\right),\\ &= \sum_{i=0}^{m-1}\sum_{\ell=0}^{m-1}\sum_{k=0}^{\ell} \frac{c_{i\ell}(-1)^{\ell+k}(\ell+k)!}{(k!)^2(\ell-k)!}\psi_i(x)\left({}_0^{CF}D_t^\alpha t^k\right),\\ &= \sum_{i=0}^{m-1}\sum_{\ell=0}^{m-1}\sum_{k=0}^{\ell} \frac{c_{i\ell}(-1)^{\ell+k}(\ell+k)!}{(k!)^2(\ell-k)!}\psi_i(x),\prod\nolimits_k(t,\alpha), \end{aligned} \tag{39}$$

where,

$$\prod\nolimits_k(t,\alpha) = \frac{B(\alpha)\Gamma(1+k)}{\lceil\alpha\rceil-\alpha}\left(\sum_{r=0}^{k-n-1}\frac{(-1)^r t^{k-n-1-r}}{\Gamma(k-n-r)(\gamma)^{r+1}} + \frac{(-1)^{k-n}}{\gamma^{k-n}}e^{-\gamma t}\right), \tag{40}$$

with $\gamma(\alpha) = \alpha/(\lceil\alpha\rceil-\alpha)$.

Integrating Eq. (36) with respect to α after multiplying *s(α)* in both sides, we get the following

$$\int_0^1 s(\alpha)\,{}_0^{CF}D_t^\alpha u(t,x)d\alpha = \sum_{i=0}^{m-1}\sum_{\ell=0}^{m-1}\sum_{k=0}^{\ell} \frac{c_{i\ell}(-1)^{\ell+k}(\ell+k)!}{(k!)^2(\ell-k)!}\psi_i(x)\int_0^1 s(\alpha)\prod\nolimits_k(t,\alpha)d\alpha. \tag{41}$$

Now to calculate the integral term $\int_0^1 s(\alpha)\prod_k(t,\alpha)d\alpha$, we shall proceed as follows

$$\int_0^1 s(\alpha)\prod_k(t,\alpha)d\alpha=\int_0^1 s(\alpha)\frac{B(\alpha)\Gamma(1+k)}{\lceil\alpha\rceil-\alpha}\left(\sum_{r=0}^{k-n-1}\frac{(-1)^r t^{k-n-1-r}}{\Gamma(k-n-r)\gamma^{r+1}}+\frac{(-1)^{k-n}}{\gamma^{k-n}}e^{-\gamma t}\right)d\alpha,$$

$$\Gamma(1+k)\left(\sum_{r=0}^{k-n-1}\frac{(-1)^r t^{k-n-1-r}}{\Gamma(k-n-r)}\int_0^1\frac{s(\alpha)B(\alpha)\Gamma(1+k)}{(\lceil\alpha\rceil-\alpha)(\gamma(\alpha))^{r+1}}d\alpha+(-1)^{k-n}\int_0^1\frac{s(\alpha)B(\alpha)}{(\lceil\alpha\rceil-\alpha)(\gamma(\alpha))^{k-n}}e^{-\gamma(\alpha)t}d\alpha\right), \tag{42}$$

$$\Gamma(1+k)\left(\sum_{r=0}^{k-n-1}\frac{(-1)^r t^{k-n-1-r}}{\Gamma(k-n-r)}\int_0^1 V_1(\alpha)d\alpha+(-1)^{k-n}\int_0^1 V_2(\alpha,t)d\alpha\right),$$

where $V_1(\alpha)=\dfrac{s(\alpha)B(\alpha)\Gamma(1+k)}{(\lceil\alpha\rceil-\alpha)(\gamma(\alpha))^{r+1}}$ and $V_2(\alpha,t)=\dfrac{s(\alpha)B(\alpha)}{(\lceil\alpha\rceil-\alpha)(\gamma(\alpha))^{k-n}}e^{-\gamma(\alpha)t}$. To evaluate these integrals, we can adopt any numerical integration scheme, here we used Simpson 1/3 rule.

$$\begin{aligned}&\int_0^1 V_1(\alpha)d\alpha=M_1,\\&\int_0^1 V_2(\alpha,t)d\alpha=M_2(\alpha,t).\end{aligned} \tag{43}$$

