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Size-dependent Nonlinear Forced Vibration Analysis of Viscoelastic/Piezoelectric Nano-beam

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Abstract: In this paper, the nonlinear forced vibration of isotropic viscoelastic/ piezoelectric Euler-Bernoulli nano-beam is investigated. For this purpose, the consistent couple stress theory is utilized for modeling the viscoelastic/piezoelectric nano-beam. Hamilton's principle is also employed to obtain the governing equations of motion. Further, the Galerkin method is used in order to convert the governing partial differential equations to a nonlinear second-order ordinary differential one, and then multiple scale method is used to solve motion equation.

Keywords: Nonlinear forced vibration; Piezoelectric; Viscoelastic; Consistent couple stress theory; Multiple scale method.

1. Introduction

Nowadays, with the advancement of nano-science, nano-beams are one of the most important engineering members that are used in various types of structures such as nano electromechanical systems. Beam vibration in transverse mode and its natural frequencies is very important because of the failure phenomenon at high-frequency vibration amplitude. Also, the nonlinear terms have an important role in the vibration analysis of the nano-beam [1-4].

Nano-sized structures have a high volume surface and atoms in the surface layer have a different structure, as compared to the volume structure. Hence, the relation between surface and physical and mechanical behavior depends on properties such as size. Although it is difficult to conduct experimental studies and control them at the nano-scale and the molecular techniques are time-consuming and costly, researchers have used the continuum mechanics theories in order to find the properties of nanostructures [5-7]. Since the classical continuum mechanics is unable to model structures with a small size, various higher-order continuum theories have been suggested to describe the sizedependency of structures. It is reported by various researchers such as Eringen's nonlocal elasticity theory [8, 9], surface elasticity theory [10], modified couple stress theory (MCST) [11-14] and strain gradient theory [15-18]. Among the higher-order theories, couple stress theory has been widely considered. Voigt first proposed couple stress in materials [19], and Cosserat brothers were the first to introduce the mathematical model for materials with couple stress [20]. Couple stress theory was modified by Topin [21] and many studies have been done on this research theory; for example, Asghari [22] used a non-classic couple stress theory and examined the static and free vibration behavior of the simply-supported Timoshenko beam. He solved the equations using the perturbation method. Hadjesfandiari and Dargush developed a new couple of stress theory [23]. Poorjamshidian [24] studied the nonlinear vibration for a simply-supported flexible beam with a constant velocity carrying a moving mass. The results have shown that due to the increase in the velocity concentrated mass, the nonlinear vibration frequency is reduced. On the other hand, whatever the mass moves into the



middle of the beam, beam frequency decreases. Akgoz and Civalek [25] applied the non-classical MCST to simulate Euler-Bernoulli micro-beam modeling and used the Rayleigh-Ritz method to solve the vibration equation. Hadjesfandiari [26] developed model the isotropic homogeneous nano-beam for the nano-scale size-dependent piezoelectric material. Ansari [27] conducted the nonlinear forced vibration analysis on the nano-beam of Timoshenko based on the nonlinear theory.

Arefi et al. [28] investigated the nonlinear free and forced vibration analysis of an embedded functionally graded sandwich micro-beam with a moving mass. Ghorbanpour Arani [29], on the other hand, investigated the nonlinear vibration of the nano-beam coupled with the piezoelectric nano-beam and solved the vibration equations using the differential quadrature method (DQM) and the strain gradient theory. Damping and vibration absorption by viscoelastic materials can be important for different applications of beams. Ghayesh et al. [30], for instance, introduced the size-effect into the equations for the viscoelastic micro-beam and used the couple stress theory for analysis. In another study, Faraji Oskouyi et al. [31] investigated the nonlinear vibrational viscoelastic Euler-Bernoulli nano-beam vibrations.

A literature survey indicates that the vibration behavior of viscoelastic/piezoelectric Euler-Bernoulli nano-beams modeled by consistent couple stress theory has not been studied. Therefore, in this paper, the nonlinear free and forced vibration analysis of the isotropic piezoelectric/viscoelastic nano-beam in a direct piezoelectric process investigated. The consistent couple stress theory is employed to consider the size-effect and final equation of motion solved by multiple scale methods. The effects of the coefficient of viscosity, size- effect and piezoelectric property have been studied in the nonlinear forced vibration.

2. Equation of motion and related boundary conditions

Figure 1 shows the schematic representation of a nano-beam at length L, thickness h and width b. subjected to a transverse harmonic excitation force per unit length F(x,t). The beam is clamped-clamped at both ends, homogeneous, isotropic, piezoelectric film with a considerable thickness against the aluminum substrate used electrode. The substrate has a slight thickness, which can be ignored by the displacement effect.



Fig. 1. Nano-beam-type piezoelectric energy harvester

For the Euler-Bernoulli beam model, the displacement field is written as follows:

$$u_1 = u(x,t) - z \frac{\partial w(x,t)}{\partial x} \quad , u_2 = 0 \quad , u_3 = w(x,t)$$
(1)

According to consistent couple stress theory, size-dependent piezoelectricity proposed by Hadjesfandiari was obtained by strain energy stored in the piezoelectric material on volume V as follows [26]:

$$U = \frac{1}{2} \int_{V} (\sigma_{ij} e_{ij} + \mu_{ij} \kappa_{ij} - D_i E_i) dV$$
⁽²⁾

In equation (2), σ_{ij} is classical stress tensor (Cauchy stress) and e_{ij} is the strain tensor. μ_{ij} is the couple stress tensor and κ_{ij} is curvature tensor. D_i and E_i were used for the components of vector of electric displacement and electric field, respectively. For viscoelastic and isotropic piezoelectric materials, the stress, couple stress, and electrical displacement relationship are written as follows [26]:

$$\sigma_{ij} = \lambda e_{kk} \delta_{ij} + 2\mu e_{ij} + 2\mu l^2 \nabla^2 \omega_{ij} + \eta \frac{\partial e_{ij}}{\partial t}$$
(3)

$$\mu_i = -8\mu l^2 \kappa_i + 2f E_i + \eta \frac{\partial \kappa_i}{\partial t}$$
(4)

$$D_i = \varepsilon E_i + 4f\kappa_i \tag{5}$$

In equation (3), λ and μ are Lamé parameters and η is the viscoelastic coefficient. Moreover, *f*, *l* are the piezoelectric coefficient and material length scale parameter, respectively. In the classical equations, the size-effect coefficient is zero. For a size-dependent piezoelectric model described above, Hamilton's principle is as the following:



