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Laplace Variational Iteration Method for Modified Fractional Derivatives with Non-singular Kernel

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Abstract. A universal approach by Laplace transform to the variational iteration method for fractional derivatives with the nonsingular kernel is presented; in particular, the Caputo-Fabrizio fractional derivative and the Atangana-Baleanu fractional derivative with the non-singular kernel is considered. The analysis elaborated for both non-singular kernel derivatives is shown the necessity of considering the modified Caputo-Fabrizio fractional derivative and the analogous modifications for the Atangana-Baleanu fractional derivative with non-singular Mittag-Leffler kernel in order to satisfy the initial conditions for some fractional differential equations.

Keywords: Variational iteration method; Fractional calculus; Laplace transform; Modified Caputo-Fabrizio fractional derivative; Modified Atangana-Baleanu fractional derivative.

1. Introduction

Fractional differential equations are generalizations of classical differential equations of integer order. In recent years, fractional differential equations (FDE's) have attracted considerable interest. It is caused by both the development of the theory of fractional calculus itself and by the applications of such constructions in various sciences such as physics, engineering, biology, signal processing, system identification, control theory, finance dynamics and other areas [1]-[6].

The Lagrange multiplier technique [7]-[9] was widely used to solve a number of nonlinear problems, which arise in mathematical physics and other, related areas, and it was developed into a powerful analytical method, i.e., the variational iteration method [7]-[11] for solving differential equations. The method has been applied to initial and boundary value problems [12]-[17] fractal initial value problems [18]-[19], q-difference equations [20], fuzzy equations [21]-[23], among others.

Generally, in applications of variational iteration method (VIM) to initial value problems of differential equations, one usually considers the next three steps: (a) first it is established the functional, which is going to be under the variational procedure; (b) identifying the Lagrange multipliers; (c) determining the initial iteration. The step (b) is very crucial. Applications of the method to fractional differential equations (FDE's) have considered the Lagrange multipliers and implemented directly to the FDE's and the series solutions have been turned into a poor convergence solution.

We will consider the Laplace variational iterative method (LVIM), following the main steps as usually has been done in the literature [7]-[9] and applied to solve some FDE's where the Caputo-Fabrizio fractional derivative [24]-[25] and the Atangana-Baleanu fractional derivative with non-singular kernel have been considered. The LVIM can be easily applied to overcome the drawback that appears in the VIM avoiding the poor convergence problems. Then we will show

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the necessity of considering the modified Caputo-Fabrizio fractional derivative [26], and the analogous modifications for the Atangana-Baleanu fractional derivative with non-singular kernel [27], in order to satisfy the initial conditions for some special types of fractional differential equations.

The work is organized as follows: In the second section, we present the basic definitions of the fractional Caputo-Fabrizio derivative and the Atangana-Baleanu fractional derivative with the non-singular kernel; next, we will present a brief review of the LVIM taking into account the specific properties of both types of the fractional derivative with non-singular kernel. In section 3, the LVIM will be applied together with the Caputo-Fabrizio fractional derivative and the Atangana-Baleanu fractional derivative to obtain an analytical solution for some FDE's. In section 4, the results obtained in the previous section will be analyzed and we will consider the fractional modified Caputo-Fabrizio fractional derivative, in order to solve the problems associated with the solutions obtained when the original Caputo-Fabrizio fractional derivative was applied, also the analogous modifications for the Atangana-Baleanu fractional derivative, will be presented in order to satisfy the initial conditions. The last section is devoted to the conclusions.

2. Preliminaries

In this section, first, we summarize the basic properties of the Caputo-Fabrizio fractional derivative based on the exponential decay law, for this fractional operator the waiting time distribution is non-singular exponential [24]. Next, we will discuss the basic properties of the Atangana-Baleanu fractional derivative, which has been introduced, by Atangana and Baleanu as a generalization of the exponential distribution by introducing the non-local and non-singular Mittag-Leffler kernel [26]. Non-singular kernel appearing in both fractional derivatives are crucial at the time of describing different type of memory effects, furthermore, the Caputo-Fabrizio and the Atangana-Baleanu fractional derivatives do not obey the index-law which makes them physically stronger and able to capture more complex natural behaviors [24]-[27]. Finally, the main steps in order to consider the LVIM to solve FDE's are analyzed when considering both non-singular fractional derivatives.

2.1. The Caputo-Fabrizio fractional derivative without singular kernel applied to the LVIM

The Caputo-Fabrizio derivative is defined as follows [24]

$$ {}_{0}^{CF}D_t^{\alpha}\phi(t) = \frac{1}{1-\alpha}\int_0^t \phi'(\tau)\exp\left(-\frac{\alpha}{1-\alpha}(t-\tau)\right)d\tau, \quad 0<\alpha\le 1. \tag{1} $$

For the main goal of the current work of applying the LVIM for the Caputo-Fabrizio fractional derivative to obtain some approximated solutions for fractional differential equations we consider the Laplace transform for the fractional derivative introduced by Caputo and Fabrizio (1), i.e.,

$$ L_t\left[{}_{0}^{CF}D_t^{\alpha}\phi(t)\right](s) = \frac{sL_t\left[\phi(t)\right](s)-\phi\left[t=0\right]}{s+\alpha(1-s)} = \frac{s\phi(s)-\phi(0)}{s+\alpha(1-s)}, \tag{2} $$

see for example [24]-[27].

Now let us introduce the main aspects of the LVIM. In this method, one usually assumes that the solution can be expressed as [7]-[9]

$$ \begin{aligned} \phi(t) &= \lim_{p\to 1}\left(\phi_0(\mathrm{t}) + p\phi_1(\mathrm{t}) + p^2\phi_2(\mathrm{t}) + \ldots\right) \\ &= \sum_{i=0}^{\infty}\phi_i(\mathrm{t}), \end{aligned} \tag{3} $$

now by considering the FDE

$$ {}_{0}^{CF}D_t^{\alpha}\phi(t) + R\left[\phi(t)\right] + N\left[\phi(t)\right] = g(t), \tag{4} $$

we proceed to illustrate the basic ideas of the LVIM, where R is a linear operator, N is a nonlinear operator, $g(t)$ is a given function and ${}_{0}^{CF}D_t^{\alpha}\phi(t)$ is the Caputo-Fabrizio fractional derivative of the function $\phi(t)$. Next, we consider the Laplace transform of the above equation with respect to the variable t, i.e.:

$$ \begin{aligned} & L_t\left[{}_{0}^{CF}D_t^{\alpha}\phi(t) + R\left[\phi(t)\right] + N\left[\phi(t)\right] - g(t)\right](s), \\ & = \frac{s\phi(s)-\phi(0)}{s+\alpha(1-s)} + L_t\left[R\left[\phi(t)\right] + N\left[\phi(t)\right] - g(t)\right](s) = 0, \end{aligned} \tag{5} $$

where $L_t\left[{}_{0}^{CF}D_t^{\alpha}\phi(t)\right](s)$ it is given by Eq. (4) and $\phi(\mathrm{s}) = L_t\left[\phi(t)\right](s)$, now by taking the functional (5) into account and introducing the Lagrange multiplier $\lambda(\mathrm{s})$ to the iterative procedure of the LVIM just in the same way as in the previous works where this method has been applied [7]-[9], then Eq. (5) can be transformed into the following iterative method for the transformed series solution (3)

$$\phi_{n+1}(s)=\phi_n(s)+\lambda\left[\frac{s\phi_n(s)-\phi(0)}{s+\alpha(1-s)}+R[\phi_n(s)]+N[\phi_n(s)]-g(t)\right], \tag{6}$$

now if we regard, as usual in the LVIM, the terms $L_t\left[R[\phi(t)]+N[\phi(t)]-g(t)\right](s)$ as restricted terms, one can derive the value of the Lagrange multiplier as

$$\frac{\delta\phi_{n+1}(s)}{\delta\phi_n(s)}=0 \Rightarrow \lambda(s)=-\frac{(1-\alpha)s+\alpha}{s}, \tag{7}$$

with this result for $\lambda(s)$ we obtain the following expressions for the transformed series solution