Thus

$$\int_0^1 s(\alpha)\prod_k(t,\alpha)d\alpha=\Gamma(1+k)\left(\sum_{r=0}^{k-n-1}\frac{(-1)^r t^{k-n-1-r}}{\Gamma(k-n-r)}M_1+(-1)^{k-n}M_2(\alpha,t)\right). \tag{44}$$

Putting this value of integral in Eq. (41), we get the following

$$\int_0^1 s(\alpha)\,{}_0^{CF}D_t^{\alpha}u(t,x)d\alpha=\sum_{i=0}^{m-1}\sum_{\ell=0}^{m-1}\sum_{k=0}^{\ell}\frac{c_{i\ell}(-1)^{\ell+k}(\ell+k)!}{(k!)^2(\ell-k)!}\psi_i(x)\times\Gamma(1+\mathrm{k})\left(\sum_{r=0}^{k-n-1}\frac{(-1)^r t^{k-n-1-r}}{\Gamma(k-n-r)}M_1+(-1)^{k-n}M_2(\alpha,t)\right). \tag{45}$$

We have approximated the time derivative considering the Legendre spectral method. To approximate the unknown function *u(x,t)* and the time derivative we consider the quasi wavelet-based numerical method. We know a function and its all derivatives can be approximated by

$$u^{(n)}(x)=\sum_{k=-W}^{W}\delta_{\Delta,\sigma}^{n}(x-x_k)u(x_k),\quad n=0,1,\ldots, \tag{46}$$

where the superscript (n) denotes the n[th] order derivative with respect to x. At spatial point $x=x_j$, we can rewrite the above equation as

$$u^{(n)}(x_j,t)=\sum_{s=-W}^{W}\delta_{\Delta,\sigma}^{n}(-s\Delta_x)u(x_{j+s}),\quad n=0,1,\ldots, \tag{47}$$

where Δx is the spatial step. Putting the value of $u(x,t)$ and their space and time derivatives in the model given by Eq. (34), we get the following residual

$$\begin{aligned}\xi_1(\mathrm{x},\mathrm{t})=&\sum_{i=0}^{m-1}\sum_{\ell=0}^{m-1}\sum_{k=0}^{\ell}\frac{c_{i\ell}(-1)^{\ell+k}(\ell+k)!}{(k!)^2(\ell-k)!}\psi_i(x)\times\Gamma(1+\mathrm{k})\left(\sum_{r=0}^{k-n-1}\frac{(-1)^r t^{k-n-1-r}}{\Gamma(k-n-r)}M_1+(-1)^{k-n}M_2(\alpha,t)\right)-\\&\left(\sum_{i=0}^{m-1}\sum_{\ell=0}^{m-1}\sum_{k=-W}^{W}\delta_{\Delta,\sigma}^{2}(x-x_k)\psi_i(x_k)\times a_{i\ell}\psi_\ell(t)\right)-\mu\left(\sum_{i=0}^{m-1}\sum_{\ell=0}^{m-1}\sum_{k=-W}^{W}\delta_{\Delta,\sigma}^{1}(x-x_k)\psi_i(x_k)\times a_{i\ell}\psi_\ell(t)\right)\times\\&\left(1-\sum_{i=0}^{m-1}\sum_{\ell=0}^{m-1}\sum_{k=-W}^{W}\delta_{\Delta,\sigma}^{0}(x-x_k)\psi_i(x_k)\times a_{i\ell}\psi_\ell(t)\right)-f(x,t).\end{aligned} \tag{48}$$

The initial and boundary conditions take the following form in view of Eq. (33)

$$\begin{aligned}&\sum_{i=0}^{m-1}\sum_{\ell=0}^{m-1}a_{i\ell}\psi_i(x)\psi_\ell(0)=f_1(x),\\&\sum_{i=0}^{m-1}\sum_{\ell=0}^{m-1}a_{i\ell}\psi_i(0)\psi_\ell(t)=f_2(t),\\&\sum_{i=0}^{m-1}\sum_{\ell=0}^{m-1}a_{i\ell}\psi_i(1)\psi_\ell(t)=f_3(t).\end{aligned} \tag{49}$$