$$\int_{t_1}^{t_2} \left(\delta T - \delta U + \delta W\right) dt = 0 \tag{6}$$

In equation (6), δ shows variation operation, *W*, *U* are the work of external forces, strain energy respectively and *T* is the kinetic energy. By substituting the equations (1), (3)-(5) into equation (2) and utilizing the variation method with integration by parts, the variation of strain energy after long mathematical calculations can be written as follows:

$$\begin{split} \delta U &= (\lambda + 2\mu)\mathcal{A}(\frac{\partial u_{0}}{\partial x} + \frac{1}{2}(\frac{\partial w}{\partial x})^{2})\delta u_{0}|_{0}^{L} - \int_{0}^{r} (\lambda + 2\mu)\mathcal{A}(\frac{\partial^{2}u_{0}}{\partial x^{2}} + (\frac{\partial w}{\partial x})(\frac{\partial^{2}w}{\partial x^{2}}))(\delta u_{0})dx \\ &+ \eta\mathcal{A}\frac{\partial}{\partial t}(\frac{\partial u_{0}}{\partial x} + \frac{1}{2}(\frac{\partial w}{\partial x})^{2})(\delta u_{0})|_{0}^{L} - \int_{0}^{l} \eta\mathcal{A}\frac{\partial}{\partial t}(\frac{\partial^{2}u_{0}}{\partial x^{2}} + (\frac{\partial w}{\partial x})(\frac{\partial^{2}w}{\partial x^{2}}))(\delta u_{0})dx \\ &+ (\lambda + 2\mu)I(\frac{\partial^{2}w}{\partial x^{2}})(\frac{\partial \delta w}{\partial x})|_{0}^{L} - (\lambda + 2\mu)I(\frac{\partial^{3}w}{\partial x^{3}})(\delta w)|_{0}^{L} + \int_{0}^{l} (\lambda + 2\mu)I(\frac{\partial^{4}w}{\partial x^{4}})(\delta w)dx \\ &+ \eta I\frac{\partial}{\partial t}(\frac{\partial^{2}w}{\partial x^{2}})(\frac{\partial \delta w}{\partial x})|_{0}^{L} - \eta I\frac{\partial}{\partial t}(\frac{\partial^{3}w}{\partial x^{3}})(\delta w)|_{0}^{L} + \int_{0}^{l} \eta I\frac{\partial}{\partial t}(\frac{\partial^{4}w}{\partial x^{4}})(\delta w)dx \\ &+ (\lambda + 2\mu)\mathcal{A}(\frac{\partial u_{0}}{\partial x} + \frac{1}{2}(\frac{\partial w}{\partial x})^{2})(\frac{\partial w}{\partial x})(\delta w)|_{0}^{L} - \int_{0}^{L} (\lambda + 2\mu)\mathcal{A}(\frac{\partial^{2}u_{0}}{\partial x^{2}} + (\frac{\partial w}{\partial x})(\frac{\partial^{2}w}{\partial x^{2}}))(\frac{\partial w}{\partial x})(\delta w)dx \\ &+ (\lambda + 2\mu)\mathcal{A}(\frac{\partial u_{0}}{\partial x} + \frac{1}{2}(\frac{\partial w}{\partial x})^{2})(\frac{\partial w}{\partial x})(\delta w)|_{0}^{L} - \int_{0}^{L} (\lambda + 2\mu)\mathcal{A}(\frac{\partial^{2}u_{0}}{\partial x^{2}} + (\frac{\partial w}{\partial x})(\frac{\partial^{2}w}{\partial x^{2}}))(\frac{\partial w}{\partial x})(\delta w)dx \\ &+ \eta \mathcal{A}\frac{\partial}{\partial t}(\frac{\partial u_{0}}{\partial x} + \frac{1}{2}(\frac{\partial w}{\partial x})^{2})(\frac{\partial w}{\partial x})(\delta w)|_{0}^{L} - \int_{0}^{L} (\lambda + 2\mu)\mathcal{A}(\frac{\partial^{2}u_{0}}{\partial x^{2}} + (\frac{\partial w}{\partial x})(\frac{\partial^{2}w}{\partial x^{2}}))(\frac{\partial w}{\partial x})(\delta w)dx \\ &+ \eta \mathcal{A}\frac{\partial}{\partial t}(\frac{\partial u_{0}}{\partial x} + \frac{1}{2}(\frac{\partial w}{\partial x})^{2})(\frac{\partial w}{\partial x})(\delta w)|_{0}^{L} - \int_{0}^{L} (\lambda + 2\mu)\mathcal{A}(\frac{\partial^{2}u_{0}}{\partial x^{2}} + (\frac{\partial w}{\partial x})(\frac{\partial^{2}w}{\partial x^{2}}))(\frac{\partial w}{\partial x})dx \\ &- \mathcal{L}_{0}^{L}\eta\mathcal{A}\left[(\frac{\partial}{\partial t}, \frac{\partial}{\partial x} + \frac{1}{2}(\frac{\partial}{\partial x})^{2})(\frac{\partial}{\partial x})\right]_{0}^{L} - 4\mu \mathcal{A}^{2}(\frac{\partial^{2}u_{0}}{\partial x^{2}} + (\frac{\partial}{\partial x})(\frac{\partial}{\partial x})\right]_{0}^{L} - 2\beta\mathcal{B}\int_{-h/2}^{h/2}\frac{\partial^{2}w}{\partial x^{2}}dz(\partial w)|_{0}^{L} + 2\beta\mathcal{B}\int_{0}^{L} \frac{\partial^{2}w}{\partial x^{2}}(\partial w)dx \\ &+ 2\beta\mathcal{A}\int_{0}^{L}(\frac{\partial^{2}w}{\partial x})\left|_{0}^{L} - 2\beta\mathcal{B}\int_{0}^{h/2}\frac{\partial^{2}w}{\partial x^{2}}dz(\partial w)|_{0}^{L} + 2\beta\mathcal{A}\int_{0}^{L}(\frac{\partial^{2}w}{\partial x^{2}})(\partial w)dx \\ &+ 2\beta\mathcal{A}\int_{0}^{L}(\frac{\partial^{2}w}{\partial x})(\frac{\partial}{\partial w})|_{0}^{L} - 2\beta\mathcal{A}\int_{0}^{h/2}\frac{\partial^{2}w}{\partial x^{2}}(\partial w)dz \\ &+ 2\beta\mathcal{A}\int_{0}^{h/2}\frac{\partial^{2}w}{\partial x^{2}}(\partial w)dx|_{0}^{L} - 2\beta\mathcal{A}(\frac{\partial^{2}w}{\partial$$

