J. Appl. Comput. Mech., 6(4) (2020) 735-740
DOI: 10.22055/JACM.2019.14813
ISSN: 2383-4536
jacm.scu.ac.ir

M&MoCS
Shahid Chamran University of Ahvaz

Journal of Applied and Computational Mechanics

Research Paper

Variational Principle for the Generalized KdV-Burgers Equation with Fractal Derivatives for Shallow Water Waves

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Received September 11 2019; Revised October 12 2019; Accepted for publication October 12 2019.
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Abstract. The unsmooth boundary will greatly affect motion morphology of a shallow water wave, and a fractal space is introduced to establish a generalized KdV-Burgers equation with fractal derivatives. The semi-inverse method is used to establish a fractal variational formulation of the problem, which provides conservation laws in an energy form in the fractal space and possible solution structures of the equation.

Keywords: Continuum assumption, Two scale transform, Fractal dimension, Variational derivative.

1. Introduction

This paper considers the following generalized KdV-Burgers equation [1-4]

$$\frac{\partial u}{\partial T}+au\frac{\partial u}{\partial X}+b\frac{\partial^2 u}{\partial X^2}+c\frac{\partial^3 u}{\partial X^3}=0 \tag{1}$$

where *a, b* and *c* are constants.

When a=1,b=0 and c=1, Eq. (1) is the KdV equation, and when a=1,b=1and c=0 , we obtain the Burgers equation. There are many analytical methods to solve Eq. (1), among which the exp-function method [5-7], the semi-inverse variational method [8-11], the Taylor series method [12], He's frequency formulation for fast insight into the periodic property of a nonlinear equation [13], the homotopy perturbation method [14-19] and the variational iteration method [20-22] have been caught much attention.

Eq. (1) describes a shallow water wave, however an unsmooth boundary will greatly affect the solitary properties, so the smooth space (X,T) should be replaced by a fractal space (X^β, T^α), where β and α are, respectively, fractal dimensions in space and time. In the fractal space Eq. (1) can be modified as

$$\frac{\partial u}{\partial T^\alpha}+au\frac{\partial u}{\partial X^\beta}+b\frac{\partial^2 u}{\partial X^{2\beta}}+c\frac{\partial^3 u}{\partial X^{3\beta}}=0 \tag{2}$$

where the fractal derivatives are defined as [23, 24]

$$\frac{\partial u}{\partial T^{\alpha}}(T_0,X)=\Gamma(1+\alpha)\lim_{\substack{T-T_0\to\Delta T\\ \Delta T\neq 0}}\frac{u(T,X)-u(T_0,X)}{(T-T_0)^{\alpha}} \tag{3}$$

$$\frac{\partial u}{\partial X^{\beta}}(T,X_0)=\Gamma(1+\alpha)\lim_{\substack{X-X_0\to\Delta X\\ \Delta X\neq 0}}\frac{u(T,X)-u(T,X_0)}{(X-X_0)^{\beta}} \tag{4}$$

We have the following chain rule

$$\frac{\partial^2}{\partial X^{2\beta}}=\frac{\partial}{\partial X^{\beta}}\frac{\partial}{\partial X^{\beta}} \tag{5}$$

$$\frac{\partial^3 u}{\partial X^{3\beta}}=\frac{\partial}{\partial X^{\beta}}\frac{\partial}{\partial X^{\beta}}\frac{\partial}{\partial X^{\beta}} \tag{6}$$

In the definitions given in Eqs. (3) and (4), ΔX and ΔT are, respectively, the smallest spatial scale for discontinuous boundary and the smallest temporal scale for watching the solitary wave . When the spatial scale is larger than ΔX , the boundary is considered as a smooth one, and traditional continuum mechanics works, on the scale of ΔX , the boundary is discontinuous, and it is considered a fractal curve. When we watch the solitary wave on a scale larger than ΔT , a smooth wave morphology is predicted, however, when we observe the wave on the scale of ΔT , discontinuous wave morphology can be found [24].

