Journal of
Applied and Computational Mechanics

Research Paper

Numerical Solution of Time Fractional Cable Equation via the Sinc-Bernoulli Collocation Method

Nasrin Moshtaghi[1] , Abbas Saadatmandi[2]

[1] Department of Applied Mathematics, Faculty of Mathematical Sciences, University of Kashan, Kashan 87317-53153, Iran, Email: nasrin.moshtaghi@gmail.com

[2] Department of Applied Mathematics, Faculty of Mathematical Sciences, University of Kashan, Kashan 87317-53153, Iran, Email: saadatmandi@kashanu.ac.ir

Abstract. An important equation usually used in modeling neuronal dynamics is cable equation. In this work, a numerical method for the fractional cable equation which involves two Riemann-Liouville fractional derivatives is proposed. Our computational technique is based on collocation idea where a combination of Bernoulli polynomials and Sinc functions are used to approximate the solution to this problem. The constructed approximation by our method convert the fractional cable equation into a set of algebraic equations. Also, we provide two numerical examples to confirm the accuracy and effectiveness of the present method.

Keywords: Fractional cable equation, Bernoulli polynomials, Riemann-Liouville fractional derivative, Sinc function, Numerical solution.

1. Introduction

During the last few decades, a lot of attention has been paid to the fractional differential equations (FDEs) because of their extensive engineering applications. For example, these equations frequently appear in modeling many areas of fluid mechanics, viscoelasticity, biology, pharmacy, and control systems [16, 24]. Today, different numerical methods have been used to solve FDEs (see for example [2, 3, 4, 5, 13, 24, 26, 27, 28, 32-34] and references therein).

The standard model that explains electrodiffusion of ions in nerve cells is given by the following Nernst–Planck equation [17],

$$\frac{\partial C_k}{\partial t} = D_k \frac{\partial^2 C_k}{\partial x^2} + D_k \frac{Fz_k}{RT} \frac{\partial}{\partial x}\left(C_k \frac{\partial V_m}{\partial x}\right) - \frac{4}{dFz_k} i_{m,k},$$

where C_k is the concentration of ionic species with diffusivity D_k and charge z_k. Also, V_i is the membrane voltage. We refer to [17] for more details. If we consider slowly varying ionic concentrations along the axial direction, $\partial C_k / \partial t \approx 0$, then the standard cable equation (SCE) is obtained. To cover the anomalous diffusion in the movement of the ions, SCE needs to be modified. In recent years, to model anomalous electrodiffusion of ions in spiny dendrites, the fractional cable equation (FCE) from the fractional Nernst-Planck equation is derived (see [17, 36] and references therein).

In this study, we consider the following FCE [35]:

$$\frac{\partial u(x,t)}{\partial t} = D_t^{1-\gamma_1}\left(\kappa \frac{\partial^2 u(x,t)}{\partial x^2}\right) - \mu^2 D_t^{1-\gamma_2} u(x,t) + g(x,t), \qquad (x,t) \in \Omega \times [0,T], \tag{1}$$

with initial and boundary conditions

$$u(x,0) = \theta(x), \qquad x \in \Omega, \tag{2}$$

$$u(x,t) = 0, \qquad (x,t) \in \partial\Omega \times [0,T]. \tag{3}$$

Here $\Omega = (0,L); 0 < \gamma_1, \gamma_2 < 1; \kappa > 0$ and μ^2 are constants. Also, $g(x,t)$ is a given function and $u(x,t)$ is an unknown function. In Equation (1) the fractional derivatives are Riemann-Liouville type and are defined by:

Definition 1.1. Suppose that $u(t)$ be in $C[a,b]$ and $n-1 < \gamma < n$, then the Riemann-Liouville fractional derivative of order γ is defined by [23]

$$D_t^{\gamma}u(t) = \frac{1}{\Gamma(n-\gamma)}\frac{d^n}{dt^n}\int_0^t (t-s)^{n-\gamma-1}u(s)ds, \qquad\qquad t \in [0,T],$$

where, Γ denoting the Gamma function and we have [16, 23]

$$D_t^{\gamma}(\tau_1 u_1(t) + \tau_2 u_2(t)) = \tau_1 D_t^{\gamma}u_1(t) + \tau_2 D_t^{\gamma}u_2(t), \qquad (\tau_1 \text{ and } \tau_2 \text{ are constants}), \qquad\qquad (4)$$

$$D_t^{\gamma}t^n = \frac{\Gamma(n+1)}{\Gamma(n+1-\gamma)}t^{n-\gamma}, \qquad n > -1. \qquad\qquad (5)$$

In recent years, a lot of numerical methods to solve problem (1)-(3) have been devised such as homotopy analysis method [12], implicit numerical methods [19], finite element method [35], implicit compact difference scheme [11], Galerkin finite element method [36], finite difference/ Legendre spectral schema [18], meshless method [8], explicit numerical method [25], and discontinuous Galerkin finite element method [33]. We also refer the interested reader to [21, 31].

The main purpose of this work is to solve problem (1)-(3) by the Sinc-Bernoulli collocation method. In our method, $u(x,t)$ is approximated by the Bernoulli polynomials in time direction and the Sinc functions in space direction. Our method has the advantage of converting the solution of FCE given (1)-(3) into the solution of algebraic equations. Therefore, the computation becomes very simple and the corresponding algorithm is computer-oriented. Numerical methods related to the Sinc functions was developed as an efficient method in [20, 30]. These methods are widely used for solving many problems arising in applied sciences. Also, these methods are used as an efficient and effective tool for solving fractional or ordinary differential equations, the main reason being two: first, their desirable behavior towards singularity problems associated with the simple implementation of the method, and second, because of their exponential convergence rate [20, 30]. For some recently published papers on Sinc methods, we refer the interested reader to [1, 6, 9, 22, 23, 27, 28, 29] and references therein.