Similarly, the variation of kinetic energy and external works can be written as follows:

$$\delta T = \rho A \int_{0}^{L} \left[\frac{\partial u_{0}}{\partial t} \delta(\frac{\partial u_{0}}{\partial t}) + \frac{\partial w}{\partial t} \delta(\frac{\partial w}{\partial t}) \right] dx + \rho I \int_{0}^{L} \frac{\partial^{2} w}{\partial x \partial t} (\frac{\partial^{2} \delta w}{\partial x \partial t}) dx$$

$$\delta W = \int_{0}^{L} (F(x,t) \cdot \delta w) dx$$
(8)

In order to obtain the governing equations of the system, equations (7)–(9) are substituted into Hamilton's principle resulting in:

$$\rho A \frac{\partial^2 u}{\partial t} - \left(\lambda + 2\mu\right) A \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}\right) - \eta A \left(\frac{\partial^3 u}{\partial x^2 \partial t} + \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial x \partial t} + \frac{\partial w}{\partial x}\frac{\partial^3 w}{\partial x^2 \partial t}\right) = 0$$
(10)

$$\rho A \frac{\partial^2 w}{\partial t} + \left(\left(\lambda + 2\mu \right) I - 4\mu I^2 A \right) \frac{\partial^4 w}{\partial x^4} + \left(\eta I + \frac{\eta A}{2} \right) \frac{\partial^5 w}{\partial x^4 \partial t} - \left(\lambda + 2\mu \right) A \left(\frac{\partial^2 u}{\partial x^2} \frac{\partial w}{\partial x} + \frac{\partial u}{\partial x} \frac{\partial^2 w}{\partial x^2} + \frac{3}{2} \left(\frac{\partial w}{\partial x} \right)^2 \frac{\partial^2 w}{\partial x^2} \right) - \eta A \left(\frac{\partial^3 u}{\partial x^2 \partial t} \frac{\partial w}{\partial x} + 2 \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 u}{\partial x \partial t} + 3 \frac{\partial w}{\partial x} \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial x \partial t} + \left(\frac{\partial w}{\partial x} \right)^2 \frac{\partial^3 w}{\partial x^2 \partial t} \right) + \rho I \frac{\partial^4 w}{\partial x^2 \partial t^2} + 2 f b \frac{\partial^2}{\partial x^2} \left(\varphi \left(x, \frac{h}{2} \right) - \varphi \left(x, \frac{h}{2} \right) \right) - F(x, t) = 0$$

$$(11)$$

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$$\varepsilon b(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial z^2}) = 0 \tag{12}$$

Also, the following boundary conditions are obtained for a clamped-clamped nano-beam as follows:

$$\left(\lambda+2\mu\right)A\left(\frac{\partial u_0}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right)+\eta A\left(\frac{\partial^2 u_0}{\partial x \partial t}+\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x \partial t}\right)\Big|_{x=0,L}=0 \quad (13)$$

$$\left(-\left(\lambda+2\mu\right)I+4\mu I^{2} A\right)\frac{\partial^{3} w}{\partial x^{3}}+\left(-\eta I-\frac{\eta A}{2}\right)\frac{\partial^{4} w}{\partial x^{3} \partial t}+\left(\lambda+2\mu\right) A\left(\frac{\partial u_{0}}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^{2}\right)\left(\frac{\partial w}{\partial x}\right)$$

$$+\eta A\left(\frac{\partial^{2} u_{0}}{\partial x \partial t}\frac{\partial w}{\partial x}+\left(\frac{\partial w}{\partial x}\right)^{2}\frac{\partial^{2} w}{\partial x \partial t}\right)-\rho I\frac{\partial^{3} w}{\partial x^{2} \partial t}-2fb\int_{-h/2}^{h/2}\frac{\partial^{2} \varphi}{\partial x \partial z}dz\Big|_{x=0,L}=0 \text{ or } \delta w\Big|_{x=0,L}=0$$

$$(14)$$

$$\left(\left((\lambda+2\mu)I-4\mu l^2A\right)\frac{\partial^2 w}{\partial x^2}+\left(\frac{\eta A}{2}\right)\frac{\partial^3 w}{\partial x^2 \partial t}+\eta I\left(\frac{\partial^3 w}{\partial x^2 \partial t}\right)+2fb\int_{-h/2}^{h/2}\frac{\partial \varphi}{\partial z}dz\right)\Big|_{x=0,L}=0 \quad (15)$$

$$\varepsilon b \int_{-h/2}^{h/2} \frac{\partial \varphi}{\partial x} dz \Big|_{x=0,L} + \int_{0}^{L} (\varepsilon b \frac{\partial \varphi}{\partial z} + 2 f b \frac{\partial^2 w}{\partial x^2}) dx \Big|_{z=-h/2,h/2} = 0 \text{ or } \delta \varphi \Big|_{x=0,L} = 0$$
(16)

Since it is easier to solve with a dimensionless form of the equations, the nonlinear partial differential equation of motion can be rewritten in the dimensionless form by assuming as:

$$\hat{x} = \frac{x}{L}, \hat{u} = \frac{u}{h}, \hat{w} = \frac{w}{h}, s = \frac{L}{h}, \beta = \frac{Ah^2}{I}, \hat{t} = t\tau$$

$$\tau = \sqrt{\frac{(\lambda + 2\mu)I}{\rho AL^4}}, \eta_f = \frac{\eta\tau}{(\lambda + 2\mu)}, \xi = \frac{4\mu l^2 A}{(\lambda + 2\mu)I}, \eta_s = \frac{\eta A\tau}{2(\lambda + 2\mu)I}$$

$$\alpha = \frac{I}{AL^2}, \gamma = \frac{2 fbL^2}{h(\lambda + 2\mu)I}, \hat{F} = \frac{FL^4}{h(\lambda + 2\mu)I}, \vartheta = \frac{2 fb}{\varepsilon L}$$
(17)