$$\begin{aligned}
&\phi_0(s)=\frac{\phi(0)}{s}+\lambda(s)L_t[-g(\mathrm{t})](s)\\
&\phi_1(s)=\lambda(s)L_t\left[R[\phi(t)]+N[\phi(t)]\right](s)\\
&\vdots\\
&\phi_{n+1}(s)=\lambda(s)L_t\left[R[\phi_0(t),\phi_1(t),\ldots,\phi_n(t)]+N[\phi_0(t),\phi_1(t),\ldots,\phi_n(t)]\right](s),
\end{aligned} \tag{8}$$

with the application of inverse Laplace transform, the series solution of Eq. (8) is given by

$$\begin{aligned}
&\phi_0(\mathrm{t})=L_t^{-1}\left[\frac{\phi(0)}{s}+\lambda(s)L_t[-g(\mathrm{t})](s)\right](\mathrm{t})\\
&\phi_1(\mathrm{t})=L_t^{-1}\left[\lambda(s)L_t\left[R[\phi(t)]+N[\phi(t)]\right](s)\right](\mathrm{t})\\
&\vdots\\
&\phi_{n+1}(\mathrm{t})=L_t^{-1}\left[\lambda(s)L_t\left[R[\phi_0(t),\phi_1(t),\ldots,\phi_n(t)]+N[\phi_0(t),\phi_1(t),\ldots,\phi_n(t)]\right](s)\right](\mathrm{t}).
\end{aligned} \tag{9}$$

2.2. The Atangana-Baleanu fractional derivative without singular kernel applied to the LVIM

Now we will proceed in an analogous way for the Atangana-Baleanu fractional derivative of order α in the Caputo sense, which is given as follows [26]:
The Caputo-Fabrizio derivative is defined as follows [24]

$${}_0^{ABC}D_t^{\alpha}\phi(t)=\frac{B(\alpha)}{1-\alpha}\int_0^t\phi'(\tau)E_\alpha\left(-\frac{\alpha}{1-\alpha}(t-\tau)^\alpha\right)d\tau,\quad 0<\alpha\le 1. \tag{10}$$

next, we consider the Laplace transform for the fractional derivative introduced by Atangana-Baleanu in the Caputo sense

$$L_t\left[{}_0^{ABC}D_t^{\alpha}\phi(t)\right](s)=\frac{B(\alpha)}{1-\alpha}\frac{s^\alpha}{s^\alpha+\dfrac{\alpha}{1-\alpha}}\left(L_t[\phi(t)](\mathrm{s})-\frac{\phi(0)}{s}\right), \tag{11}$$

see for example [26], where for simplicity we will consider the case

$$B(\alpha)=1. \tag{12}$$

Again one assumes that the solution can be expressed as

$$\phi(t)=\lim_{p\to 1}\left(\phi_0(\mathrm{t})+p\phi_1(\mathrm{t})+p^2\phi_2(\mathrm{t})+\ldots\right)=\sum_{i=0}^{\infty}\phi_i(\mathrm{t}), \tag{13}$$

now by considering the FDE

$${}_0^{ABC}D_t^{\alpha}\phi(t)+R[\phi(t)]+N[\phi(t)]=g(t), \tag{14}$$

we proceed to illustrate the basic ideas of the LVIM, where R is a linear operator, N is a nonlinear operator, $g(t)$ is a given function and ${}_0^{ABC}D_t^{\alpha}\phi(t)$ is the Atangana-Baleanu-Caputo fractional derivative of the function $\phi(t)$. Next, we consider the Laplace transform of the above equation with respect to the variable t, i.e.:

$$\begin{aligned}&L_t\left[{}_0^{ABC}D_t^{\alpha}\phi(t)+R[\phi(t)]+N[\phi(t)]-g(t)\right](s),\\&=\frac{B(\alpha)}{1-\alpha}\frac{s^{\alpha}}{s^{\alpha}+\dfrac{\alpha}{1-\alpha}}\left(L_t[\phi(t)](\mathrm{s})-\frac{\phi(0)}{s}\right)+L_t\left[R[\phi(t)]+N[\phi(t)]-g(t)\right](s)=0,\end{aligned}\tag{15}$$

where $L_t\left[{}_0^{ABC}D_t^{\alpha}\phi(t)\right](s)$ it is given by Eq. (11) and $\phi(\mathrm{s})=L_t[\phi(t)](s),$ now by taking the functional (15) into account and introducing the Lagrange multiplier $\lambda(\mathrm{s})$ to the iterative procedure of the LVIM just in the same way as in the previous works where this method has been applied [7]-[9], then Eq. (15) can be transformed into the following iterative method for the transformed series solution (13)

$$\theta_{n+1}(s)=\theta_n(s)+\lambda\Big[L_t\left[{}_0^{ABC}D_t^{\alpha}\phi_n(t)\right](s)+R[\phi_n(s)]+N[\phi_n(s)]-g(t)\Big],\tag{16}$$

now if we regard, as usual in the LVIM, the terms $L_t\left[R[\phi(t)]+N[\phi(t)]-g(t)\right](s)$ as restricted terms, one can derive the value of the Lagrange multiplier as

$$\begin{aligned}&\frac{\delta\phi_{n+1}(s)}{\delta\phi_n(s)}=0\\&\Rightarrow\\&\lambda(s)=-\frac{(1-\alpha)s^{\alpha}+\alpha}{s^{\alpha}},\end{aligned}\tag{17}$$

with this result for $\lambda(\mathrm{s})$ we obtain the following expressions for the transformed series solution

$$\begin{aligned}&\phi_0(s)=\frac{\phi(0)}{s}+\lambda(s)L_t[-g(\mathrm{t})](s)\\&\phi_1(s)=\lambda(s)L_t\left[R[\phi(t)]+N[\phi(t)]\right](s)\\&\vdots\\&\phi_{n+1}(s)=\lambda(s)L_t\left[R[\phi_0(t),\phi_1(t),\ldots,\phi_n(t)]+N[\phi_0(t),\phi_1(t),\ldots,\phi_n(t)]\right](s),\end{aligned}\tag{18}$$

with the application of inverse Laplace transform, the series solution of Eq. (18) is given by

$$\begin{aligned}&\phi_0(\mathrm{t})=L_t^{-1}\left[\frac{\phi(0)}{s}+\lambda(s)L_t[-g(\mathrm{t})](s)\right](\mathrm{t})\\&\phi_1(\mathrm{t})=L_t^{-1}\left[\lambda(s)L_t\left[R[\phi(t)]+N[\phi(t)]\right](s)\right](\mathrm{t})\\&\vdots\\&\phi_{n+1}(\mathrm{t})=L_t^{-1}\left[\lambda(s)L_t\left[R[\phi_0(t),\phi_1(t),\ldots,\phi_n(t)]+N[\phi_0(t),\phi_1(t),\ldots,\phi_n(t)]\right](s)\right](\mathrm{t}).\end{aligned}\tag{19}$$

3. Applications of the LVIM

Now we will consider the LVIM in order to obtain analytical solutions for some fractional differential equations with fractional derivatives with non-singular kernel.

Example 1. As a first application of the LVIM, we will consider the fractional differential equation

$${}_0^{CF}D_t^{\alpha}\phi(\mathrm{t})=\kappa\phi(\mathrm{t}),\tag{20}$$

with the initial condition

$$\phi(\mathrm{t}=0)=\phi(0),\tag{21}$$

this kind of fractional differential equation has been the object of many researchers working with the anomalous diffusion [28] with $\kappa<0$ as it can be seen as follows

$${}_0^{CF}D_t^{\alpha}\psi(x,t)=\delta\frac{\partial^2\psi(x,t)}{\partial x^2},\tag{22}$$

using the method of variable separation with $\psi(x,t)=T(x)\phi(t),$ we obtain

$$\frac{{}_{0}^{CF}D_{t}^{\alpha}\phi(t)}{\delta\phi(t)}=\frac{\dfrac{\partial^{2}T(x)}{\partial x^{2}}}{T(x)}=-\lambda, \tag{23}$$

From this last equation, we found

$$\begin{aligned}&{}_{0}^{CF}D_{t}^{\alpha}\phi(t)=-\lambda\delta\phi(\mathrm{t})=\kappa\phi(\mathrm{t}),\\&\Rightarrow\\&{}_{0}^{CF}D_{t}^{\alpha}\phi(t)=\kappa\phi(\mathrm{t}),\end{aligned} \tag{24}$$

and

$$\frac{\partial^{2}T(x)}{\partial x^{2}}=-\lambda T(\mathrm{x}), \tag{25}$$

with the boundary conditions, *T'(0)=0* and *T'(L)=0*, with the initial condition $\phi(t=0)=\phi(0)$. In the present work we will consider for simplicity only the solution of the temporal part of Eq. (22), and we will find some problems that appear in order to satisfy the initial condition $\phi(t=0)=\phi(0)$ no matters if we consider the non-singular kernel fractional derivative of Caputo-Fabrizio type or Atangana-Baleanu type. In section 4, we will show that these inconsistencies will be solved when the modified Caputo-Fabrizio fractional derivative and the modified Atangana-Baleanu fractional derivative are considered.