This paper is organized as follows: The next section is devoted to the some basic definitions and results of Bernoulli polynomials and Sinc functions. In Section 3, we construct the Sinc-Bernoulli collocation method for solving the problem (1)-(3). In Section 4, we obtain the error bound. Numerical simulations are reported in Section 5.

2. Mathematical Preliminaries

2.1 Sinc functions

The Sinc function is defined by [30]

$$\text{Sinc}(x) = \frac{\sin(\pi x)}{\pi x}, \quad x \in \mathbb{R}.$$

Also, for a given $h > 0$ and $i = 0, \pm 1, \pm 2, \ldots$, the i th Sinc functions are defined as

$$S(i,h)(x) = \text{Sinc}\left(\frac{x - ih}{h}\right).$$

Let $h > 0$. Then $g(x)$, which is defined on $\mathbb{R}$, has the cardinal series representation of the form

$$C(g,h)(x) = \sum_{i=-\infty}^{\infty} g_i S(i,h)(x), \ g_i = g(ih),$$

whenever this series converges. $C(g,h)(x)$ is called the whittaker cardinal expansion of g and its properties have been broadly studied in [20]. Our aim in this paper, is to construct the approximation over Ω. For this aim, we use the conformal one-to-one map

$$\phi(z) = \log\left(\frac{z}{L-z}\right),$$

which maps $D = \left\{z \in \mathbb{C}, \left|\arg(z/L-z)\right| < d \leq \pi/2\right\}$, onto the strip

$$D_d = \left\{z \in \mathbb{C} : \left|\xi(z)\right| < d\right\}.$$

Note that, $\phi(\Omega) = \mathbb{R}$, $\phi(0) = -\infty$ and $\phi(L) = \infty$. The Sinc basis functions on $(0,L)$ are defined as

$$S_i(x) = S(i,h) o \phi(x) = \text{Sinc}\left(\frac{\phi(x) - ih}{h}\right).$$

Here $S(i,h) o \phi(x)$ is defined by $S(i,h)(\phi(x))$.

Also, for $h > 0$, a set of Sinc points x_r on Ω is defined by

$$\mathbf{X}_r = \phi^{-1}(rh) = \frac{L \exp(rh)}{1 + \exp(rh)}, \qquad r = 0, \pm 1, \pm 2, \ldots. \qquad\qquad (6)$$

Definition 2.1. ([30]**).** The space $L_{\alpha}(D)$ is the family of all analytic functions u in D, such that

$$|u(x)| \leq C_1 \frac{|\rho(x)|^\alpha}{[1+|\rho(x)|]^{2\alpha}}, \qquad\qquad x \in D, \ 0 < \alpha \leq 1,$$

where C_1 is a constant and $\rho(x) = e^{\phi(x)}$.

The following theorem tells us that Sinc interpolation on $L_\alpha(D)$ converge exponentially.

Theorem 2.2. ([30]). Let $u(x)$ be in $L_\alpha(D)$, $0 < \alpha \leq 1$ and $d > 0$, let m be a positive integer, and let $h = \sqrt{\pi d / \alpha m}$, then there exists a constant C_2, which is independent of m such that

$$\sup_{x \in \Omega} \left| u(x) - \sum_{i=-m}^{m} u(x_i) S_i(x) \right| \leq C_2 \exp\left(\sqrt{-\pi d \alpha m}\right).$$

Also, we need the following relationships [30].

$$\delta_{i,r}^{(0)} = [S_i(x)]_{x=x_r} = \begin{cases} 1, & i = r, \\ 0, & i \neq r, \end{cases} \tag{7}$$

$$\delta_{i,r}^{(1)} = \frac{d}{d\phi}[S_i(x)]_{x=x_r} = \frac{1}{h}\begin{cases} 0, & i = r, \\ \dfrac{(-1)^{r-i}}{r-i}, & i \neq r, \end{cases} \tag{8}$$

$$\delta_{i,r}^{(2)} = \frac{d^2}{d\phi^2}[S_i(x)]_{x=x_r} = \frac{1}{h^2}\begin{cases} \dfrac{-\pi^2}{3}, & i = r, \\ \dfrac{-2(-1)^{r-i}}{(r-i)^2}, & i \neq r. \end{cases} \tag{9}$$

2.2 The shifted Bernoulli polynomials

The Bernoulli basis polynomials of order i are defined as [15]

$$\beta_i(x) = \sum_{l=0}^{i} \frac{i!}{l!(i-l)!} B_l x^{i-l}. \qquad x \in [0,1],$$

Here, $B_l = \beta_l(0)$, $l = 0,1,2,\ldots,i$ are called the Bernoulli numbers. These numbers can be obtained by using the generating function

$$\frac{x}{e^x - 1} = \sum_{l=0}^{\infty} B_l \frac{x^l}{l!},$$

where $B_{2l+1} = 0$ for $l \in \mathbb{N}$ (for more details see [15]). As mentioned in [14], the Bernoulli polynomials belong to the space of $L^2[0,1]$ and $Y = span\{\beta_0(x),\ldots,\beta_n(x)\}$ is a complete subspace of $L^2[0,1]$. Now suppose that $f(x) \in L^2[0,1]$, since Y is a finite subspace of $L^2[0,1]$, according to the best approximation, there exist the unique coefficients $\{c_i\}_{i=0}^{n}$ such that [14]

$$f(x) \approx f_n(x) = \sum_{i=0}^{n} c_i \beta_i(x). \tag{10}$$

In fact, $f_n(x)$ is the best approximation to f out of Y.