Hence, the governing equations in the dimensionless case can be converted to the following form:

$$\frac{\partial^2 \hat{u}_0}{\partial \hat{t}^2} - \beta s \left(s \frac{\partial^2 \hat{u}_0}{\partial \hat{x}^2} + \frac{\partial \hat{w}}{\partial \hat{x}} \frac{\partial^2 \hat{w}}{\partial \hat{x}^2}\right) - \beta s \eta_f \left(s \frac{\partial^3 \hat{u}_0}{\partial \hat{x}^2 \partial \hat{t}} + \frac{\partial^2 \hat{w}}{\partial \hat{x}^2} \frac{\partial^2 \hat{w}}{\partial \hat{x} \partial \hat{t}} + \frac{\partial \hat{w}}{\partial \hat{x}} \frac{\partial^3 \hat{w}}{\partial \hat{x}^2 \partial \hat{t}}\right) = 0$$
(18)

$$\frac{\partial^{2} \widehat{w}}{\partial \widehat{t}^{2}} + (1+\xi) \frac{\partial^{4} \widehat{w}}{\partial \widehat{x}^{4}} + (\eta_{f} + \eta_{s}) \frac{\partial^{5} \widehat{w}}{\partial \widehat{x}^{4} \partial \widehat{t}} - \beta \left(s \frac{\partial \widehat{w}}{\partial \widehat{x}} \frac{\partial^{2} \widehat{u}_{0}}{\partial \widehat{x}^{2}} + s \frac{\partial \widehat{u}_{0}}{\partial \widehat{x}} \frac{\partial^{2} \widehat{w}}{\partial \widehat{x}^{2}} + \frac{3}{2} \frac{\partial^{2} \widehat{w}}{\partial \widehat{x}^{2}} \left(\frac{\partial \widehat{w}}{\partial \widehat{x}}\right)^{2}\right) - \beta \eta_{f} \left(s^{2} \frac{\partial \widehat{w}}{\partial \widehat{t}} \frac{\partial^{3} \widehat{u}_{0}}{\partial \widehat{x}^{2} \partial \widehat{t}} + 2s \frac{\partial^{2} \widehat{u}_{0}}{\partial \widehat{x} \partial \widehat{t}} \frac{\partial^{2} \widehat{w}}{\partial \widehat{x}^{2}} + 3 \frac{\partial^{2} \widehat{w}}{\partial \widehat{x} \partial \widehat{t}} \frac{\partial^{2} \widehat{w}}{\partial \widehat{x}^{2}} \frac{\partial^{3} \widehat{w}}{\partial \widehat{x}} + \frac{\partial^{3} \widehat{w}}{\partial \widehat{x}^{2} \partial \widehat{t}} \left(\frac{\partial \widehat{w}}{\partial \widehat{x}}\right)^{2}\right) + \alpha \frac{\partial^{4} \widehat{w}}{\partial \widehat{x}^{2} \partial \widehat{t}^{2}} + \gamma \frac{\partial^{2}}{\partial \widehat{x}^{2}} (\varphi(\widehat{x}, 1/2) - \varphi(\widehat{x}, -1/2)) - \widehat{F} = 0$$

$$(19)$$

$$\frac{\partial^2 \varphi}{\partial \hat{x}^2} + s^2 \frac{\partial^2 \varphi}{\partial \hat{z}^2} = 0$$
⁽²⁰⁾

Also, the following boundary conditions are obtained as follows:

$$\beta s(s\frac{\partial \hat{u}_0}{\partial \hat{x}} + \frac{1}{2}(\frac{\partial \hat{w}}{\partial \hat{x}})^2) + \beta s\eta_f(s\frac{\partial^2 \hat{u}_0}{\partial \hat{x}\partial \hat{t}} + \frac{\partial \hat{w}}{\partial \hat{x}}\frac{\partial^2 \hat{w}}{\partial \hat{x}\partial \hat{t}})\Big|_{\hat{x}=0,1} = 0 \text{ or } \delta \hat{u}_0\Big|_{\hat{x}=0,1} = 0$$

$$(21)$$

$$\beta s \left(\frac{\partial \hat{u}_{0}}{\partial \hat{t}} + \frac{1}{2} \left(\frac{\partial \hat{w}}{\partial \hat{x}}\right)^{2}\right) \left(\frac{\partial \hat{w}}{\partial \hat{x}}\right) + \beta \eta_{f} \left(s \left(\frac{\partial^{2} \hat{u}_{0}}{\partial \hat{x} \partial \hat{t}} \frac{\partial \hat{w}}{\partial \hat{x}}\right) + \left(\frac{\partial^{2} \hat{w}}{\partial \hat{x} \partial \hat{t}}\right) \left(\frac{\partial \hat{w}}{\partial \hat{x}}\right)^{2}\right) + (-1 + \xi) \frac{\partial^{3} \hat{w}}{\partial \hat{x}^{3}} - (\eta_{f} + \eta_{s}) \left(\frac{\partial^{4} \hat{w}}{\partial \hat{x}^{3} \partial \hat{t}}\right) - \alpha \left(\frac{\partial^{3} \hat{w}}{\partial \hat{x} \partial \hat{t}^{2}}\right) - \gamma \left(\frac{\partial}{\partial \hat{x}} \left(\varphi(\hat{x}, 1/2) - \varphi(\hat{x}, -1/2)\right)\right) \Big|_{\hat{x}=0,1} = 0 \text{ or } \delta \hat{w} \Big|_{\hat{x}=0,1} = 0$$
(22)



$$(1+\xi)\frac{\partial^{2}\hat{w}}{\partial\hat{x}^{2}} + \eta_{f}\left(\frac{\partial^{3}\hat{w}}{\partial\hat{x}^{2}\partial\hat{t}}\right) + \beta s^{2}\eta_{f}\left(\frac{\partial^{3}\hat{w}}{\partial\hat{x}^{2}\partial\hat{t}}\right) + \gamma\left(\left(\varphi(\hat{x},1/2) - \varphi(\hat{x},-1/2)\right)\right)\Big|_{\hat{x}=0,1} = 0$$

$$\text{or } \delta\left(\frac{\partial\hat{w}}{\partial\hat{x}}\right)\Big|_{\hat{x}=0,1} = 0$$
(23)

$$\left[\left(\frac{L}{h}\right)\frac{\partial\varphi}{\partial\hat{z}} + 9\frac{\partial^{2}\hat{w}}{\partial\hat{x}^{2}}\right]d\hat{x}\Big|_{\hat{z}=\frac{1}{2},-\frac{1}{2}} = 0 \quad \text{or} \quad \delta\varphi\Big|_{x=0,1} = 0$$
(24)