In the fractal space, all variables depend upon the scales used for observation and the fractal dimensions of the discontinuous boundary. For example, the velocity difference (Δu) across a distance (ΔX) or a period (ΔT) can be written in the forms [24]

$$\Delta u \propto (\Delta X)^{\beta} \tag{7}$$

$$\Delta u \propto (\Delta T)^{\alpha} \tag{8}$$

The fractal derives are widely used in applications [24-30] for discontinuous media.

2. Variation Principle

In a fractal space, the physical laws should be also be followed. Wang et al. [30] established a variational principle for traveling wave in a fractal space by the semi-inverse method [31].
According to the basic properties given in Eqs. (7) and (8), we have the following two-scale transform [32, 33]

$$t=T^{\alpha} \tag{9}$$

$$x=X^{\beta} \tag{10}$$

Eq. (2) becomes

$$u_t+auu_x+bu_{xx}+cu_{xxx}=0\,, \tag{11}$$

In order to use the semi-inverse method [31] to establish a variational formulation for Eq. (11), we write Eq. (11) in the form

$$u_t+(\frac{1}{2}au^2+bu_x+cu_{xx})_x=0 \tag{12}$$

According to Eq. (12), we can introduce a function φ satisfying

$$\varphi_x=u \tag{13}$$

$$\varphi_t = -(\frac{1}{2}au^2 + bu_x + cu_{xx}) \tag{14}$$

We want to establish a variational formulation for the problem

$$J(u,\varphi) = \iint L(u,u_t,u_x,u_{xx},u_{xxx},\varphi,\varphi_x,\varphi_{xx},\varphi_{xxx})dxdt \tag{15}$$

where L is the trial-Lagrange function.
By the semi-inverse method [31], we assume that the trial-Lagrange function can be written in the form

$$L = u\varphi_t + (\frac{1}{2}au^2 + bu_x + cu_{xx})\varphi_x + F \tag{16}$$

where F is an unknown function of u and/or φ and/or their derivatives. If F is free from φ and its derivatives, the stationary condition with respect to φ is Eq. (2). The semi-inverse method is a useful mathematical tool to establishment of a needed variational formulation from governing equations [8-11, 34-39].
The stationary condition with respect to u reads

$$\varphi_t + au\varphi_x - b\varphi_{xx} + c\varphi_{xxx} + \frac{\delta F}{\delta u} = 0 \tag{17}$$

where $\delta F / \delta u$ is the variational derivative defined as

$$\frac{\delta F}{\delta u} = \frac{\partial F}{\partial u} - \frac{\partial}{\partial t}(\frac{\partial F}{\partial u_t}) - \frac{\partial}{\partial x}(\frac{\partial F}{\partial u_x}) + \frac{\partial^2}{\partial t^2}(\frac{\partial F}{\partial u_{tt}}) + \frac{\partial^2}{\partial t \partial x}(\frac{\partial F}{\partial u_{tx}}) + \frac{\partial^2}{\partial x^2}(\frac{\partial F}{\partial u_{xx}}) - \cdots \tag{18}$$

In view of Eqs. (13) and (14), we have

$$\begin{aligned} \frac{\delta F}{\delta u} &= -\varphi_t - au\varphi_x + b\varphi_{xx} - c\varphi_{xxx} \\ &= \frac{1}{2}au^2 + bu_x + cu_{xx} - au^2 + bu_x - cu_{xx} \\ &= -\frac{1}{2}au^2 + 2bu_x \end{aligned} \tag{19}$$

From Eq. (19), we cannot identify F, so we have to modify the trial-Lagrange function in the form [37,38]

$$L = Au\varphi_t + B\varphi_x\varphi_t + (\frac{1}{2}au^2 + bu_x + cu_{xx})\varphi_x + F \tag{20}$$