From Eq. (24), the direct application of the LVIM suggest the construction of the following iterative procedure

$$\begin{aligned}&\phi_{0}(\mathrm{s})=\frac{\phi(0)}{s}\Rightarrow\phi_{0}(\mathrm{t})=\phi(0),\\&\phi_{1}(\mathrm{s})=\kappa\left[\lambda(\mathrm{s})\phi_{0}(\mathrm{s})\right]=\kappa\left[(1-\alpha)+\frac{\alpha}{s}\right]\frac{\phi(0)}{s}\ \Rightarrow\ \phi_{1}(\mathrm{t})=L_{t}^{-1}\left[\kappa\left[(1-\alpha)+\frac{\alpha}{s}\right]\frac{\phi(0)}{s}\right]\\&=(1-\alpha)\kappa\phi(0)+\alpha\kappa\phi(0)\frac{t}{1!},\\&\phi_{2}(\mathrm{s})=\kappa\left[\lambda(\mathrm{s})\phi_{1}(\mathrm{s})\right]=\kappa\left[(1-\alpha)+\frac{\alpha}{s}\right]\left(\kappa\left[(1-\alpha)+\frac{\alpha}{s}\right]\frac{\phi(0)}{s}\right)\\&\phi_{2}(\mathrm{s})=\kappa^{2}\left[(1-\alpha)+\frac{\alpha}{s}\right]^{2}\frac{\phi(0)}{s}\Rightarrow\phi_{2}(\mathrm{t})=\left((1-\alpha)^{2}+\frac{(1-\alpha)\alpha 2\mathrm{t}}{1!}+\frac{\alpha^{2}\mathrm{t}^{2}}{2!}\right)\kappa^{2}\phi(0),\end{aligned} \tag{26}$$

$$\begin{aligned}&\phi_{3}(s)=\kappa^{3}\left[(1-\alpha)+\frac{\alpha}{s}\right]^{3}\frac{\phi(0)}{s}\Rightarrow\\&\phi_{3}(t)=\left(1-3\alpha+3\alpha^{2}-\alpha^{3}\right)\kappa^{2}\phi(0)+\left(\frac{3\alpha(1-\alpha)^{2}}{1!}\right)\kappa^{3}t\phi(0)+\left(\frac{3\alpha^{2}(1-\alpha)}{2!}\right)\kappa^{3}t^{2}\phi(0)+\frac{\kappa^{3}\alpha^{3}t^{3}\phi(0)}{3!}\\&\vdots\end{aligned} \tag{27}$$

when $n\to\infty$, the series solution tends to the exact solution

$$\begin{aligned}&\phi(\mathrm{t})=\lim_{n\to\infty}\sum_{i=0}^{n}\left(\phi_{i}(\mathrm{t})\right)\approx\frac{1}{6}\left(-6\alpha^{3}\kappa^{3}+18\alpha^{2}\kappa^{3}+6\alpha^{2}\kappa^{2}-18\alpha\kappa^{3}-12\alpha\kappa^{2}-6\alpha\kappa+6\kappa^{3}+6\kappa^{2}+6\kappa+6\right)\phi(0)+\\&\frac{1}{6}\left(18\alpha^{3}\kappa^{3}-36\alpha^{2}\kappa^{3}-12\alpha^{2}\kappa^{2}+18\alpha\kappa^{3}+12\alpha\kappa^{2}+6\alpha\kappa\right)t\phi(0)+\\&\frac{1}{6}\left(-9\alpha^{3}\kappa^{3}+9\alpha^{2}\kappa^{3}+3\alpha^{2}\kappa^{2}\right)t^{2}\phi(0)+\\&\frac{1}{6}\alpha^{3}\kappa^{3}t^{3}\phi(0)+\dots\\&=\frac{\phi(0)}{1-\kappa(1-\alpha)}+\frac{\alpha\kappa t\phi(0)}{\left(1-\kappa(1-\alpha)\right)^{2}}+\frac{\alpha^{2}\kappa^{2}t^{2}\phi(0)}{2!\left(1-\kappa(1-\alpha)\right)^{3}}+\dots\\&=\frac{\phi(0)}{1-\kappa(1-\alpha)}\exp\left(\frac{\alpha\kappa t}{1-\kappa(1-\alpha)}\right).\end{aligned} \tag{28}$$

As it can be noticed, we have obtained a very fast convergent solution of Eq. (20) only by considering the direct application of the LVIM. This fast convergence process can be easily seen in Figure 1, where we have plotted the relative error difference for the exact analytical solution given by Eq. (28) and the analytical series solution (27), for the special case *n=10* and *n=15* iterative series solution.

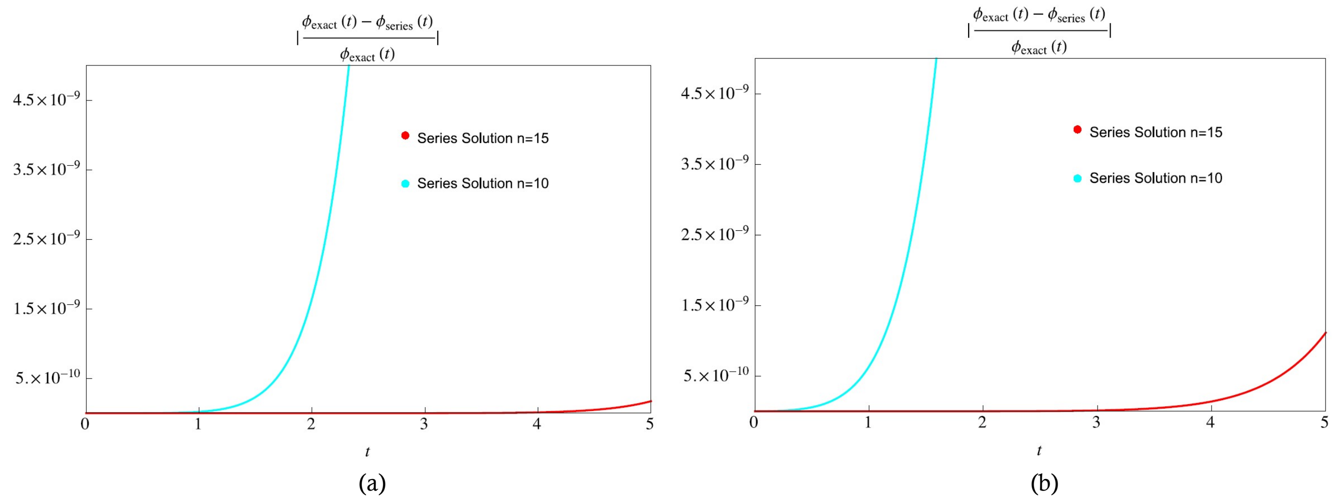


Fig. 1. The behavior of the relative error difference for the exact analytical solution (28) and the analytical series solution (27). In Fig. (1a), we have considered the following values $\alpha = 0.85,\ \kappa = -0.25$ with $\phi(0) = 1$; additionally in Fig. (1b), we have considered the same values for κ and $\phi(0)$, we have taken $\alpha = 0.7$.