3. Solution method

To analyze the forced vibration of the nonlinear equations in (18) and (20), transverse, longitudinal displacements and electrical field are introduced as the following series expansions:

$$\mathbf{u}(\hat{x},\hat{t}) = \sum_{k=1}^{\infty} U_k(\hat{x}) \cdot r_k(\hat{t})$$
(25)

$$\mathbf{w}(\hat{x},\hat{t}) = \sum_{k=1}^{\infty} W_k(\hat{t}).q_k(\hat{t})$$
(26)

$$\varphi(\hat{x},\hat{z},\hat{t}) = \sum_{k=1}^{\infty} \varphi(\hat{x},\hat{z})q_k(\hat{t})$$
(27)

where $r_k(t)$ and $q_k(t)$ represent the k_{th} function for the longitudinal and transverse motions, respectively. Also, U_k and W_k are the k_{th} dimensionless eigenfunctions for the longitudinal and transverse motions of clamped- clamped boundary conditions that can be defined as:

$$U_k(\hat{x}) = \sin(k\pi\hat{x}) \tag{28}$$

$$W_k(\hat{x}) = \cosh(\alpha_k \hat{x}) - \cos(\alpha_k \hat{x}) - \frac{\cosh \alpha_k - \cos \alpha_k}{\sinh \alpha_k - \sin \alpha_k} (\sinh(\alpha_k \hat{x}) - \sin(\alpha_k \hat{x}))$$
(29)

where α_k is the k_{th} root of the transcendental equation. Only one mode considered (k = 1) here; then the first root of the transcendental equation is equal to $\alpha_1 = 4.733$.

By solving the Laplace equation, equation (20), with related boundary conditions in equation (24), the electrical potential function can be obtained as follows:

$$\varphi(\hat{x},\hat{z},\hat{t}) = \sum_{k=1}^{\infty} \frac{\int_{0}^{1} \mathcal{G}\frac{\partial^{2}\bar{w}}{\partial\hat{x}^{2}} \sin(\pi\hat{x})d\hat{x}}{(k\pi)\cosh\left(\frac{k\pi h}{2L}\right)} \sinh\left(\frac{k\pi h}{L}\hat{z}\right)\sin(k\pi\hat{x}) \cdot q(\hat{t})$$
(30)

By Substituting of the series expansions are given in equations (28) and (29) into the dimensionless nonlinear equations of motion (equations (18) and (19)), and using the Galerkin method, following reduced-order differential equations can be obtained:

$$\int_{0}^{1} (U^{2} d\hat{x}) \ddot{r}(\hat{t}) - \beta s \left[s \int_{0}^{1} (UU'' d\hat{x}) r(\hat{t}) + \int_{0}^{1} (UW'W''d\hat{x}) q(\hat{t})^{2} \right] -\beta \eta_{f} s \left[s \left(\int_{0}^{1} (UU'' d\hat{x}) \dot{r}(\hat{t}) + 2 \int_{0}^{1} (UW'W''d\hat{x}) q(\hat{t}) \dot{q}(\hat{t}) \right) \right] = 0$$
(31)

$$\begin{aligned} & \left(\int_{0}^{1} W^{2} d\bar{x}\right) \ddot{q}(\hat{t}) + (1+\xi) \left(\int_{0}^{1} W W^{(4)} d\bar{x}\right) q(\hat{t}) + (\eta_{f} + \eta_{s}) \left(\int_{0}^{1} W W^{(4)} d\bar{x}\right) \dot{q}(\hat{t}) \\ & -\beta \left[s \left[\left(\int_{0}^{1} W U'' W' d\bar{x}\right) r(\hat{t}) q(\hat{t}) + \left(\int_{0}^{1} W U' W'' d\bar{x}\right) r(\hat{t}) q(\hat{t}) + \frac{3}{2} \left(\int_{0}^{1} W W'^{2} W'' d\bar{x}\right) q(\hat{t})^{3} \right) \right] \\ & -\beta \eta_{f} \left[s \left[\left(2\int_{0}^{1} W U' W'' d\bar{x}\right) \dot{r}(\hat{t}) q(\hat{t}) + \left(\int_{0}^{1} W U'' W' d\bar{x}\right) \dot{r}(\hat{t}) q(\hat{t}) + 4 \left(\int_{0}^{1} W W' W'' W' d\bar{x}\right) q(\hat{t})^{2} \dot{q}(\hat{t}) \right) \right] \\ & + \alpha \left(\int_{0}^{1} W W'' d\bar{x}\right) \ddot{q}(\hat{t}) + \gamma \left(\int_{0}^{1} W \left(\frac{\partial^{2}}{\partial \bar{x}^{2}} \left(\varphi(\hat{x}, 1/2) - \varphi(\hat{x}, -1/2)\right)\right) d\bar{x} \right) q(\hat{t}) - \hat{F} \int_{0}^{1} W d\bar{x} = 0 \end{aligned}$$

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By solving the above integrals, equations (31) and (32) are only the function of time and the location terms are eliminated (the non-linear PDE equation is reduced to the ODE one). It is assumed that the effect of the nano-beam longitudinal inertia is ignored ($\ddot{r}(t) = \dot{r}(t) = 0$). Finally, by obtaining r(t) as function of q(t) from equation (31) and substituting it into equation (32), the governing motion equation along the transverse direction of the beam is extracted as follows:

$$Y_{1}\ddot{q}(\hat{t}) + Y_{2}q(\hat{t}) + Y_{3}q(\hat{t})^{3} + Y_{4}\dot{q}(\hat{t}) + Y_{5}q^{2}(\hat{t})\dot{q}(\hat{t}) - \hat{F}\int_{0}^{1}Wd\hat{x} = 0$$
(33)

In equation (33), the coefficients will be as follows:

$$Y_1 = \int_0^1 \left(\alpha W W'' + W^2 \right) d\hat{x}$$
(34)

$$Y_{2} = \int_{0}^{1} \left[(1+\xi)WW^{(4)} + W\gamma \left[\frac{\partial^{2}}{\partial \hat{x}^{2}} \left(\varphi(\hat{x}, 1/2) - \varphi(\hat{x}, -1/2) \right) \right] \right] d\hat{x}$$
(35)

$$Y_{3} = \int_{0}^{1} \left(\frac{3}{2}\beta W''(W')^{2}W + \beta W''(W')^{2}W + \beta W'(W'')^{2}W\frac{U'}{U''}\right)d\hat{x}$$
(36)

$$Y_{4} = \int_{0}^{1} (\eta_{f} + \eta_{s}) (WW^{(4)}) d\hat{x}$$
(37)

$$Y_{5} = \int_{0}^{1} \left(4\eta_{f} \beta \left(W' \right)^{2} W'' W + 2\eta_{f} \beta \left(W' \right)^{2} W'' W + 2\eta_{f} \beta \left(W'' \right)^{2} W' W \frac{U}{U''} \right) d\bar{x}$$
(38)