In the following theorem, an explicit formula for Bernoulli numbers is given. This formula specifies the relationship between Bernoulli numbers and Striling numbers.

Theorem 2.3. ([10]). For $l \in \mathbb{N}$, the Bernoulli numbers B_{2l} can be computed by

$$B_{2l} = 1 + \sum_{j=1}^{2l-1} \frac{S(1+2l, j+1)S(2l, 2l-j)}{\dfrac{(2l)!}{j!(2l-j)!}} - \frac{2l}{2l+1}\sum_{j=1}^{2l} \frac{S(2l,j)S(1+2l, 2l+1-j)}{\dfrac{(2l)!}{(j-1)!(2l+1-j)!}}$$

where

$$S(r,s) = \frac{1}{s!}\sum_{k=1}^{s}(-1)^{s-k}\frac{s!}{k!(s-k)!}k^r, \qquad 1 \leq s \leq r,$$

are Striling numbers of the second kind.

To apply the Bernoulli polynomials $\beta_i(x)$ on $[0,T]$, we use the change of variable $t = xT$. Then the shifted Bernoulli polynomials $\tilde{\beta}_i(t) = \beta_i(t/T)$, for $t \in [0,T]$, are obtained as:

$$\tilde{\beta}_i(t) = \sum_{l=0}^{i} \frac{i!}{l!(i-l)!}\left(\frac{1}{T}\right)^{i-l} B_l t^{i-l}. \tag{11}$$

Now, we obtain the Riemann-Liouville fractional derivative of $\tilde{\beta}_i(t)$.

Theorem 2.4. Assume that $\tilde{\beta}_i(t)$ is a shifted Bernoulli polynomial of degree i as (11) and also let $\gamma > 0$, then

$$D_t^\gamma \tilde{\beta}_i(t) = \sum_{l=0}^{i} Z_{il}^{(\gamma)} t^{i-l-\gamma}, \qquad i = 0,1,\ldots,$$

where

$$Z_{il}^{(\gamma)} = \frac{i!}{l!\,\Gamma(1+i-l-\gamma)}\left(\frac{1}{T}\right)^{i-l} B_l.$$

Proof. Using Equations (4), (5) and (11), the proof is straightforward.

It is worth to mention here that, the ordinary first-order derivative of $\tilde{\beta}_i(t)$ can be determined as

$$\frac{d\tilde{\beta}_i(t)}{dt} = \sum_{l=1}^{i} Z_{il}^{(1)} t^{i-l-1}, \qquad i = 1,2,\ldots. \tag{12}$$

3. The Sinc-Bernoulli Collocation Method

To solve the FCE as given in (1)-(3), we use $(n+1)$ terms of shifted Bernoulli polynomials, in time, and $(2m+1)$ terms of Sinc functions, in space, to approximate $u(x,t)$. Thus, we have

$$u(x,t) \approx u_{m,n}(x,t) = \sum_{i=-m}^{m} \sum_{j=0}^{n} u_{ij} S_i(x) \tilde{\beta}_j(t), \tag{13}$$

where $\{u_{ij}\}$ are $(n+1)(2m+1)$ unknown coefficients. Note that $u_{m,n}(x,t)$ satisfies the boundary conditions in (3), since when x tend to 0 and L then $S_i(x)$ tend to zero.

Theorem 3.1. Let $u(x,t)$ is approximated by $u_{m,n}(x,t)$ as (13). Also, let $0 < \gamma_1, \gamma_2 < 1$ and x_r be Sinc points. Then we have

$$\frac{\partial u_{m,n}(x_r,t)}{\partial t} = \sum_{j=1}^{n} \sum_{k=1}^{j} u_{rj} Z_{jk}^{(1)} t^{j-k-1},$$

$$D_t^{1-\gamma_2} u_{m,n}(x_r,t) = \sum_{j=0}^{n} \sum_{k=0}^{j} u_{rj} Z_{jk}^{(1-\gamma_2)} t^{j-k-1+\gamma_2},$$

$$D_t^{1-\gamma_1}\left(\frac{\partial^2 u_{m,n}(x_r,t)}{\partial x^2}\right) = \sum_{i=-m}^{m} \sum_{j=0}^{n} \sum_{k=0}^{j} u_{ij} E_{ir} Z_{jk}^{(1-\gamma_1)} t^{j+\gamma_1-1-k},$$

where $E_{ir} = \phi''(x_r)\delta_{i,r}^{(1)} + (\phi'(x_r))^2 \delta_{i,r}^{(2)}$.