Now considering Eqs. (44) and (45) at suitable collocation points (x_j,t_j) in equation (44), we consider the discrete sampling points $x_k = x_j$ equal to the collocation points and using Eq. (43), we get the following non-linear system of algebraic equations

$$\xi_1(x_j,t_j)=\sum_{i=0}^{m-1}\sum_{\ell=0}^{m-1}\sum_{k=0}^{\ell}\frac{c_{i\ell}(-1)^{\ell+k}(\ell+k)!}{(k!)^2(\ell-k)!}\psi_i(x_j)\times\Gamma(1+k)\left(\sum_{r=0}^{k-n-1}\frac{(-1)^r t^{k-n-1-r}}{\Gamma(k-n-r)}M_1+(-1)^{k-n}M_2(\alpha,t_j)\right)-$$
$$\left(\sum_{i=0}^{m-1}\sum_{\ell=0}^{m-1}\sum_{k=-W}^{W}\delta^2_{\Delta,\sigma}(-s\Delta_x)\psi_i(x_{j+s})a_{i\ell}\psi_\ell(t_j)\right)-\mu\left(\sum_{i=0}^{m-1}\sum_{\ell=0}^{m-1}\sum_{k=-W}^{W}\delta^1_{\Delta,\sigma}(-s\Delta_x)\psi_i(x_{j+s})\times a_{i\ell}\psi_\ell(t_j)\right)\times \tag{50}$$
$$\left(1-\sum_{i=0}^{m-1}\sum_{\ell=0}^{m-1}\sum_{k=-W}^{W}\delta^0_{\Delta,\sigma}(-s\Delta x)\psi_i(x_{j+s})\times a_{i\ell}\psi_\ell(t_j)\right)-f(x_j,t_j).$$

5. Results and Discussion

In this section, our aim is to show the accuracy and validity of our new derived method by solving some numerical examples which have C- F time-fractional derivative. All numerical computations are done with Wolfram Mathematica version-11.3.

Example 1: Considering the following distributed order diffusion equation with $s(\alpha)=(1-\alpha)\alpha^5$, and $g(u,x,t)=0$

$$\int_0^1(1-\alpha)\alpha^5\,{}_0^{CF}D_t^\alpha u(t,x)d\alpha=\frac{\partial^2u(x,t)}{\partial x^2}+f(x,t), \tag{51}$$

with the following initial and boundary conditions

$$\begin{aligned}&u(x,0)=0,\\&u(0,t)=0,\\&u(1,t)=t^2,\end{aligned} \tag{52}$$

the force function is $f(x,t)$ and the exact analytical solution of the above problem is $u(x,t)=t^2(2x-x^2)$.
We plot the 3D graph of the numerical and exact solution for $m=4$, $W=20$, $\Delta x=1/10$ and $\Delta x=1/20$ for $\alpha=1$ (classical case) which is depicted by Fig. 1. The absolute error for various m values at $t=0.1$ is presented in Table 1.