For the vibration analysis of the governing equation, Eq. (33), which includes piezoelectric, viscoelastic coefficients, and the size-effect, first, the status of the applied force must be determined. Thus, the governing equation could be rewritten as follows:

$$\ddot{q}(\hat{t}) + \gamma_1 q(\hat{t}) + \gamma_2 q(\hat{t})^3 + \gamma_3 \dot{q}(\hat{t})(1 + q^2(\hat{t})) - F = 0$$
(39)

In the above equation, γ_1 is the sum of the size-effect coefficient and the piezoelectric coefficient and γ_3 is the viscoelastic coefficient. In the next section, the method of multiple scales has been used to the solution of equation (39) for free and forced vibration.

3.1. Nonlinear forced vibration analysis

The method of multiple scales has been used for obtaining the nonlinear forced vibration of the system. Here, it is considered that the excitation is applied an external force to the structure and it is assumed harmonic as follows:

$$F = f \cos\left(\Omega \hat{t}\right) \tag{40}$$

where *f* and Ω are amplitude and frequency of the external excitation, respectively. Approximate solution of equation (39) as a second-order expansion in terms of the positive and small parameter ε , is as follows:

$$q(T_0, T_1, T_2) = q_0(T_0, T_1, T_2, ...) + \varepsilon q_1(T_0, T_1, T_2, ...) + \varepsilon^2 q_2(T_0, T_1, T_2, ...)$$
(41)

where q_0 , q_1 and q_2 are three unknown functions. To obtain a second-order uniform expansion by using the method of multiple scales, we need the three-time scales T_0 , T_1 , and T_2 as follow:

$$T_n = \varepsilon^n \hat{t}, n = 0, 1, 2, \dots$$
 (42)

In the above terms of the time scales, the time derivatives become:

$$\frac{d}{dt} = \frac{\partial}{\partial T_0} + \varepsilon \left(\frac{\partial}{\partial T_1}\right) + \varepsilon^2 \left(\frac{\partial}{\partial T_2}\right)$$

$$\frac{d^2}{dt^2} = \left[\frac{\partial^2}{\partial T_0^2} + \varepsilon \left(\frac{\partial^2}{\partial T_0 \partial T_1}\right) + \varepsilon^2 \left(\frac{\partial^2}{\partial T_0 \partial T_2}\right)\right] + \varepsilon \left[\frac{\partial^2}{\partial T_1 \partial T_0} + \varepsilon \left(\frac{\partial^2}{\partial T_1^2}\right) + \varepsilon^2 \left(\frac{\partial^2}{\partial T_1 \partial T_2}\right)\right] + \varepsilon^2 \left[\frac{\partial^2}{\partial T_2 \partial T_1} + \varepsilon \left(\frac{\partial^2}{\partial T_2^2}\right)\right]$$
(43)



The excitation frequency is $\Omega = \omega_0 + \sigma \varepsilon^2$ where ω_0 is the linear natural frequency of the system and σ is the detuning parameter. By substituting equations (43) and (41) into the equation of motion (equation 39) and equate coefficients of like powers of ε^0 , ε^1 , ε^2 to zero, the following equations are obtained:

$$O(\varepsilon^{0}): \gamma_{1}q_{0}(T_{0}, T_{1}, T_{2}) + \frac{\partial^{2}}{\partial T_{0}^{2}}q_{0}(T_{0}, T_{1}, T_{2}) = 0$$
(44a)

$$O(\varepsilon^{1}):\gamma_{1}q_{1}(T_{0},T_{1},T_{2})+\frac{\partial^{2}}{\partial T_{0}^{2}}q_{1}(T_{0},T_{1},T_{2})=-2\frac{\partial^{2}}{\partial T_{0}\partial T_{1}}q_{0}(T_{0},T_{1},T_{2})$$
(44b)

$$O(\varepsilon^{2}): \gamma_{1}q_{2}(T_{0},T_{1},T_{2}) + \frac{\partial^{2}}{\partial T_{0}^{2}}q_{2}(T_{0},T_{1},T_{2}) = -\gamma_{3}(\frac{\partial}{\partial T_{0}}q_{0}(T_{0},T_{1},T_{2}))q_{0}^{2}(T_{0},T_{1},T_{2})$$

$$-2\frac{\partial^{2}}{\partial T_{0}\partial T_{1}}q_{1}(T_{0},T_{1},T_{2}) - 2(\frac{\partial^{2}}{\partial T_{0}\partial T_{2}}q_{0}(T_{0},T_{1},T_{2})) - \gamma_{2}q_{0}^{3}(T_{0},T_{1},T_{2})$$

$$-\frac{\partial^{2}}{\partial T_{1}^{2}}q_{0}(T_{0},T_{1},T_{2}) - \frac{\partial^{2}}{\partial T_{0}}q_{0}(T_{0},T_{1},T_{2}) - f\cos(\Omega t)$$
(44c)

The general solution for the equation (44.a) is as follows:

$$q_0(T_0, T_1, T_2) = A(T_1, T_2)e^{i\sqrt{(\gamma_1)}T_0} + \overline{A}(T_1, T_2)e^{-i\sqrt{(\gamma_1)}T_0}$$
(45)

where $A(T_1, T_2)$, $\overline{A}(T_1, T_2)$ are the complex conjugate functions. By replacing equation (45) in (44.a) following equation is obtained:

$$\gamma_{1}q_{1}(T_{0},T_{1},T_{2}) + \frac{\partial^{2}}{\partial T_{0}^{2}}q_{1}(T_{0},T_{1},T_{2}) = 2i((\gamma_{1})(-\frac{\partial}{\partial T_{1}}A(T_{1},T_{2})e^{i\sqrt{(\gamma_{1})}T_{0}} + \frac{\partial}{\partial T_{1}}\overline{A}(T_{1},T_{2})e^{-i\sqrt{(\gamma_{1})}T_{0}})$$
(46)