Proof. Using Equations (7), (12) and (13), we obtain

$$\frac{\partial u_{m,n}(x_r,t)}{\partial t} = \sum_{i=-m}^{m} \sum_{j=0}^{n} u_{ij} \delta_{ir}^{(0)} \frac{d\tilde{\beta}_j(t)}{dt}$$

$$= \sum_{j=1}^{n} \sum_{k=1}^{j} u_{rj} Z_{jk}^{(1)} t^{j-k-1}.$$

Similarly, by using Theorem 2.4 and Equation (7), we have

$$D_t^{1-\gamma_2} u_{m,n}(x_r,t) = \sum_{i=-m}^{m} \sum_{j=0}^{n} u_{ij} \delta_{ir}^{(0)} D_t^{1-\gamma_2}\tilde{\beta}_j(t)$$

$$= \sum_{j=0}^{n} \sum_{k=0}^{j} u_{rj} Z_{jk}^{(1-\gamma_2)} t^{j-k-1+\gamma_2}.$$

Finally, employing Equations (8), (9) and Theorem 2.4 we obtain

$$D_t^{1-\gamma_1}\left(\frac{\partial^2 u_{m,n}(x_r,t)}{\partial x^2}\right) = \sum_{i=-m}^{m} \sum_{j=0}^{n} u_{ij}\left(\frac{d^2 S_i(x)}{dx^2}\right)_{x=x_r} D_t^{1-\gamma_1}\tilde{\beta}_j(t)$$

$$= \sum_{i=-m}^{m} \sum_{j=0}^{n} \sum_{k=0}^{j} u_{ij}\left(\phi'' \frac{dS_i(x)}{d\phi} + \phi'^2 \frac{d^2 S_i(x)}{d\phi^2}\right)_{x=x_r} Z_{jk}^{(1-\gamma_1)} t^{j-k-1+\gamma_1}$$

$$= \sum_{i=-m}^{m} \sum_{j=0}^{n} \sum_{k=0}^{j} u_{ij}\left(\phi''(x_r)\delta_{i,r}^{(1)} + (\phi'(x_r))^2 \delta_{i,r}^{(2)}\right) Z_{jk}^{(1-\gamma_1)} t^{j+\gamma_1-1-k}.$$

The proof is complete.

We now describe the numerical collocation scheme to problem (1)-(3). By substituting $u_{m,n}(x,t)$ into (1) we have

$$\frac{\partial u_{m,n}(x,t)}{\partial t} = D_t^{1-\gamma_1}\left(\kappa \frac{\partial^2 u_{m,n}(x,t)}{\partial x^2}\right) - \mu^2 D_t^{1-\gamma_2} u_{m,n}(x,t) + g(x,t). \tag{14}$$

Next, we collocate (14) at the Sinc points $\{x_r\}_{r=-m}^m$ and the shifted Legendre roots $\{t_l\}_{l=0}^n$ of $L_{n+1}(2(t/T)-1)$. Here $L_n(t)$, $-1 \le t \le 1$ is the well-known Legendre polynomial of order n (see[4]). Then by applying Theorem 3.1, for $r = -m,...,m$ and $l = 1,...,n$, we obtain

$$\sum_{j=1}^n \sum_{k=1}^j u_{rj} Z_{jk}^{(1)} t_l^{j-k-1} - \kappa \sum_{i=-m}^m \sum_{j=0}^n \sum_{k=0}^j u_{ij} E_{ir} Z_{jk}^{(1-\gamma_1)} t_l^{j-k-1+\gamma_1} + \mu^2 \sum_{j=0}^n \sum_{k=0}^j u_{rj} Z_{jk}^{(1-\gamma_2)} t_l^{j-k-1+\gamma_2} = g(x_r, t_l). \tag{15}$$

Moreover, by substituting $u_{m,n}(x,t)$ into (2) we have

$$\sum_{i=-m}^m \sum_{j=0}^n u_{ij} S_i(x) \tilde{\beta}_j(0) = \theta(x).$$

Collocating the above equation at $\{x_r\}_{r=-m}^m$ and using (7) we obtain

$$\sum_{j=0}^n u_{rj} \tilde{\beta}_j(0) = \theta(x_r), \quad r = -m,...,m \ . \tag{16}$$

Finally, the FCE (1)-(3) is reduced to linear algebraic equations (15) and (16) which can be solved for the unknown coefficients $\{u_{ij}\}$. So by using (13), $u_{m,n}(x,t)$ can be found. The results of this section can be summarized in the following algorithm.

BEGIN
Input m and n.
Step 1:   Choose the roots of order $n+1$ of shifted Legendre polynomials as $n+1$ collocation points: $\{t_l\}_{l=0}^n$
Step 2:   Select the $2m+1$ Sinc collocation points $\{x_r\}_{r=-m}^m$ by (6).
Step 3:   Create the $(2m+1) \times n$ equations by (15).
Step 4:   Create the $2m+1$ equations by (16).
Step 5:   Solve the $(2m+1) \times (n+1)$ linear algebraic equations given in step 3 and step 4 for the unknown coefficients $\{u_{ij}\}$.

Step 6: Find the numerical solution as $u_{m,n}(x,t) = \sum_{i=-m}^m \sum_{j=0}^n u_{ij} S_i(x) \tilde{\beta}_j(t)$.

END.

4. Error Bounds

In this section, we will find an upper bound for the truncated Sinc-Bernoulli series of $u(x,t)$. For simplicity, we set $T = 1$.

Lemma 4.1. ([7]). Suppose $\psi(t) \in C^{(n+1)}[0,1]$ and $\psi_n(t)$ is the approximation of $\psi(t)$ by using Bernoulli polynomials as given in (10), then, we have

$$\|\psi(t) - \psi_n(t)\|_2 \le \frac{\Lambda}{(n+1)!\sqrt{3+2n}},$$

where $\Lambda = \max_{t \in [0,1]} |\psi^{(n+1)}(t)|$.

In order to estimate $\|u(x,t) - u_{m,n}(x,t)\|_2$ we need to introduce the space

$$H_r^\alpha(I) = \left\{ u(x,t) : u(x,t) = \sum_{l=1}^r v_l(x) w_l(t), \text{ such that } u(x,.) \in L_\alpha(D) \text{ and } u(.,t) \in C[0,1] \right\}$$

where $\alpha \in (0,1]$, $r \in \mathbb{N}$, and $I = [0,1] \times [0,1]$.