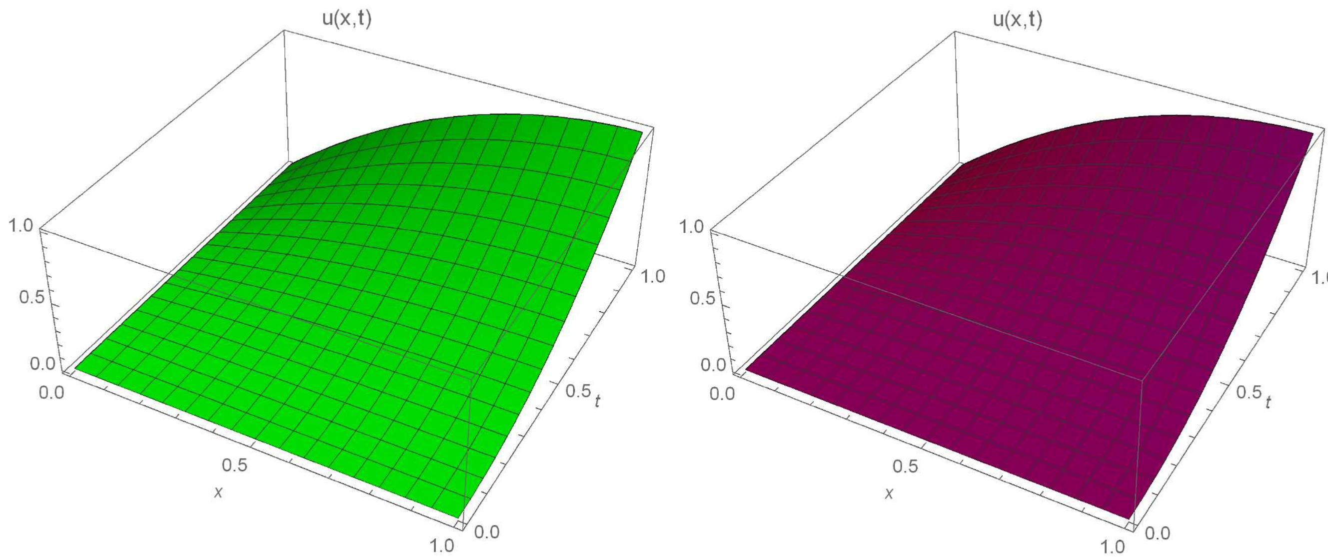

Fig. 1. Plots of $u(x;t)$ for m = 4, W = 20, Δx =1/10 and Δx =1/20 for numerical and exact solutions, respectively.

Example 2: Considering $s(\alpha)=(1-\alpha)\alpha^7$, and $g(u,x,t)=u^2$, we get the following C-F time-fractional distributed order reaction-diffusion equation with variable coefficients

$$\int_0^1 (1-\alpha)\alpha^7 \, {}_0^{CF}D_t^{\alpha} u(t,x)d\alpha = (x+t)\frac{\partial^2 u(x,t)}{\partial x^2}u(x,t) + (xt)(u(x,t)^2) + f(x,t), \tag{53}$$

Table 1. Variations of absolute error for different x values at $t=0.1$ and various m values.

$x\downarrow$	$m=4,\ \Delta x=\frac{1}{10}$	$m=4,\ \Delta x=\frac{1}{20}$
$\frac{1}{9}$	7.0×10^{-5}	9.1×10^{-13}
$\frac{2}{9}$	1.0×10^{-4}	1.7×10^{-12}
$\frac{3}{9}$	1.1×10^{-4}	2.5×10^{-12}
$\frac{4}{9}$	1.0×10^{-4}	3.1×10^{-12}
$\frac{5}{9}$	7.4×10^{-4}	3.4×10^{-12}
$\frac{6}{9}$	4.4×10^{-4}	3.2×10^{-12}
$\frac{7}{9}$	1.6×10^{-5}	2.7×10^{-12}
$\frac{8}{9}$	9.2×10^{-5}	1.7×10^{-12}

which under the prescribed initial and boundary conditions

$$\begin{aligned} &u(0,x) = x^2, \\ &u(\mathrm{t},0) = 0, \\ &u(\mathrm{t},1) = e^t, \end{aligned} \tag{54}$$

gives the exact solution of above the problem is $u(x,t)=x^2e^t$ with suitable force function $f(x,t)$.
We plot the 3D graph of the numerical and exact solution for $m=4$, $W=20$, $\Delta x=1/10$, and $\Delta x=1/20$ for $\alpha=1$ (classical case) which is depicted by Fig. 2. The absolute error for various m values at $t=0.1$ is presented in Table 2, which clearly predicts that our numerical results are in complete agreement with the existing results.