By eliminating the secular and small-divisor terms in equation (46), and solving it, we can write:

$$-2i((\gamma_1)(\frac{\partial}{\partial T_1}A(T_1,T_2)) = 0 \to A(T_1,T_2) = A(T_2), \overline{A}(T_1,T_2) = \overline{A}(T_2)$$

$$(47)$$

The general solution for the equation (44.b) by eliminating the secular term is as follows:

$$q_1(T_0, T_1, T_2) = B(T_1, T_2)e^{i\sqrt{(\gamma_1)}T_0} + \overline{B}(T_1, T_2)e^{-i\sqrt{(\gamma_1)}T_0}$$
(48)

Finally, after substituting the equations (43) and (45) into equation (44.c), to eliminate the secular terms from equation (44.c), we obtain:

$$i(\sqrt{\gamma_1})\left(\gamma_3 A^2(T_2)\overline{A}(T_2) + \gamma_3 A(T_2) + 2\frac{\partial}{\partial T_2} A(T_2) + 2\frac{\partial}{\partial T_1} B(T_1, T_2)\right) + \frac{\partial^2}{\partial T_1^2} A(T_2) - \frac{fe^{iT_2\sigma}}{2} + 3\gamma_2 A^2(T_2)\overline{A}(T_2) = 0$$

$$(49)$$

Since $A(T_1, T_2) = A(T_2)$, it follows from equation (49) that $B(T_1, T_2) = B(T_2)$ and $\partial B(T_1, T_2) / \partial T_1 = 0$, therefore q_0, q_1 have exactly the same form. Consequently taking $A(T_2)$ in polar form as $A(T_2) = 1/2 a(T_2) \exp(i\beta(T_2))$ and then separating the result into real and imaginary parts, the following equations are obtained.

$$\dot{a} = \frac{-1}{\sqrt{\gamma_1}} \left(\frac{-f\sin(\sigma T_2 - \beta)}{2} + \frac{a\sqrt{\gamma_1}\gamma_3}{2} + \frac{a^3\sqrt{\gamma_1}\gamma_3}{8} \right)$$
(50)

$$a\dot{\beta} = \frac{1}{\sqrt{\gamma_1}} (\frac{3}{8}a^3\gamma_2 - \frac{f}{2}\cos(\sigma T_2 - \beta))$$
(51)

By substituting $\Psi = \sigma T_2 - \beta$ in the above equations, Eq. (50) and (51) can be rewritten as follows:

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$$\dot{a} = \frac{-1}{\sqrt{\gamma_1}} \left(\frac{-f\sin(\Psi)}{2} + \frac{a\sqrt{\gamma_1}\gamma_3}{2} + \frac{a^3\sqrt{\gamma_1}\gamma_3}{8} \right)$$
(52)

$$a\dot{\Psi} = a\sigma - \frac{1}{\sqrt{\gamma_1}} \left(\frac{3}{8}a^3\gamma_2 - \frac{f}{2}(\cos(\Psi))\right)$$
(53)

Corresponds to the singular point of the system and the steady-state motion when $\dot{a} = \dot{\Psi} = 0$, the frequency-response equation can be written as follows:

$$\sigma = \frac{1}{a\sqrt{\gamma_1}} \left(\frac{3}{8}a^3\gamma_2 \pm \frac{f}{2}\left(\sqrt{1 - \left(\frac{1}{f}\left(a\sqrt{\gamma_1}\gamma_3 + \frac{a^3\sqrt{\gamma_1}\gamma_3}{4}\right)\right)^2}\right)\right)$$
(54)

Substituting equation (45) and equation (48) into equation (41), the dynamic response of nano-beam is obtained as follows:

$$q(\hat{t}) = a\cos(\Omega \hat{t} + \Psi) + O(\varepsilon^2)$$
(55)

3.2. Nonlinear free vibration analysis

By replacing F=0 in relation (39) could investigate the nonlinear free vibration. By replacing f = 0 in the Eqs. (50) and (51), the following equations are obtained:

$$\dot{a} = \frac{-1}{\sqrt{\gamma_1}} \left(\frac{a\sqrt{\gamma_1}\gamma_3}{2} + \frac{a^3\sqrt{\gamma_1}\gamma_3}{8} \right)$$
(56)

$$\dot{\beta} = \frac{1}{\sqrt{\gamma_1}} (\frac{3}{8}a^2 \gamma_2) \tag{57}$$

The solutions of Eqs. (56) and (57) are as follows:

$$a = \pm \frac{2}{\sqrt{4a_0 e^{\gamma_3 T_2} - 1}} \tag{58}$$

$$\beta = -\frac{3\gamma_2}{2\gamma_1}T_2 + \frac{3\gamma_2\ln(4a_0e^{\gamma_3T_2} - 1)}{2\gamma_1\gamma_3} + \beta_0$$
(59)

Finally, by the solution of equation (39), the time response for the free vibration of the nano-beam is as follows:

$$q(\hat{t}) = \pm \frac{1}{\sqrt{4a_0 e^{\gamma_3 T_2} - 1}} \cos(\sqrt{\gamma_1} t - \frac{3\gamma_2}{2\gamma_1} T_2 + \frac{3\gamma_2 \ln(4a_0 e^{\gamma_3 T_2} - 1)}{2\gamma_1 \gamma_3} + \beta_0) + O(\varepsilon^2)$$
(60)

4. Results and discussion

In this section, the numerical results of the free and forced vibration of a piezoelectric/viscoelastic nano-beam made of polyvinylidene fluoride (PVDF) film [32] in the clamped-clamped boundary conditions are presented. The nano-beam dimension is: $100nm \times 5nm \times 5nm$. The properties of PVDF film used are: E=238.24 GPa, $\rho=1.74$ g/cm³. First, the results of forced vibration are investigated, considering the effects of different factors on the frequency response of the nano-beam excited by the force *f* and the frequency of Ω ; then, the time response of the forced vibration of the nano-beam is presented. Finally, the results of the free vibration are investigated and the effects of various factors on the natural frequency are investigated as well.

4.1. Validation of the results

The nonlinear forced vibration of isotropic viscoelastic/piezoelectric Euler-Bernoulli nano-beam investigated for the first time in this research, so there aren't any theoretical or experimental results for validation of the results. Therefore, in this part, validation of the results has been done in the special case for classic continuum theory (l=0, $\eta=0$). The geometrical and mechanical properties of the beam have been utilized according to Hoseini et al. [33] and the comparison between the results is shown in figure 2. Figure 2 shows the results of this paper and ref. [33] have a supreme agreement.