Theorem 4.2. Suppose $u(x,t) \in H_r^\alpha(I)$ and $u_{m,n}(x,t)$ is the approximation of $u(x,t)$ as in Equation (13). Also, suppose $h = \sqrt{\pi d / \alpha m}$, then

$$\|u(x,t) - u_{m,n}(x,t)\|_2 \le K\sqrt{m} \exp(-\sqrt{\pi d \alpha m}) + \frac{(2m+1)K}{(n+1)!\sqrt{2n+3}}$$

where K is a constant.

Proof. Employing triangle inequality, we get

$$\left\| u(x,t) - \sum_{i=-m}^m \sum_{j=0}^n u_{ij} S_i(x) \tilde{\beta}_j(t) \right\|_2 \le \left\| u(x,t) - \sum_{i=-m}^m u(x_i,t) S_i(x) \right\|_2 + \left\| \sum_{i=-m}^m u(x_i,t) S_i(x) - \sum_{i=-m}^m \sum_{j=0}^n u_{ij} S_i(x) \tilde{\beta}_j(t) \right\|_2. \tag{17}$$

Since $u(x,t) \in H_r^\alpha(I)$, we obtain

$$\left\| u(x,t) - \sum_{i=-m}^{m} u(x_i,t)S_i(x) \right\|_2 = \left\| \sum_{l=1}^{r} w_l(t)\left(v_l(x) - \sum_{i=-m}^{m} v_l(x_i)S_i(x) \right) \right\|_2$$
$$\leq \sum_{l=1}^{r} \left\| w_r(t) \right\|_\infty \left\| v_l(x) - \sum_{i=-m}^{m} v_l(x_i)S_i(x) \right\|_2 \tag{18}$$

According to Theorem 2.2, one obtains

$$\left\| v_l(x) - \sum_{i=-m}^{m} v_l(x_i)S_i(x) \right\|_2 = \left(\int_0^1 \left(v_l(x) - \sum_{i=-m}^{m} v_l(x_i)S_i(x) \right)^2 dx \right)^{\frac{1}{2}}$$
$$\leq \left\| v_l(x) - \sum_{i=-m}^{m} v_l(x_i)S_i(x) \right\|_\infty$$
$$\leq \zeta_l \sqrt{m} \exp(-\sqrt{\pi d\alpha m}), \qquad l = 1,2,...,r,$$

where $\zeta_l, l = 1,...,r$, are constant. Also, since $w_l(t) \in C([0,1])$, there exist real numbers $\{W_l\}_{l=1}^{r}$ such that $|w_l(t)| < W_l$. Thus, from (18), we conclude that

$$\left\| u(x,t) - \sum_{i=-m}^{m} u(x_i,t)S_i(x) \right\|_2 \leq rW\overline{\zeta}\sqrt{m} \exp(-\sqrt{\pi d\alpha m}), \tag{19}$$

where $W = \max_{1 \leq l \leq r}\{W_l\}$ and $\overline{\zeta} = \max_{1 \leq l \leq r}\{\zeta_l\}$. Now, since $\|S_i(x)\|_\infty \leq 1$, we get

$$\left\| \sum_{i=-m}^{m} u(x_i,t)S_i(x) - \sum_{i=-m}^{m}\sum_{j=0}^{n} u_{ij}S_i(x)\tilde{\beta}_j(t) \right\|_2 \leq \sum_{i=-m}^{m} \left\| u(x_i,t) - \sum_{j=0}^{n} u_{ij}\tilde{\beta}_j(t) \right\|_2. \tag{20}$$

Also, using Lemma 4.1 we have

$$\left\| u(x_i,t) - \sum_{j=0}^{n} u_{ij}\tilde{\beta}_j(t) \right\|_2 \leq \frac{\eta_i}{(n+1)!\sqrt{2n+3}}, \qquad i = -m,...,m. \tag{21}$$

where $\eta_i, i = -m,...m$, are constant. Finally, by using (20) and (21), we have

$$\left\| \sum_{i=-m}^{m} u(x_i,t)S_i(x) - \sum_{i=-m}^{m}\sum_{j=0}^{n} u_{ij}S_i(x)\tilde{\beta}_j(t) \right\|_2 \leq \frac{(2m+1)\eta}{(n+1)!\sqrt{2n+3}}, \tag{22}$$

where $\eta = \max\{\eta_i, -m \leq i \leq m\}$. Let $K = \max\{rW\overline{\zeta}, \eta\}$. The desired result can be achieved by inserting (19) and (22) into (17).

5. Numerical Examples

In this section, we show the effectiveness of our method with two numerical examples. We report the results based on the following discrete $l^\infty - error$ and $l^2 - error$, which are defined respectively as:

$$l^\infty - error = \max\left\{ \left| u_{exact}(x_i,t_j) - u_{m,n}(x_i,t_j) \right|, -m \leq i \leq m, 0 \leq j \leq n \right\},$$

$$l^2 - error = \left(\sum_{i=-m}^{m}\sum_{j=0}^{n} \left(u_{exact}(x_i,t_j) - u_{m,n}(x_i,t_j) \right)^2 \right)^{\frac{1}{2}}.$$

Here, $\{x_i, t_j\}$ are the collocation points as given in Section 3. Also, in all examples we choose $h = \pi/\sqrt{m}$. It is worth mentioning that all the numerical calculations of this paper are done by Maple software.