However, it is important to see that the solution obtained for the fractional differential equation (20)

$$\phi(\mathrm{t}) = \lim_{n\to\infty} \sum_{i=0}^{n} (\phi_i(\mathrm{t})) = \frac{\phi(0)}{1-\kappa(1-\alpha)} \exp\left(\frac{\alpha\kappa t}{1-\kappa(1-\alpha)}\right), \tag{29}$$

does not satisfy the initial condition

$$\phi(\mathrm{t}=0) = \frac{\phi(0)}{1-\kappa(1-\alpha)} \neq \phi(0). \tag{30}$$

Therefore, is not possible to obtain the solution of Eq. (20) with the initial condition (21) by considering the fractional Caputo-Fabrizio derivative. In order to obtain the proper solution to this problem, in the next section, we will consider the differential equation (20) considering the modified Caputo-Fabrizio fractional derivative with a non-singular kernel [27].

Example 2. Before we try to solve the inconsistencies of the above example, related with the initial conditions, let us to consider the following example, where we will considerer the same physical phenomena, the solution of the time-dependent fractional equation related with the diffusion process given by Eq. (22), but in this new example we will consider the Atangana-Baleanu fractional derivative in the Caputo sense, i.e.:

$$ {}_{0}^{ABC}D_t^{\alpha} \phi(\mathrm{t}) = \kappa\phi(\mathrm{t}), \tag{31}$$

with the initial condition

$$\phi(\mathrm{t}=0) = \phi(0), \tag{32}$$

the direct application of the LVIM suggest the construction of the following iterative procedure

$$\begin{aligned}
&\phi_0(\mathrm{s}) = \frac{\phi(0)}{s} \Rightarrow \phi_0(\mathrm{t}) = \phi(0),\\
&\phi_1(\mathrm{s}) = \kappa\left[\lambda(\mathrm{s})\phi_0(\mathrm{s})\right] = \kappa\left[(1-\alpha)+\frac{\alpha}{s^{\alpha}}\right]\frac{\phi(0)}{s} \Rightarrow \phi_1(\mathrm{t}) = L_t^{-1}\left[\kappa\left[(1-\alpha)+\frac{\alpha}{s^{\alpha}}\right]\frac{\phi(0)}{s}\right]\\
&= (1-\alpha)\kappa\phi(0) + \alpha\kappa\phi(0)\frac{t^{\alpha}}{\Gamma(1+\alpha)},\\
&\phi_2(\mathrm{s}) = \kappa^2\left[(1-\alpha)+\frac{\alpha}{s^{\alpha}}\right]^2\frac{\phi(0)}{s} \Rightarrow \phi_2(\mathrm{t}) = \left((1-\alpha)^2 + \frac{(1-\alpha)\alpha 2\mathrm{t}^{\alpha}}{\Gamma(1+\alpha)} + \frac{\alpha^2\,\mathrm{t}^{2\alpha}}{\Gamma(1+2\alpha)}\right)\kappa^2\phi(0),
\end{aligned} \tag{33}$$

$$\phi_3(s)=\kappa^3\left[(1-\alpha)+\frac{\alpha}{s^\alpha}\right]^3\frac{\phi(0)}{s}\ \Rightarrow$$
$$\phi_3(t)=\left(1-3\alpha+3\alpha^2-\alpha^3\right)\kappa^2\phi(0)+\left(\frac{3\alpha}{\Gamma(1+\alpha)}-\frac{6\alpha}{\Gamma(1+\alpha)}+\frac{3\alpha^3}{\Gamma(1+\alpha)}\right)\kappa^3 t\phi(0)$$
$$+\left(\frac{3\alpha^2}{\Gamma(1+2\alpha)}-\frac{3\alpha^3}{\Gamma(1+2\alpha)}\right)\kappa^3 t^{2\alpha}\phi(0)+\frac{\kappa^3\alpha^3 t^{3\alpha}\phi(0)}{\Gamma(1+3\alpha)}$$
$$\vdots$$

when $n\to\infty$, the series solution tends to the exact solution

$$\begin{aligned}
\phi(\mathrm{t})&=\lim_{n\to\infty}\sum_{i=0}^{n}\left(\phi_i(\mathrm{t})\right)\approx\phi(0)\left(1+\kappa(1-\alpha)+\kappa^2(1-\alpha^2)+\kappa^3(1-\alpha^3)+\ldots\right)\\
&+\phi(0)\frac{t^\alpha}{\Gamma(1+\alpha)}\alpha\kappa\left(1+2\kappa(1-\alpha)+3\kappa^2(1-\alpha^2)+\ldots\right)\\
&+\phi(0)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\alpha\kappa\left(1+3\kappa(1-\alpha)+6\kappa^2(1-\alpha^2)+\ldots\right)\\
&=\frac{\phi(0)}{1-\kappa(1-\alpha)}+\frac{\phi(0)}{(1-\kappa(1-\alpha))}\frac{t^\alpha}{\Gamma(1+\alpha)}\left(\frac{\alpha\kappa}{1-\kappa(1-\alpha)}\right)\\
&+\frac{\phi(0)}{(1-\kappa(1-\alpha))}\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\left(\frac{\alpha\kappa}{1-\kappa(1-\alpha)}\right)^2\\
&\ldots\\
&=\frac{\phi(0)}{1-\kappa(1-\alpha)}\mathrm{E}_\alpha\left(\frac{\alpha\kappa t^\alpha}{1-\kappa(1-\alpha)}\right),
\end{aligned}\tag{34}$$

where $E_\alpha(\cdot)\forall\alpha>0$ is the Mittag-Leffler function with one parameter. As can be seen, we have obtained a very fast convergent solution for the FDE given by Eq. (31) only by considering the direct application of the LVIM. This fast convergence process can be easily seen in Figure 2, where we have plotted the relative error difference for the exact analytical solution given by Eq. (34) and the analytical series solution (33), for the special case of *n=10* and *n=15* iterative series solution.

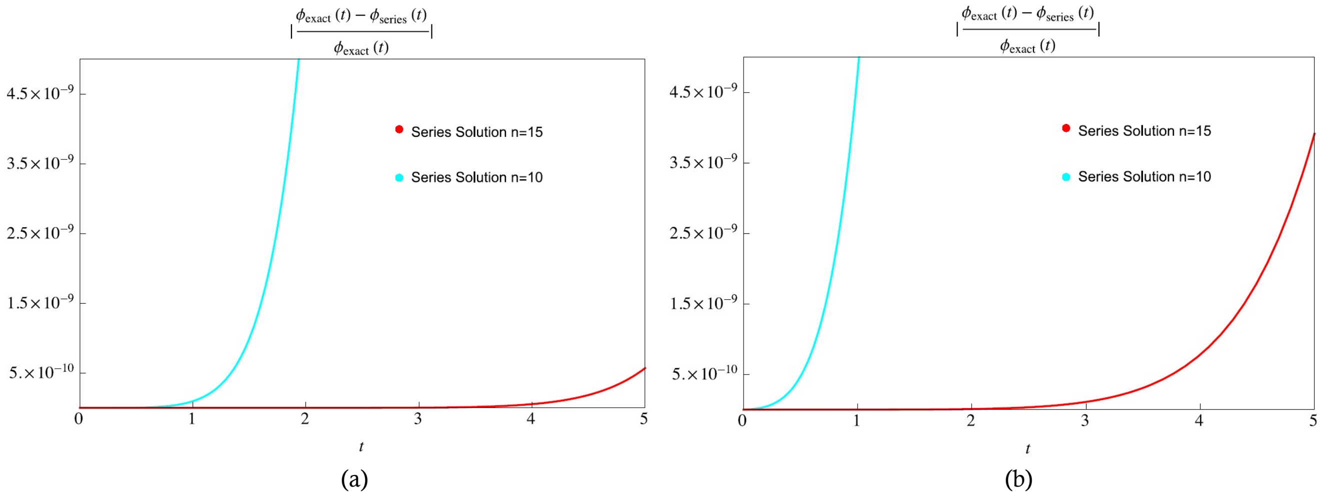


Fig. 2. The behavior of the relative error difference for the exact analytical solution (34) and the analytical series solution (33). In Fig. (2a), we have considered the following values $\alpha=0.85$, $\kappa=-0.25$ with $\phi(0)=1$; additionally in Fig. (2b), we have considered the same values for κ and $\phi(0)$, we have taken $\alpha=0.7$.