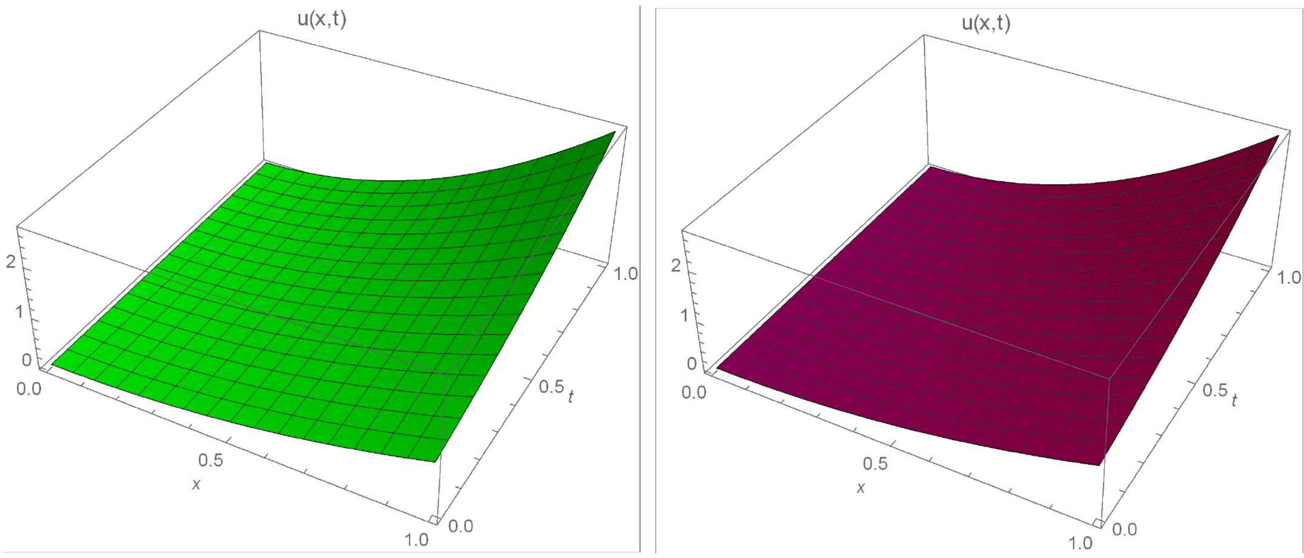

Fig. 2. Plots of $u(x;\,t)$ for m = 4, W = 20, Δx =1/10 and Δx =1/20 for numerical and exact solutions, respectively.

Example 3: Considering $s(\alpha)=(1-\alpha^2)\alpha^8$, and $g(u(\mathrm{x,t}),x,t)=\mathrm{u}(1-\mathrm{u})$, we get the following C-F time-fractional distributed order reaction-diffusion equation

$$\int_0^1 (1-\alpha^2)\alpha^8\, {}_0^{CF}D_t^{\alpha} u(t,x)d\alpha = \frac{\partial^2 u(x,t)}{\partial x^2} u(1-\mathrm{u}) + f(x,t), \tag{55}$$

Table 2. Variations of absolute error for different x values at $t=0.1$ and various m values.

$x\downarrow$	$m=4,\ \Delta x=\frac{1}{10}$	$m=4,\ \Delta x=\frac{1}{20}$
$\frac{1}{9}$	1.2×10^{-3}	3.7×10^{-6}
$\frac{2}{9}$	2.2×10^{-3}	1.9×10^{-5}
$\frac{3}{9}$	3.2×10^{-3}	4.7×10^{-5}
$\frac{4}{9}$	3.3×10^{-3}	8.8×10^{-5}
$\frac{5}{9}$	3.0×10^{-3}	1.4×10^{-4}
$\frac{6}{9}$	2.4×10^{-3}	2.0×10^{-4}
$\frac{7}{9}$	1.6×10^{-3}	3.7×10^{-4}
$\frac{8}{9}$	4.8×10^{-4}	2.8×10^{-4}

with the following initial and boundary conditions

$$\begin{aligned} &u(x,0)=0,\\ &u(0,t)=0,\\ &u(1,t)=t^2,\\ &\frac{\partial}{\partial t}u(x,0)=0, \end{aligned} \tag{56}$$

gives the exact solution $u(x,t)=t^2x^2$ with suitable force function *f(x,t)*.

We plot the 3D graph of the numerical and exact solution for $m=4$, $W=20$, $\Delta x=1/10$ and $\Delta x=1/20$ for $\alpha=1$ (classical case) which is depicted by Fig. 3. The absolute error for various m values at $t=0.1$ is presented in Table 3, which clearly predicts that our numerical results are in complete agreement with the existing results.