Example 5.1. Consider the following FCE [12, 8, 35]

$$\frac{\partial u(x,t)}{\partial t} - D_t^{1-\gamma_1}\left(\frac{\partial^2 u(x,t)}{\partial x^2} \right) + D_t^{1-\gamma_2}u(x,t) = 2\left(t + \frac{\pi^2 t^{1+\gamma_1}}{\Gamma(2+\gamma_1)} + \frac{t^{1+\gamma_2}}{\Gamma(2+\gamma_2)} \right)\sin(\pi x),$$

with the initial condition $u(x,0) = 0$ and homogeneous conditions. Its exact solution is given as $u(x,t) = t^2 \sin(\pi x)$. This problem is solved with the Sinc-Bernoulli collocation method described in Section 3. Surface of the error function $u_{exact}(x,t) - u_{m,n}(x,t)$ with $n = 2$ and $m = 50$ is shown in Figure 1. Also, Table 2 shows the discrete $l^\infty - error$ and $l^2 - error$ for $n = 2$ and various m, γ_1, and γ_2. It is found that in Table 2, as m increases, $l^\infty - error$ and $l^2 - error$ decreases.
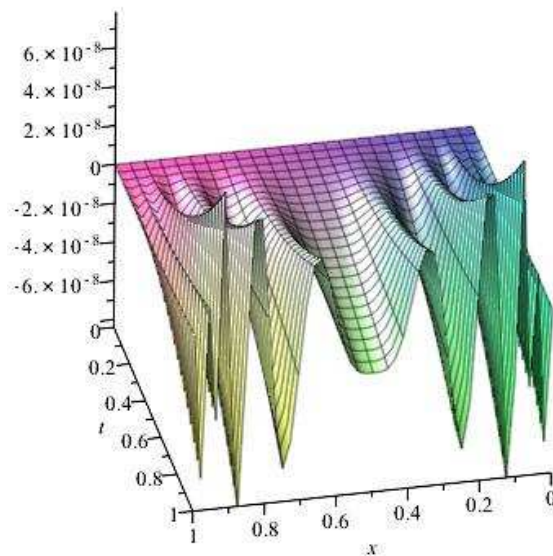
Fig. 1. Surface of the error function obtained for Example 5.1 with $n = 2$, $m = 50$, $\gamma_1 = 0.2$ and $\gamma_2 = 0.7$.
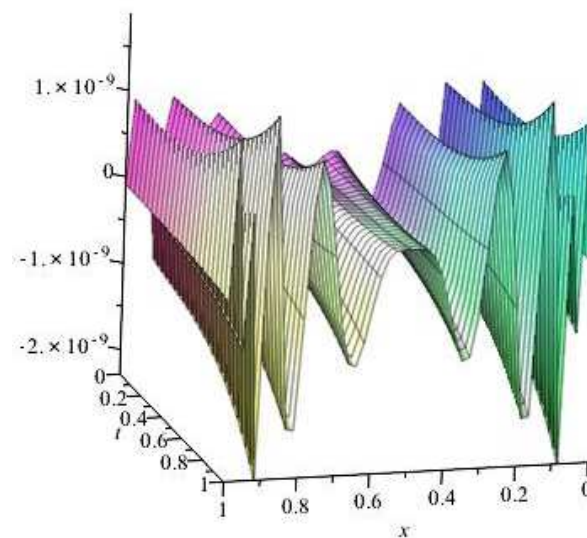

Fig. 2. Surface of the error function obtained for Example 5.2 with $n = 3$, $m = 50$, $\gamma_1 = \gamma_2 = 0.5$.

Table 1. Values of $l^\infty - error$ and $l^2 - error$ with $n = 2$, for Example 1.

m	$\gamma_1 = \gamma_2 = 0.5$		$\gamma_1 = \gamma_2 = 0.8$		$\gamma_1 = 0.1, \gamma_2 = 0.9$	
	$l^\infty - error$	$l^2 - error$	$l^\infty - error$	$l^2 - error$	$l^\infty - error$	$l^2 - error$
5	9.34×10^{-3}	2.57×10^{-2}	9.30×10^{-3}	2.56×10^{-2}	9.47×10^{-3}	2.59×10^{-2}
10	1.01×10^{-3}	2.08×10^{-3}	9.94×10^{-4}	2.06×10^{-3}	1.06×10^{-3}	2.11×10^{-3}
15	1.17×10^{-4}	3.67×10^{-4}	1.15×10^{-4}	3.66×10^{-4}	1.21×10^{-4}	3.70×10^{-4}
20	2.40×10^{-5}	6.68×10^{-5}	2.38×10^{-5}	6.67×10^{-5}	2.48×10^{-5}	6.73×10^{-5}
25	5.32×10^{-6}	1.87×10^{-5}	5.26×10^{-6}	1.87×10^{-5}	5.49×10^{-6}	1.88×10^{-5}
30	1.45×10^{-6}	4.77×10^{-6}	1.44×10^{-6}	4.76×10^{-6}	1.49×10^{-6}	4.79×10^{-6}
40	1.39×10^{-7}	5.07×10^{-7}	1.38×10^{-7}	5.07×10^{-7}	1.42×10^{-7}	5.09×10^{-7}
70	4.86×10^{-10}	2.17×10^{-9}	4.83×10^{-10}	2.16×10^{-9}	4.96×10^{-10}	2.17×10^{-9}

Table 2. Values of $l^\infty - error$ and $l^2 - error$ with $n = 3$, for Example 2.