However, it is important to see that the solution obtained for the fractional differential equation (20)

$$\phi(\mathrm{t})=\lim_{n\to\infty}\sum_{i=0}^{n}\left(\phi_i(\mathrm{t})\right)=\frac{\phi(0)}{1-\kappa(1-\alpha)}E_\alpha\left(\frac{\alpha\kappa t^\alpha}{1-\kappa(1-\alpha)}\right),\tag{35}$$

does not satisfy the initial condition (32)

$$\phi(\mathrm{t}=0)=\frac{\phi(0)}{1-\kappa(1-\alpha)}\neq\phi(0). \tag{36}$$

Therefore, is not possible to obtain the solution of Eq. (31) with the initial condition (32) by considering the fractional Atangana-Baleanu-Caputo derivative. In order to obtain the proper solution to this problem, in the next section, we will consider the differential equation (31) considering the modified Atangana-Baleanu fractional derivative with non-singular kernel.

4. Applications of the LVIM by considering the modified Caputo-Fabrizio and the modified Atangana-Baleanu fractional derivatives

In this section, we will analyze the problems that have been pointed out in the previous section for the solutions obtained by the application of the LVIM to the FDE's (24) and (31). The problems associated with these solutions will be solved when we will consider the modified Caputo-Fabrizio fractional derivative, and the analogous modifications for the Atangana-Baleanu fractional derivative; these fractional differential operators will be presented in order to satisfy the initial conditions considered in both previous examples.

4.1 The modified Caputo-Fabrizio fractional derivative applied to the LVIM.

In order to overcome this strange inconsistency observed in Eq. (30), the modified Caputo-Fabrizio fractional derivative will be considered. This fractional derivative is given by [27]

$${}_0^{Modified-CF}D_t^{\alpha}\phi(t)=\frac{1}{1-\alpha}\frac{d}{dt}\left\{\int_0^t[\phi(\tau)-CF_{const}]\exp\left(-\frac{\alpha}{1-\alpha}(t-\tau)\right)d\tau\right\},\quad 0<\alpha\leq 1, \tag{37}$$

where ${}_0^{Modified-CF}D_t^{\alpha}\phi(t)$ is the modified Caputo-Fabrizio fractional derivative with respect to t and CF_{const} is some constant introduced with the propose of fulfilling the initial value conditions, but not necessarily will be equal to the value of initial condition of the function, i.e.: $CF_{const}(0)\neq\phi(0)=\phi(\mathrm{t}=0)$.

Now, we consider the following equation in order to obtain an expression for the Laplace transform of the modified Caputo-Fabrizio fractional derivative given by Eq. (37)

$$\begin{aligned}
&{}_0^{Modified\text{-}CF}D_t^{\alpha}\phi(t)=f(t),\quad t\geq 0\\
&\Rightarrow\\
&L_t\left[{}_0^{Modified\text{-}CF}D_t^{\alpha}\phi(t)\right](s)=\frac{s\phi(s)-CF_{const}}{s+\alpha(1-s)}=f(s)\\
&\Rightarrow\\
&f(s)=\frac{CF_{const}}{s}+(1-\alpha)f(s)+\frac{\alpha f(s)}{s},
\end{aligned} \tag{38}$$

and the associated integral operator for the modified Caputo-Fabrizio fractional derivative will be given by

$$\phi(t)=CF_{const}+(1-\alpha)f(t)+\alpha\int_0^t f(y)dy, \tag{39}$$

therefore the associated integral operator for the Modified-CF derivative is given by

$$\phi(t)={}_0^{Modified-CF}I_t^{\alpha}\left[f(t)\right]=CF_{const}+(1-\alpha)f(t)+\alpha\int_0^t f(y)dy. \tag{40}$$

Let us consider a differential equation similar to Eq. (20) for which it will be proposed the new fractional derivative with non-singular kernel (37)

$${}_0^{Modified-CF}D_t^{\alpha}\phi(t)+R\left[\phi(t)\right]+N\left[\phi(t)\right]=g(t), \tag{41}$$

now by Eq. (38) and the direct application of the LVIM, the following condition is established

$$\frac{\delta\phi_{n+1}(s)}{\delta\phi_n(s)}=0, \tag{42}$$

the Lagrange multiplier can be derived from obtaining the following result

$$\lambda(s) = -\frac{(1-\alpha)s+\alpha}{s}. \tag{43}$$

Example 3. We consider the fractional differential equation

$$ {}_{0}^{Modified-CF}D_t^{\alpha}\phi(\mathrm{t}) = \kappa\phi(\mathrm{t}), \tag{44}$$

with the initial condition

$$\phi(\mathrm{t}=0) = \phi(0), \tag{45}$$

the direct application of the LVIM suggest the construction of the following iterative procedure

$$\begin{aligned}
&\phi_0(\mathrm{s}) = \frac{CF_{const}}{s} \Rightarrow \phi_0(\mathrm{t}) = CF_{const},\\
&\phi_1(\mathrm{s}) = \kappa\left[\lambda(\mathrm{s})CF_{const}\right] = \kappa\left[(1-\alpha)+\frac{\alpha}{s}\right]\frac{CF_{const}}{s} \Rightarrow \phi_1(\mathrm{t}) = L_t^{-1}\left[\kappa\left[(1-\alpha)+\frac{\alpha}{s}\right]\frac{CF_{const}}{s}\right]\\
&= (1-\alpha)\kappa CF_{const} + \alpha\kappa CF_{const}\frac{t}{1!},\\
&\phi_2(\mathrm{s}) = \kappa\left[\lambda(\mathrm{s})\phi_1(\mathrm{s})\right] = \kappa\left[(1-\alpha)+\frac{\alpha}{s}\right]\left(\kappa\left[(1-\alpha)+\frac{\alpha}{s}\right]\frac{CF_{const}}{s}\right)\\
&\phi_2(\mathrm{s}) = \kappa^2\left[(1-\alpha)+\frac{\alpha}{s}\right]^2\frac{CF_{const}}{s} \Rightarrow \phi_2(\mathrm{t}) = \left((1-\alpha)^2 + \frac{(1-\alpha)\alpha 2\mathrm{t}}{1!} + \frac{\alpha^2\mathrm{t}^2}{2!}\right)\kappa^2 CF_{const},\\
&\phi_3(s) = \kappa^3\left[(1-\alpha)+\frac{\alpha}{s}\right]^3\frac{CF_{const}}{s} \Rightarrow\\
&\phi_3(t) = \left(1-3\alpha+3\alpha^2-\alpha^3\right)\kappa^2 CF_{const} + \left(\frac{3\alpha(1-\alpha)^2}{1!}\right)\kappa^3 tCF_{const} + \left(\frac{3\alpha^2(1-\alpha)}{2!}\right)\kappa^3 t^2 CF_{const} + \frac{\kappa^3\alpha^3 t^3 CF_{const}}{3!}\\
&\vdots
\end{aligned} \tag{46}$$

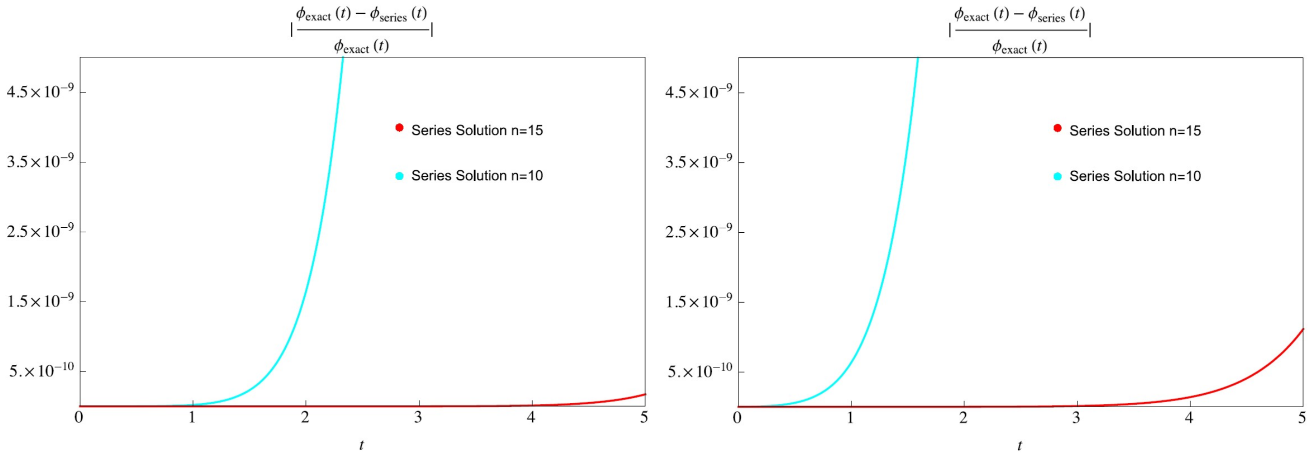
when $n \to \infty$, the series solution tends to the exact solution