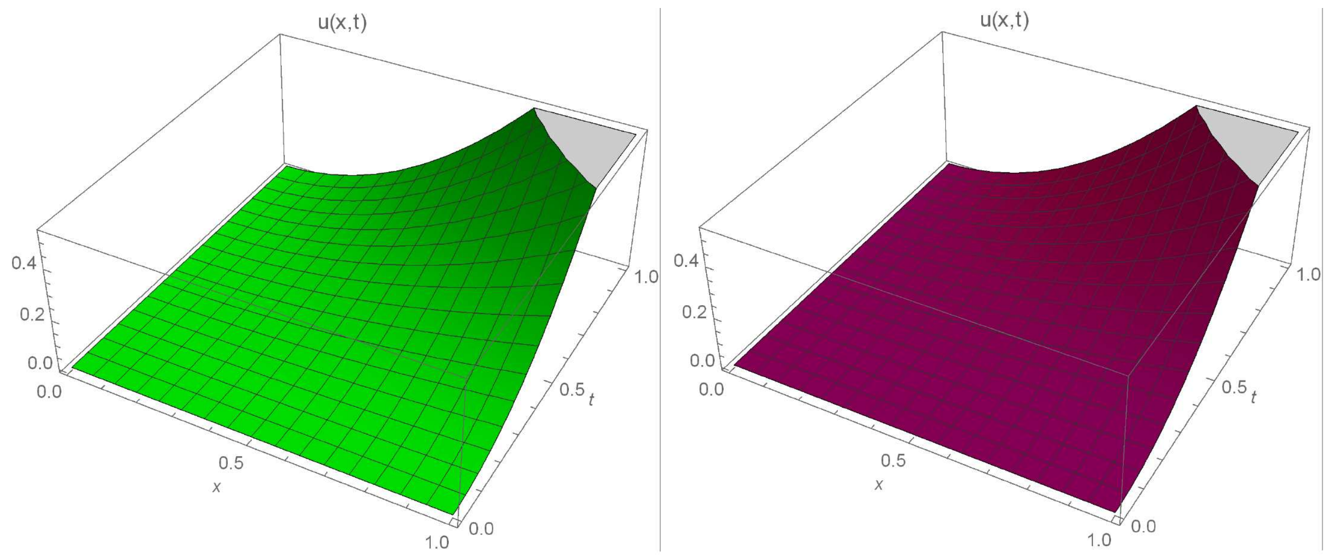

Fig. 3. Plots of $u(x;\,t)$ for m = 4, W = 20, Δx =1/10 and Δx =1/20 for numerical and exact solutions, respectively.

Table 3. Variations of absolute error for different x values at $t=0.1$ and various m values.

$x\downarrow$	$m=4,\ \Delta x=\frac{1}{10}$	$m=4,\ \Delta x=\frac{1}{20}$
$\frac{1}{9}$	1.7×10^{-4}	4.8×10^{-13}
$\frac{2}{9}$	3.1×10^{-4}	8.9×10^{-12}
$\frac{3}{9}$	4.0×10^{-4}	1.2×10^{-12}
$\frac{4}{9}$	4.4×10^{-4}	1.4×10^{-12}
$\frac{5}{9}$	4.5×10^{-4}	1.5×10^{-12}
$\frac{6}{9}$	2.0×10^{-4}	1.4×10^{-12}
$\frac{7}{9}$	3.2×10^{-4}	6.8×10^{-13}
$\frac{8}{9}$	1.8×10^{-4}	8.3×10^{-13}

6. Conclusion

In this research work, first, we found out the approximation formula of C-F fractional derivative of the function t^k. We developed a new numerical algorithm with the combination of the Legendre spectral method and the quasi wavelet-based numerical method to solve fractional PDEs having a C-F fractional derivative. We implemented this new algorithm to solve the new class of C-F fractional distributed order diffusion equation. The Legendre spectral method was used to approximate the C-F time-fractional derivative and distributed the integral term. For the approximation of $u(x,t)$ and their spatial derivatives we used the quasi wavelet-based theory. We showed the successful implementation of this method to solve the C-F time-fractional distributed order FPDEs. We easily concluded that our proposed method has good accuracy and also valid different type of FPDEs. The graphical and tabular exhibitions were presented to validate the effectiveness of the proposed method used for solving various linear/nonlinear problems having C-F fractional derivative.