Fig. 2. Validation of present work results with Ref. [33] in special case $l=0, \eta=0$.



Fig. 3. Frequency-response curve of the Euler- Bernoulli viscoelastic/piezoelectric nano-beam for different γ_1 coefficient

4.2. Nonlinear forced vibration results

4.2.1. Effect of size-effect and piezoelectric coefficients on the nano-beam frequency response

As noted in the preceding sections, the effects of size-effect and piezoelectric coefficients are applied by the coefficient of γ_i . The simultaneous effect of size and piezoelectric coefficient on the nonlinear vibrational amplitude is plotted in Fig. 3. As can be seen, by increasing the effect of the size on the piezoelectric coefficient, the frequency response amplitude is decreased and the graph tends to the right; in other words, it shows the hardening behavior. In fact, this behavior indicates that by considering the effect of the size of piezoelectric material on the consistent couple stress theory equations, the nano-beam becomes more hardened, resulting in the reduced frequency response of the system. The important conclusion that can be inferred from this graph is that the size-effect and the piezoelectric-effect play an important role in the frequency response of the system. As these coefficients increase, that is, the effects of nano-scale are considered in structural equations, the system vibration behavior gets closer to reality.

4.2.2. The effect of viscoelastic coefficient on the nano-beam frequency response

As we know, the viscoelastic coefficient causes the damping of the system. Figure 4 shows the effect of the damping coefficient on the frequency response of the system. It is quite clear in this figure that as the damping factor is increased, the amplitude of the frequency response is decreased; as a result, with exciting the nano-beam with a higher damping coefficient, the frequency response is decreased; then with exciting nano-beam with a higher damping coefficient, the frequency is damped and the body vibration disappears; however, at lower coefficients, it takes a longer time for the nano-beam vibration to damp.

4.2.3. The effect of excitation force amplitude on the nano-beam frequency response

Figure 5 shows the effect of excitation force amplitude on the frequency response. As the amplitude of the force is increased, the output frequency is raised and the graph deviates from the axis $\sigma = 0$, exhibiting a hardening behavior; in fact, the higher the amplitude, the greater the output response.

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Fig. 4. Frequency-response curve of the Euler- Bernoulli viscoelastic/piezoelectric nano-beam for different damping coefficients



Fig. 5. Frequency-response curve of the Euler-Bernoulli viscoelastic/piezoelectric nano-beam for different values of forcing amplitude



Fig. 6. Time response curve of nonlinear forced vibration of the Euler- Bernoulli viscoelastic/piezoelectric nano-beam at different force amplitudes and $\Omega = 2$





Fig. 7. Time response curve of nonlinear forced vibration of the Euler-Bernoulli viscoelastic/piezoelectric nano-beam at different frequency and f = 5



Fig. 8. Time response curve of nonlinear free vibration of the Euler- Bernoulli viscoelastic/piezoelectric nano-beam for different values of damping coefficient

4.2.4. Time response of nano-beam excited with external force

The time response of the forced vibration of the nano-beam is shown in Figures 6 and 7. Figure 6 shows the time response of the forced vibration of the nano-beam at four different force amplitudes (f) and the constant frequency of Ω = 2. As shown in Fig. 6, the vibrational response increases as the amplitude of the applied force is raised. As shown in Fig. 7, for 3 different frequencies at a constant force amplitude of f = 5, the time response of the nonlinear vibrations of the nano-beam is plotted. As shown, when the frequency is increased, the number of nano-beam vibrations is raised over a given period. It should be noted that at higher frequencies, the piezoelectric output response is increased as a larger number of molecules contribute to the current generation in exchange for more dipole body vibration.

4.3. Nonlinear-free vibration results

4.3.1. Effect of damping coefficient on the time response of nano-beam

Figure 8 shows the nonlinear time response of the free vibrations of the nano-beam; as shown in Fig. 8, the higher the damping coefficient, the shorter the time taken for the frequency of the system to be damped; the vibrational response disappears too. It should be noted that unlike the forced response in figures 6 and 7, the response will be eliminated despite the viscosity effect.



Fig. 9. Variation of natural frequency for different values of size-effect coefficient



Fig. 10. Variation of natural frequency for different values of nano-beam length

4.3.2. Effect of size coefficient on the natural frequency of the nano-beam

As shown in Fig. 9, the nonlinear and linear natural frequency of the nano-beam against changes in size-effect is plotted. As can be seen, with increasing the size-effect coefficient or decreasing h/l, the natural frequency of the nano-beam is raised, indicating the direct effect of the size-effect coefficient on the nano-beam hardening. The natural frequency of nano-beams in both linear and nonlinear states is also investigated, as shown in Fig. 9, evidently showing that frequency in the nonlinear state is more than that in the linear one.

4.3.3. The effect of the length of nano-beam on the natural frequency

The nonlinear natural frequency of variation versus the length of nano-beam is shown in Figure 10. As can be seen, increasing the length of the nanobeam leads to a decrease in the natural frequency of the nano-beam. This result is according to the natural property of nanobeam in the vibration that by increasing the length of nanobeam, the stability of nanobeam decrease and the natural frequency decreases too.

5. Conclusion

In this paper, the free and forced vibration of the isotropic piezoelectric/viscoelastic Euler-Bernoulli nano-beam was investigated using the consistent couple stress theory. Hamilton's principle was applied to obtain the governing equations and boundary conditions. The Galerkin method was also used to transforms the PDE equations into ODE ones in order to solve motion equations; then, the multiple-scale method was carried out for solving equations. It should be noted that the piezoelectric nano-beam vibration analysis was studied only in the case of the direct piezoelectric effect. The frequency response of a piezoelectric/viscoelastic nano-beam for different damping coefficients, size-effect coefficients and various values of forcing function amplitude showed. The results showed that when the damping coefficient increases,



the peak amplitude of response decreases due to nonlinearity. Also, with increases of size-effect and piezoelectric coefficients, the hardening phenomena occurred in nano-beam. Finally, the time response curve for nonlinear forced and free vibration were drawn and discussed.

Authors Contribution

The manuscript was written through the contribution of all authors. All authors discussed the results, reviewed and approved the final version of the manuscript.

Conflict of Interest

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