$$\begin{aligned}
&\phi(\mathrm{t}) = \lim_{n\to\infty}\sum_{i=0}^{n}\left(\phi_i(\mathrm{t})\right) \approx\\
&\frac{1}{6}\left(-6\alpha^3\kappa^3+18\alpha^2\kappa^3+6\alpha^2\kappa^2-18\alpha\kappa^3-12\alpha\kappa^2-6\alpha\kappa+6\kappa^3+6\kappa^2+6\kappa+6\right)CF_{const} +\\
&\frac{1}{6}\left(18\alpha^3\kappa^3-36\alpha^2\kappa^3-12\alpha^2\kappa^2+18\alpha\kappa^3+12\alpha\kappa^2+6\alpha\kappa\right)tCF_{const} +\\
&\frac{1}{6}\left(-9\alpha^3\kappa^3+9\alpha^2\kappa^3+3\alpha^2\kappa^2\right)t^2 CF_{const} +\\
&\frac{1}{6}\alpha^3\kappa^3 t^3 CF_{const} + \ldots\\
&= \frac{CF_{const}}{1-\kappa(1-\alpha)} + \frac{\alpha\kappa tCF_{const}}{(1-\kappa(1-\alpha))^2} + \frac{\alpha^2\kappa^2 t^2 CF_{const}}{2!(1-\kappa(1-\alpha))^3} + \ldots\\
&= \frac{CF_{const}}{1-\kappa(1-\alpha)}\exp\left(\frac{\alpha\kappa t}{1-\kappa(1-\alpha)}\right).
\end{aligned} \tag{47}$$

As it can be noticed, we have obtained a very fast convergent solution of Eq. (44) only by considering the direct application of the LVIM. This fast convergence process can be easily seen in Figure 3, where we have plotted the relative error difference for the exact analytical solution given by Eq. (47) and the analytical series solution (46), for the special case *n=10* and *n=15* iterative series solution.

It is important to see that the solution obtained for the fractional differential equation (44)

$$\phi(\mathrm{t}) = \lim_{n\to\infty}\sum_{i=0}^{n}\left(\phi_i(\mathrm{t})\right) = \frac{CF_{const}}{1-\kappa(1-\alpha)}\exp\left(\frac{\alpha\kappa t}{1-\kappa(1-\alpha)}\right), \tag{48}$$



(a) (b)

Fig. 3. The behavior of the relative error difference for the exact analytical solution (47) and the analytical series solution (46). In Fig. (3a), we have considered the following values $\alpha = 0.85$, $\kappa = -0.25$ with $\phi(0) = 1$; additionally in Fig. (3b), we have considered the same values for κ and $\phi(0)$, we have taken $\alpha = 0.7$.

does satisfy the initial condition (45)

$$\phi(\mathrm{t}=0) = \frac{CF_{const}}{1-\kappa(1-\alpha)} = \phi(0). \tag{49}$$

only by the proper definition of CF_{const} as

$$CF_{const} = \phi(0)(1-\kappa(1-\alpha)). \tag{50}$$

Now, we will present a similar procedure for the case of Eq. (31), where some modifications need to be done in the Atangana-Baleanu fractional derivative in order to overcome the inconsistencies found in Eq. (36).

4.2 The modified Atangana-Baleanu fractional derivative applied to the LVIM.

In order to overcome this strange inconsistency observed in Eq. (36), the modified Atangana-Baleanu fractional derivative will be considered. This fractional derivative is given by

$${}_{0}^{Modified-AB}D_t^{\alpha}\phi(t) = \frac{B(\alpha)}{1-\alpha}\frac{d}{dt}\left\{\int_0^t [\phi(\tau) - AB_{const}]E_{\alpha}\left(-\frac{\alpha}{1-\alpha}(t-\tau)^{\alpha}\right)d\tau\right\}, \quad 0<\alpha\leq 1, \tag{51}$$

where again for simplicity we will consider the case [26]

$$B(\alpha) = 1, \tag{52}$$

where ${}_{0}^{Modified-AB}D_t^{\alpha}\phi(t)$ is the modified Atangana-Baleanu fractional derivative with respect to t and AB_{const} is some constant introduced with the propose of fulfilling the initial value conditions, but not necessarily will be equal to the value of initial condition of the function, i.e.: $AB_{const}(0) \neq \phi(0) = \phi(\mathrm{t}=0)$.

The main goal is to consider this new fractional derivative with a non-singular kernel of the Mittag-Leffler type and then be able to eliminate the contradiction found in Eq. (36). Now, we consider the following equation in order to obtain an expression for the Laplace transform of the modified Atangana-Baleanu fractional derivative given by Eq. (51)

$$\begin{aligned}
&{}_{0}^{Modified-AB}D_t^{\alpha}\phi(t) = f(t), \quad t \geq 0 \\
&\Rightarrow \\
&L_t\left[{}_{0}^{Modified-AB}D_t^{\alpha}\phi(t)\right](s) = \frac{s^{\alpha}\left(\phi(s) - \frac{AB_{const}}{s}\right)}{(1-\alpha)s^{\alpha}+\alpha} = f(s) \\
&\Rightarrow \\
&\phi(s) = \frac{AB_{const}}{s} + (1-\alpha)f(s) + \frac{\alpha f(s)}{s},
\end{aligned} \tag{53}$$

and the associated integral operator for the modified Atangana-Baleanu fractional derivative will be given by

$$\phi(t) = {}_0^{Modified-AB} I_t^{\alpha}\left[f(t)\right] = AB_{const} + (1-\alpha) f(t) + \frac{\alpha}{\Gamma(\alpha)} \int_0^t f(y)(\mathrm{t}-\mathrm{y})^{\alpha-1} dy. \tag{54}$$

Let us consider a differential equation similar to Eq. (20) for which it will be proposed the new fractional derivative with non-singular kernel (51)

$${}_0^{Modified-AB} D_t^{\alpha}\phi(t) + R\left[\phi(t)\right] + N\left[\phi(t)\right] = g(t), \tag{55}$$

now by Eq. (53) and the direct application of the LVIM, the following condition is established

$$\frac{\delta\phi_{n+1}(s)}{\delta\phi_n(s)} = 0, \tag{56}$$

the Lagrange multiplier can be derived from obtaining the following result

$$\lambda(s) = -\frac{(1-\alpha)s^{\alpha} + \alpha}{s^{\alpha}}. \tag{57}$$

Example 4. For our next example, let us consider a differential equation similar to Eq. (31), for which we propose the new fractional derivative with non-singular kernel (51), i.e.:

$${}_0^{Modified-AB} D_t^{\alpha}\phi(\mathrm{t}) = \kappa\phi(\mathrm{t}), \tag{58}$$