Author Contributions

Sachin Kumar and J.F. Gómez-Aguilar developed the mathematical modeling and examined the theory validation. The numerical method was worked out by Sachin Kumar and J.F. Gómez-Aguilar. Sachin Kumar analyzed the data and numerical simulations; J.F. Gómez-Aguilar polished the language and were in charge of technical checking. The manuscript was written through the contribution of all authors. All authors discussed the results, reviewed and approved the final version of the manuscript.

Acknowledgments

José Francisco Gómez Aguilar acknowledges the support provided by CONACyT: Cátedras CONACyT para jóvenes investigadores 2014 and SNI CONACyT.

Conflict of Interest

There is no competing interest among the authors regarding the publication of the article.

Funding

The authors received no financial support for the research, authorship, and publication of this article.

References

[1] R.L. Bagley, P.J. Torvik, A theoretical basis for the application of fractional calculus to viscoelasticity. *Journal of Rheology*, 27(3), 1983, 201-210.
[2] A. Kilbas, H. Srivastava, J.J.Trujillo, *Theory and Applications of the Fractional Differential Equations*, Vol. 204, Elsevier (North-Holland), Amsterdam, 2006.

[3] I. Podlubny, *Fractional differential equations, to methods of their solution and some of their applications, Fractional Differential Equations: An Introduction to Fractional Derivatives*, Academic Press, San Diego, CA, 1998.
[4] B. Karaagac, Analysis of the cable equation with non-local and non-singular kernel fractional derivative. *The European Physical Journal Plus*, 133(2), 2018, 1-15.
[5] A. Atangana, J.F. Gómez-Aguilar, Decolonisation of fractional calculus rules: Breaking commutativity and associativity to capture more natural phenomena. *The European Physical Journal Plus*, 133(4), 2018, 1-21.
[6] B. Karaagac, A study on fractional Klein Gordon equation with non-local and non-singular kernel. *Chaos, Solitons & Fractals*, 126, 2019, 218-229.
[7] K.M. Owolabi, Analysis and numerical simulation of multicomponent system with Atangana–Baleanu fractional derivative. *Chaos, Solitons & Fractals*, 115, 2018, 127-134.
[8] K.M. Owolabi, Numerical patterns in system of integer and non-integer order derivatives. *Chaos, Solitons & Fractals*, 115, 2018, 143-153.
[9] A. Atangana, Non validity of index law in fractional calculus: A fractional differential operator with Markovian and non-Markovian properties. *Physica A: Statistical Mechanics and its Applications*, 505, 2018, 688-706.
[10] K.M. Owolabi, Numerical patterns in reaction–diffusion system with the Caputo and Atangana–Baleanu fractional derivatives. *Chaos, Solitons & Fractals*, 115, 2018, 160-169.
[11] K.M. Owolabi, Z. Hammouch, Spatiotemporal patterns in the Belousov–Zhabotinskii reaction systems with Atangana–Baleanu fractional order derivative. *Physica A: Statistical Mechanics and its Applicatio*ns, 523, 2019, 1072-1090.
[12] A. Atangana, T. Mekkaoui, Trinition the complex number with two imaginary parts: Fractal, chaos and fractional calculus. *Chaos, Solitons & Fractals*, 128, 2019, 366-381.
[13] K.M. Owolabi, A. Atangana, On the formulation of Adams-Bashforth scheme with Atangana-Baleanu-Caputo fractional derivative to model chaotic problems. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 29(2), 2019, 1-13.
[14] A. Atangana, S. Qureshi, Modeling attractors of chaotic dynamical systems with fractal–fractional operators. *Chaos, Solitons & Fractals*, 123, 2019, 320-337.
[15] K.M. Owolabi, Z. Hammouch, Spatiotemporal patterns in the Belousov–Zhabotinskii reaction systems with Atangana–Baleanu fractional order derivative. *Physica A: Statistical Mechanics and its Applications*, 523, 2019, 1072-1090.
[16] A. Atangana, Z. Hammouch, Fractional calculus with power law: The cradle of our ancestors. *The European Physical Journal Plus*, 134(9), 2019, 1-12.
[17] L. Suarez, A. Shokooh, An eigenvector expansion method for the solution of motion containing fractional derivatives. *Journal of Applied Mechanics*, 64(3), 1997, 629-635
[18] P. Darania, A. Ebadian, A method for the numerical solution of the integrodifferential equations. *Applied Mathematics and Computation*, 188, 2007, 657-668.
[19] I. Hashim, O. Abdulaziz, S. Momani, Homotopy analysis method for fractional IVPS. *Communications in Nonlinear Science and Numerical Simulation*, 14(3), 2009, 674-684.
[20] K. Diethelm, N.J. Ford, A.D. Freed, A predictor-corrector approach for the numerical solution of fractional differential equations. *Nonlinear Dynamics*, 29(1-4), 2002, 3-22.
[21] Y. Li, N. Sun, Numerical solution of fractional differential equations using the generalized block pulse operational matrix. *Computers & Mathematics with Applications*, 62(3), 2011, 1046-1054.
[22] H. Jafari, S. Yousefi, M. Firoozjaee, S. Momani, C.M. Khalique, Application of Legendre wavelets for solving fractional differential equations. *Computers & Mathematics with Applications*, 62(3), 2011, 1038-1045.
[23] L. Yuanlu, Solving a nonlinear fractional differential equation using Chebyshev wavelets. *Communications in Nonlinear Science and Numerical Simulation*, 15(9), 2010, 2284-2292.
[24] Y. Li, W. Zhao, Haar wavelet operational matrix of fractional order integration and its applications in solving the fractional order differential equations. *Applied Mathematics and Computation*, 216(8), 2010, 2276-2285.
[25] Z. Odibat, On Legendre polynomial approximation with the vim or ham for numerical treatment of nonlinear fractional differential equations. *Journal of Computational and Applied Mathematics*, 235(9), 2011, 2956-2968.
[26] B. Gürbüz, M. Sezer, Laguerre polynomial solutions of a class of initial and boundary value problems arising in science and engineering fields. *Acta Physica Polonica A*, 130(1), 2016, 194-197.
[27] S. Araci, Novel identities for q-Genocchi numbers and polynomials. *Journal of Function Spaces and Applications*, 2012, 2012, 1-12.
[28] H. Zhang, X. Yang, D. Xu, A high-order numerical method for solving the 2d fourth-order reaction-diffusion equation. *Numerical Algorithms*, 80(3), 2019, 849-877.
[29] P. Couteron, O. Lejeune, Periodic spotted patterns in semi-arid vegetation explained by a propagation-inhibition model. *Journal of Ecology*, 89(4), 2001, 616-628.
[30] S. Kondo, R. Asai, A reaction-diffusion wave on the skin of the marine angelfish pomacanthus. *Nature*, 376(6543), 1995, 1-7.
[31] J.D. Murray, A pre-pattern formation mechanism for animal coat markings. *Journal of Theoretical Biology*, 88(1), 1981, 161-199.
[32] S. Kondo, *How animals get their skin patterns: fish pigment pattern as a live turing wave*, in: Systems Biology, Springer, 2009, 37-46.