m	$\gamma_1 = \gamma_2 = 0.5$		$\gamma_1 = \gamma_2 = 0.8$		$\gamma_1 = 0.1, \gamma_2 = 0.9$	
	$l^\infty - error$	$l^2 - error$	$l^\infty - error$	$l^2 - error$	$l^\infty - error$	$l^2 - error$
5	3.02×10^{-3}	9.22×10^{-3}	3.32×10^{-3}	1.07×10^{-2}	2.00×10^{-3}	5.35×10^{-3}
10	1.09×10^{-4}	3.53×10^{-4}	1.21×10^{-4}	4.12×10^{-4}	7.28×10^{-5}	2.04×10^{-4}
15	1.58×10^{-5}	7.42×10^{-5}	1.74×10^{-5}	8.66×10^{-5}	1.06×10^{-5}	4.30×10^{-5}
20	1.32×10^{-6}	5.86×10^{-6}	1.45×10^{-6}	6.84×10^{-6}	8.60×10^{-7}	3.31×10^{-6}
25	4.36×10^{-7}	2.58×10^{-6}	4.81×10^{-7}	3.01×10^{-6}	2.93×10^{-7}	1.49×10^{-6}
30	5.69×10^{-8}	2.80×10^{-7}	6.28×10^{-8}	3.27×10^{-7}	3.73×10^{-8}	1.61×10^{-7}
40	4.21×10^{-9}	2.25×10^{-8}	4.65×10^{-9}	2.63×10^{-8}	2.77×10^{-9}	1.30×10^{-8}
70	7.54×10^{-12}	5.33×10^{-11}	8.33×10^{-12}	6.21×10^{-11}	5.00×10^{-12}	3.07×10^{-11}

Example 5.2. Consider the problem (1)-(3) with $\Omega = (0,1), T = \kappa = \mu = 1$ and

$$g(x,t) = 3t^2(x^2 - x) - \frac{12t^{2+\gamma_1}}{\Gamma(3+\gamma_1)} - \frac{2t^{\gamma_1-1}}{\Gamma(\gamma_1)} + \left(\frac{6t^{2+\gamma_2}}{\Gamma(3+\gamma_2)} + \frac{t^{\gamma_2-1}}{\Gamma(\gamma_2)}\right)(x^2 - x),$$

$$\theta(x) = x^2 - x.$$

This problem has exact solution $u(x,t) = (t^3 + 1)(x^2 - x)$. Surface of the error function $u_{exact}(x,t) - u_{m,n}(x,t)$ with $n = 3$ and $m = 50$ has been shown in Figure 2. Also, Table 2 represents the $l^\infty - error$ and $l^2 - error$ for $n = 3$ and various m, γ_1, and γ_2. This table shows that the errors decrease as m increases.

6. Conclusion

In the present work, the combination of shifted Bernoulli polynomials and Sinc functions together with the collocation method were used to reduce the solution of time FCE which involves two fractional derivatives to the solution of a set of algebraic equations. This method was very easy to implement and our numerical results showed that the Sinc-Bernoulli collocation method can solve FCE effectively.

Author Contributions

N. Moshtaghi planned the scheme, initiated the project, and suggested the experiments; A. Saadatmandi conducted the experiments and analyzed the empirical results. The manuscript was written through the contribution of all authors. All authors discussed the results, reviewed, and approved the final version of the manuscript.

Conflict of Interest

The authors declared no potential conflicts of interest with respect to the research, authorship, and publication of this article.

Funding

The authors received no financial support for the research, authorship, and publication of this article.