with the initial condition

$$\phi(\mathrm{t}=0) = \phi(0), \tag{59}$$

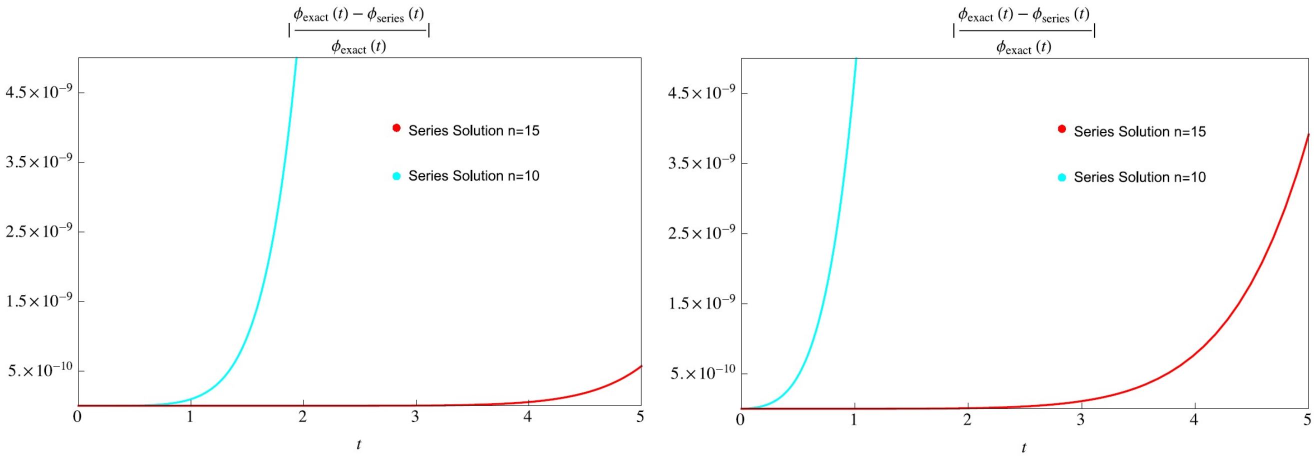
the direct application of the LVIM suggest the construction of the following iterative procedure

$$\begin{aligned}
&\phi_0(\mathrm{s}) = \frac{AB_{const}}{s}\\
&\Rightarrow\\
&\phi_0(\mathrm{t}) = AB_{const},\\
&\phi_1(\mathrm{s}) = \kappa\left[\lambda(\mathrm{s})AB_{const}\right] = \kappa\left[(1-\alpha) + \frac{\alpha}{s^{\alpha}}\right]\frac{AB_{const}}{s}\\
&\Rightarrow\\
&\phi_1(\mathrm{t}) = L_t^{-1}\left[\kappa\left[(1-\alpha) + \frac{\alpha}{s^{\alpha}}\right]\frac{AB_{const}}{s}\right]\\
&= (1-\alpha)\kappa AB_{const} + \alpha\kappa AB_{const}\frac{t^{\alpha}}{\Gamma(1+\alpha)},\\
&\phi_2(\mathrm{s}) = \kappa^2\left[(1-\alpha) + \frac{\alpha}{s^{\alpha}}\right]^2\frac{AB_{const}}{s}\\
&\Rightarrow\\
&\phi_2(\mathrm{t}) = \left((1-\alpha)^2 + \frac{(1-\alpha)\alpha 2\,\mathrm{t}^{\alpha}}{\Gamma(1+\alpha)} + \frac{\alpha^2\,\mathrm{t}^{2\alpha}}{\Gamma(1+2\alpha)}\right)\kappa^2 AB_{const},\\
&\phi_3(s) = \kappa^3\left[(1-\alpha) + \frac{\alpha}{s^{\alpha}}\right]^3\frac{AB_{const}}{s}\\
&\Rightarrow\\
&\phi_3(t) = \left(1-3\alpha+3\alpha^2-\alpha^3\right)\kappa^2 AB_{const} + \left(\frac{3\alpha}{\Gamma(1+\alpha)} - \frac{6\alpha}{\Gamma(1+\alpha)} + \frac{3\alpha^3}{\Gamma(1+\alpha)}\right)\kappa^3 t AB_{const}\\
&+\left(\frac{3\alpha^2}{\Gamma(1+2\alpha)} - \frac{3\alpha^3}{\Gamma(1+2\alpha)}\right)\kappa^3 t^{2\alpha} AB_{const} + \frac{\kappa^3\alpha^3 t^{3\alpha} AB_{const}}{\Gamma(1+3\alpha)}\\
&\vdots
\end{aligned} \tag{60}$$

when $n \to \infty$, the series solution tends to the exact solution

$$
\begin{aligned}
&\phi(\mathrm{t}) = \lim_{n\to\infty}\sum_{i=0}^{n}\left(\phi_i(\mathrm{t})\right) \approx AB_{const}\left(1+\kappa(1-\alpha)+\kappa^2(1-\alpha^2)+\kappa^3(1-\alpha^3)+\ldots\right)\\
&+AB_{const}\frac{t^{\alpha}}{\Gamma(1+\alpha)}\alpha\kappa\left(1+2\kappa(1-\alpha)+3\kappa^2(1-\alpha^2)+\ldots\right)\\
&+AB_{const}\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\alpha\kappa\left(1+3\kappa(1-\alpha)+6\kappa^2(1-\alpha^2)+\ldots\right)\\
&=\frac{AB_{const}}{1-\kappa(1-\alpha)}+\frac{AB_{const}}{\left(1-\kappa(1-\alpha)\right)}\frac{t^{\alpha}}{\Gamma(1+\alpha)}\left(\frac{\alpha\kappa}{1-\kappa(1-\alpha)}\right)\\
&+\frac{AB_{const}}{1-\kappa(1-\alpha)}\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\left(\frac{\alpha\kappa}{1-\kappa(1-\alpha)}\right)^2\\
&\ldots\\
&=\frac{AB_{const}}{1-\kappa(1-\alpha)}\mathrm{E}_{\alpha}\left(\frac{\alpha\kappa t^{\alpha}}{1-\kappa(1-\alpha)}\right).
\end{aligned}
\tag{61}
$$

As can be seen, we have obtained a very fast convergent solution for the FDE given by Eq. (61) only by considering the direct application of the LVIM. This fast convergence process can be easily seen in Figure 4, where we have plotted the relative error difference for the exact analytical solution given by Eq. (61) and the analytical series solution (60), for the special case of *n=10* and *n=15* iterative series solution.



(a) (b)

Fig. 4. The behavior of the relative error difference for the exact analytical solution (61) and the analytical series solution (60). In Fig. (4a), we have considered the following values $\alpha = 0.85$, $\kappa = -0.25$ with $\phi(0) = 1$; additionally in Fig. (4b), we have considered the same values for κ and $\phi(0)$, we have taken $\alpha = 0.7$.

It is important to see that the solution obtained for the fractional differential equation (58)

$$
\phi(\mathrm{t}) = \lim_{n\to\infty}\sum_{i=0}^{n}\left(\phi_i(\mathrm{t})\right) = \frac{AB_{const}}{1-\kappa(1-\alpha)}E_{\alpha}\left(\frac{\alpha\kappa t^{\alpha}}{1-\kappa(1-\alpha)}\right), \tag{62}
$$

does satisfy the initial condition (59)

$$
\phi(\mathrm{t}=0) = \frac{AB_{const}}{1-\kappa(1-\alpha)} = \phi(0). \tag{63}
$$

only by the proper definition of AB_{const} as

$$
AB_{const} = \phi(0)(1-\kappa(1-\alpha)). \tag{64}
$$

Now, we will present a similar procedure for the case of Eq. (31), where some modifications need to be done in the Atangana-Baleanu fractional derivative in order to overcome the inconsistencies found in Eq. (36).

5. Conclusions

The fundamental goal of this work was to apply the LVIM to solve some fractional differential equations involving the modified Caputo-Fabrizio and the modified Atangana-Baleanu fractional derivatives; these new fractional derivatives have some important properties that allowed us to introduce the proper solutions for some special family of fractional differential equations of the eigenvalue type. For these fractional differential equations, first, the LVIM with the originals Caputo-Fabrizio and the Atangana-Baleanu fractional derivatives were considered. it was showed that with these original definitions some initial conditions cannot be satisfied, even when a fast convergence solution was obtained for both fractional differential equations type. Furthermore, the direct application of the LVIM with the modified Caputo-Fabrizio and the modified Atangana-Baleanu fractional derivatives were successfully satisfied the imposed initial conditions, these initial conditions were not satisfied with the originals Caputo-Fabrizio and the Atangana-Baleanu fractional derivatives. The application of the LVIM combined with the modified Caputo-Fabrizio and the modified Atangana-Baleanu fractional derivatives were shown to be very effective to obtain fast convergent solutions and satisfy the imposed initial conditions. In further works, we will apply these new types of fractional derivatives to the modified homotopy perturbation method to solve nonlinear fractional differential equations, also with the modified Caputo-Fabrizio and the modified Atangana-Baleanu fractional derivatives. Other analytical approximated methods can be directly implemented to solve different types of FDE's to obtain fast convergent solutions that satisfy the initial and boundary conditions, for example, in [27] the multistep homotopy analysis method (MHAM) was applied to obtain analytical approximated solutions to some FDE's. Additionally, other methods like the fractional power series method [29]-[36], will be easily extended to solve FDE's and satisfy the imposed initial and boundary conditions. We believe that these novel modified fractional derivatives can be applied in the future to more complex real-world problems.