[33] J.R. Loh, A. Isah, C. Phang, Y.T. Toh, On the new properties of Caputo-Fabrizio operator and its application in deriving shifted Legendre operational matrix. *Applied Numerical Mathematics*, 132, 2018, 138-153.
[34] G. Walter, J. Blum, Probability density estimation using delta sequences. *The Annals of Statistics*, 7(2), 1979, 328-340.
[35] G. Wei, Discrete singular convolution for the solution of the Fokker-Planck equation. *The Journal of Chemical Physics*, 110(18), 1999, 8930-8942.
[36] W. De-cheng, W. Guo-Wei. The study of quasi wavelets based numerical method applied to Burgers' equations. *Applied mathematics and Mechanics*, 21(10), 2000, 1099-1110.
[37] X. Yang, D. Xu, H. Zhang, Quasi-wavelet based numerical method for fourth-order partial integro-differential equations with a weakly singular kernel. *International Journal of Computer Mathematics*, 88(15), 2011, 3236-3254.

ORCID iD

Sachin Kumar https://orcid.org/0000-0002-4924-0879
José Francisco Gómez-Aguilar https://orcid.org/0000-0001-9403-3767

© 2020 by the authors. Licensee SCU, Ahvaz, Iran. This article is an open-access article distributed under the terms and conditions of the Creative Commons Attribution-NonCommercial 4.0 International (CC BY-NC 4.0 license) (http://creativecommons.org/licenses/by-nc/4.0/).