The Euler-Lagrange equations are

$$-Au_t - 2B\varphi_{xt} - (\frac{1}{2}au^2 + bu_x + cu_{xx})_x + \frac{\delta F}{\delta \varphi} = 0 \tag{21}$$

$$A\varphi_t + au\varphi_x - b\varphi_{xx} + c\varphi_{xxx} + \frac{\delta F}{\delta u} = 0 \tag{22}$$

In view of Eqs. (13) and (14), we have

$$\frac{\delta F}{\delta \varphi} = Au_t + 2B\varphi_{xt} + (\frac{1}{2}au^2 + bu_x + cu_{xx})_x = (A+2B)u_t + (\frac{1}{2}au^2 + bu_x + cu_{xx})_x \tag{23}$$

$$\begin{aligned}\frac{\delta F}{\delta u} &= -A\varphi_t - au\varphi_x + b\varphi_{xx} - c\varphi_{xxx} \\ &= A(\frac{1}{2}au^2 + bu_x + cu_{xx}) - au^2 + bu_x - cu_{xx} \\ &= (\frac{1}{2}A - 1)au^2 + (A+1)bu_x + (A-1)cu_{xx}\end{aligned} \tag{24}$$

Setting

$$\frac{\delta F}{\delta \varphi} = 0 \tag{25}$$

and

$$A + 2B = 1 \tag{26}$$

Eq. (21) turns out to be Eq. (12). Setting the coefficient of u_x to be zero in Eq. (24)

$$A + 1 = 0 \tag{27}$$

we obtain

$$\frac{\delta F}{\delta u} = -\frac{3}{2}au^2 - 2cu_{xx} \tag{28}$$

From Eq. (28), F can be identified as

$$F = -\frac{1}{2}au^3 + c(u_x)^2 \tag{29}$$

Finally we obtain the following Lagrange function

$$J(u,\varphi) = \iint \left\{ -u\varphi_t + \varphi_x\varphi_t + (\frac{1}{2}au^2 + bu_x + cu_{xx})\varphi_x - \frac{1}{2}au^3 + c(u_x)^2 \right\} dxdt \tag{30}$$

which is subject to Eq. (13).

Proof. The Euler-Lagrange equations of Eq. (30) are

$$u_t - 2\varphi_{xt} - (\frac{1}{2}au^2 + bu_x + cu_{xx})_x = 0 \tag{31}$$

$$-\varphi_t + au\varphi_x - b\varphi_{xx} + c\varphi_{xxx} - \frac{3}{2}au^2 - 2cu_{xx} = 0 \tag{32}$$

In view of the constraint, Eq. (13), it is easy to prove that Eqs. (31) and (32) are equivalent to, respectively, Eq. (12) and Eq. (14).

In the fractal space (X^β, T^α), the variational formulation can be written in the form

$$J(u,\varphi) = \iint \left\{ -u\frac{\partial \varphi}{\partial T^\alpha} + \frac{\partial \varphi}{\partial X^\beta}\frac{\partial \varphi}{\partial T^\alpha} + (\frac{1}{2}au^2 + b\frac{\partial u}{\partial X^\beta} + c\frac{\partial^2 u}{\partial X^{2\beta}})\frac{\partial \varphi}{\partial X^\beta} - \frac{1}{2}au^3 + c(\frac{\partial u}{\partial X^\beta})^2 \right\} dX^\beta dT^\alpha \tag{33}$$

which is subject to Eq. (13).

3. Conclusion

This paper established a variational formulation for the generalized KdV-Burgers equation in a fractal space

(X^{β}, T^{α}) by the semi-inverse method. The variational principle suggested possible conservation laws and possible solution structures, and it provided a theoretical basis for both the numerical and analytical methods.

Conflict of Interest

The author declared no potential conflicts of interest with respect to the research, authorship and publication of this article.

Funding

The author received no financial support for the research, authorship and publication of this article.

Nomenclature

a, b, c	Constants	α	Fractal dimension in time
(X, T)	Coordinates on a large space	β	Fractal dimension in space
(x, t)	Coordinates on a small space	φ	Potential function

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