Author Contributions

H. Yépez-Martínez and J.F. Gómez-Aguilar developed the mathematical modeling and examined the theory validation. The numerical method was worked out by H. Yépez-Martínez and J.F. Gómez-Aguilar. H. Yépez-Martínez analyzed the data and numerical simulations; J.F. Gómez-Aguilar polished the language and were in charge of technical checking. The manuscript was written through the contribution of all authors. All authors discussed the results, reviewed and approved the final version of the manuscript.

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Conflict of Interest

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References

[1] A. Kilbas, H.M. Srivastava, J.J. Trujillo. *Theory and Applications of Fractional Differential Equations*, Elsevier Science, Amsterdam 2006.
[2] R. Hilfer. *Applications of Fractional Calculus in Physics*, World Scientific Publishing, River Edge, NJ, 2000.
[3] B.J. West, M. Bologna, P. *Grigolini. Physics of Fractal Operators*, Springer, New York, 2003.
[4] K.B. Oldham, J. Spanier. *The Fractional Calculus*, Academic Press, New York, 1974.
[5] I. Podlubny. *Fractional Differential Equations*, Academic Press, New York, 1999.
[6] D. Baleanu, K. Diethelm, E. Scalas, J.J. Trujillo. Fractional calculus models and numerical methods, *Series on Complexity, Nonlinearity, and Chaos*, World Scientific, 2012.
[7] G-C. Wu, D. Baleanu. Variational iteration method for fractional calculus - A universal approach by Laplace transform, *Advances in Difference Equations*, 2013(1), (2013), 1-18.
[8] T.A. Biala, Y.O. Afolabi, O.O. Asim. Laplace variational iteration method for integro-differential equations of fractional order, *International Journal of Pure and Applied Mathematics*, 95(3), (2014), 413-426.
[9] M. Inokuti, M. Sekine, T. Mura. *General use of the Lagrange multiplier in nonlinear mathematical physics*, In Variational Methods in the Mechanics of Solids, Pregman Press, New York, (1978), 156-162.
[10] J.H. He. Approximate analytical solution for seepage flow with fractional derivatives in porous media, *Computer Methods in Applied Mechanics and Engineering*, 167(1-2), (1998), 57-68.

[11] J.H. He. Variational iteration method - A kind of non-linear analytical technique: some examples, *International Journal of Non-Linear Mechanics*, 34(4), (1999), 699-708.-
[12] J.H. He. An elementary introduction to recently developed asymptotic methods and nanomechanics in textile engineering, *International Journal of Modern Physics B*, 22(21), (2008), 3487-3578.
[13] S. Abbasbandy. A new application of He's variational iteration method for quadratic Riccati differential equation by using Adomian's polynomials, *Journal of Computational and Applied Mathematics*, 207(1), (2007), 59-63.
[14] M.A. Noor, S.T. Mohyud-Din. Variational iteration technique for solving higher order boundary value problems, *Applied Mathematics and Computation*, 189(2), (2007), 1929-1942.
[15] M.A. Noor, S.T. Mohyud-Din. Variational iteration method for solving higher-order nonlinear boundary value problems using He's polynomials, *International Journal of Nonlinear Sciences and Numerical Simulation*, 9(2), (2008), 141-156.
[16] E. Yusufoglu. The variational iteration method for studying the Klein-Gordon equation, *Applied Mathematics Letters*, 21(7), (2008), 669-674.
[17] A. Yildirim, T. Özis. Solutions of singular IVPs of Lane-Emden type by the variational iteration method, *Nonlinear Analysis, Theory, Methods & Applications*, 70(6), (2009), 2480-2484.
[18] G.C. Wu. New trends in variational iteration method, *Fractional Calculus and Applied Analysis*, 2(2), (2011), 59-75.
[19] G.C. Wu, K.T. Wu. Variational approach for fractional diffusion-wave equations on Cantor sets, *Chinese Physics Letters*, 29(6), (2012), 1-9.
[20] G.C. Wu. Variational iteration method for q-difference equations of second order, *Journal of Applied Mathematics*, 1, (2012), 1-5.
[21] T. Allahviranloo, S. Abbasbandy, H. Rouhparvar. The exact solutions of fuzzy wave-like equations with variable coefficients by a variational iteration method, *Applied Soft Computing*, 11(2), (2011), 2186-2192.
[22] H. Jafari, C.M. Khalique. Homotopy perturbation and variational iteration methods for solving fuzzy differential equations, *Commun. Fract. Calc.*, 3(1), (2012), 38-48.
[23] H. Jafari, M. Saeidy, D. Baleanu. The variational iteration method for solving n-th order fuzzy differential equations, *Central European Journal of Physics*, 10(1), (2012), 76-85.
[24] M. Caputo, M. Fabrizio. A new definition of fractional derivative without singular kernel, *Progress in Fractional Differentiation and Applications*, 1(2), (2015), 1-13.
[25] J. Lozada, J.J. Nieto. Properties of a new fractional derivative without singular kernel, *Progress in Fractional Differentiation and Applications*, 1(2), (2015), 87-92.
[26] H. Yépez-Martínez, J.F. Gómez-Aguilar. A new modified definition of Caputo-Fabrizio fractional-order derivative and their applications to the multi-step homotopy analysis method (MHAM), *Journal of Computational and Applied Mathematics*, 346, (2019), 247-260.
[27] A. Atangana, D. Baleanu. New Fractional Derivatives with Nonlocal and Non-singular Kernel: Theory and Application to Heat Transfer Model, *Thermal Science*, 20(2), (2016), 763-769.
[28] X. Yu, Y. Zhang, H. Guang Sun, C. Zheng. Time fractional derivative model with Mittag-Leffler function kernel for describing anomalous diffusion: Analytical solution in bounded-domain and model comparison, *Chaos, Solitons and Fractals*, 115, (2018), 306- 312.
[29] M. Alquran, I. Jaradat. A novel scheme for solving Caputo time-fractional nonlinear equations: theory and application, *Nonlinear Dynamics*, 91(4), (2018), 2389-2395.
[30] H.M. Jaradat, I. Jaradat, M. Alquran, M.M.M. Jaradat, Z. Mustafa, K.M. Abohassan, R. Abdelkarim. Approximate solutions to the Generalized Time-Fractional Ito system, *Italian Journal of Pure and Applied Mathematics*, 37, (2017), 699-710.
[31] M. Alquran, H.M. Jaradat, M.I. Syam. Analytical solution of the time-fractional Phi-4 equation by using modified residual power series method, *Nonlinear Dynamics*, 90(4), (2017), 2525-2529.
[32] M. Alquran, K. Al-Khaled, S. Sivasundaram, H.M Jaradat. Mathematical and numerical study of existence of bifurcations of the generalized fractional Burgers-Huxley equation, *Nonlinear Studies*, 24(1), (2017), 235-244.
[33] I. Jaradat, M. Al-Dolat, K. Al-Zoubi, M. Alquran. Theory and applications of a more general form for fractional power series expansion, *Chaos, Solitons and Fractals*, 108, (2018), 107-110.
[34] M. Senol, M. Alquran, H. Daei Kasmaei. On the comparison of perturbation-iteration algorithm and residual power series method to solve fractional Zakharov-Kuznetsov equation, *Results in Physics*, 9, (2018), 321-327.
[35] I. Jaradat, M. Alquran, R. Abdel-Muhsen. An analytical framework of 2D diffusion, wave-like, telegraph, and Burgers' models with twofold Caputo derivatives ordering, *Nonlinear Dynamics*, 93(4), (2018), 1911-1922.
[36] I. Jaradata, M. Alquranb, K. Al-Khaled. An analytical study of physical models with inherited temporal and spatial memory, *The European Physical Journal Plus*, 133, (2018), 1-11.

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