References

[1] Abdella, K., Ross, G., Mohseniahouei, Y., Solutions to the Blasius and Sakiadis problems via a new Sinc-collocation approach, *Dynamic Systems and Applications,* 26 (2017) 105-120.
[2] Abu Argub, O., Application of residual power series method for the solution of time-fractional Schrodinger equations in one-dimensional space, *Fundamenta Informaticae,* 166 (2019) 87-110.
[3] Abu Argub, O., Numerical algorithm for the solutions of fractional order systems of Dirichlet function types with comparative analysis, *Fundamenta Informaticae,* 166 (2019) 111-137.
[4] Abu Argub, O., Maayah, B., Fitted fractional reproducing kernel algorithm for the numerical solutions of ABC-Fractional Volterra integro-differential equations, *Chaos, Solitons & Fractals,* 126 (2019) 394-402.
[5] Abu Argub, O., Maayah, B., Modulation of reproducing kernel Hilbert space method for numerical solutions of Riccati and Bernoulli equations in the Atangana-Baleanu fractional sense, *Chaos, Solitons & Fractals,* 125 (2019) 163-170.
[6] Aziz, M.R., Khani, A., Sinc operational matrix for solving the Bagley-Torvik equation. Computational Methods for Differential Equations, 5 (2017) 56-66.
[7] Behroozifar, M., Habibi, N., A numerical approach for solving a class of fractional optimal control problems via operational Bernoulli polynomials, *Journal of Vibration and Control,* 24 (2018) 2494-2511.
[8] Dehghan, M., Abbaszadeh, M., Analysis of the element free Galerkin (FEG)method for solving fractional cable equation with Dirichlet boundary condition, *Applied Numerical Mathematics,* 109 (2016) 208-234.
[9] Dehghan, M., Emami-Naeini, F., The Sinc-collocation and Sinc-Galerkin methods for solving the two-dimensional Schrodinger equation with nonhomogeneous boundary conditions, *Applied Mathematical Modelling,* 37 (2013) 9379-9397.
[10] Guoi, B.N., Qi, F., Some identities and an explicit formula for Bernoulli and Stirling numbers, *Journal of Computational and Applied Mathematics,* 255 (2014) 568-579.
[11] Hu, X., Zhang, L., Implicit compact difference schemes for the fractional cable equation, *Applied Mathematical Modelling,* 36 (2012) 4027-4043.
[12] Inc, M., Cavlak, E., Bayram, M., An approximate solution of fractional cable equation by homotopy analysis method, *Boundary Value Problems,* 2014 (2014) 58.
[13] Irandoust-pakchin, S., Dehghan, M., Abdi-mazraeh, S., Lakestani, M., Numerical solution for a class of fractional convection-diffusion equations using the flatlet oblique multiwavelets, *Journal of Vibration and Control,* 20 (2014) 913-924.
[14] Keshavarz, E., Ordokhani, Y., Razzaghi, M., A numerical solution for fractional optimal control problems via Bernoulli polynomials, *Journal of Vibration and Control,* 22 (2016) 3889-3903.
[15] Keshavarz, E., Ordokhani, Y., Razzaghi, M., The Bernoulli wavelets operational matrix of integration and its applications for the solution of linear and nonlinear problems in calculus of variations, *Applied Mathematics and Computation,* 351 (2019) 83-98.
[16] Kilbas, A.A., Srivastava, H.M., Trujillo, J.J., Theory and applications of fractional differential equations, *Elsevier Science Limited,* 2006.
[17] Langlands, T.A.M., Henry, B.I., Wearne, S.L., Fractional cable equation models for anomalous electrodiffusion in nerve cells: infinite domain solutions, *Journal of Mathematical Biology,* 71 (2011) 1168-1203.
[18] Liu, Y., Li, X., Xu, C., Finite difference/spectral approximations for the fractional equation, *Mathematics of Computation,* 80 (2011) 1369-1396.
[19] Liu, F., Yong, Q., Turner, L., Stability and convergence of two new implicit numerical methods for the fractional cable equation, *ASME 2009 international design engineering technical conferences and computers and information in engineering conference (pp.1015-1024).* American Society of Mechanical Engineers Digital Collection, 2009.
[20] Lund, J. Bowes, K., Sinc methods for quadrature and differential equations, *SIAM, PA, Philadelphia,* 1992.
[21] Mohebbi, A., Saffarian, M., Implicit RBF meshless method for the solution of two dimensional variable order fractional cable equation, *Journal of Applied and Computational Mechanics,* 6 (2020) 235-247.
[22] Mueller, J.L., Shores, T.S., A new Sinc-Galerkin method for convection-diffusion equations with mixed boundary conditions, *Computers and Mathematics with Applications,* 47 (2004) 803-822.
[23] Parand, K., Dehghan, M., Pirkhedri, A., The use of Sinc-collocation method for solving Falkner-Skan boundary- layer equation, *International Journal for Numerical Methods in Fluids,* 68 (2012) 36-47.

[24] Podulbny, I. Fractional differential equations, *Elsevier*, 1998.

[25] Quintana-Murillo, J., Yuste, S.B., An explicit numerical method for the fractional cable equation, *International Journal of Differential Equations,* 2011 (2011).

[26] Saadatmandi, A., Bernstein operational matrix of fractional derivatives and its applications, *Applied Mathematical Modelling*, 38 (2014) 1365-1372.

[27] Saadatmandi, A., Dehghan, M., Azizi, M.R., The Sinc-Legendre collocation method for a class of fractional convection-diffusion equations with variable coefficients, *Communications in Nonlinear Science and Numerical Simulation*, 17 (2012) 4125-4136.

[28] Saadatmandi, A., Khani, A., Azizi, M.R., A sinc-Gauss-Jacobi collocation method for solving Volterra's population growth model with fractional order, *Tbilisi Mathematical Journal,* 11 (2018) 123-137.

[29] Saadatmandi, A., Shateri, S., Sinc-collocation method for solving sodium alginate(SA) non-Newtonian nanofluid flow between two vertical flat plates, *Journal of the Brazilian Society of Mechanical Sciences and Engineering*, 41 (2019) 158.

[30] Stenger, F., Numerical methods based on Sinc and analytic functions, *Springer-Verlag*, New York, 1993.

[31] Yang, X., Jiang, X., Zhang, H., A time-space spectral tau method for the time fractional cable equation and its inverse problem, *Applied Numerical Mathematics,* 130 (2018) 95-111.

[32] Zhao, Z., Zheng, Y., Guo, P., A Galerkin finite element method for a class of time-space fractional differential equation with nonsmooth data, *Journal of Scientific Computing,* 70 (2017) 386-406.

[33] Zheng, Y., Zhao, Z., The discontinuous Galerkin finite element method for fractional cable equation, *Applied Numerical Mathematics,* 115 (2017) 32-41.

[34] Zheng, Y., Zhao, Z., The time discontinuous space-time finite element method for fractional diffusion-wave equation, *Applied Numerical Mathematics,* 150 (2020) 105-116.

[35] Zhu, P., Xie, S., Wang, X., Nonsmooth data error estimates for FEM approximations of the time fractional cable equation, *Applied Numerical Mathematics,* 121 (2017) 170-184.

[36] Zhuang, P., Liu, F., Turner, I.,Anh, V., Galerkin finite element method and error analysis for the fractional cable equation, *Numerical Algorithms,* 72 (2016) 447-466.

ORCID iD

Nasrin Moshtaghi https://orcid.org/0000-0002-3802-7546
Abbas Saadatmandi https://orcid.org/0000-0002-7744-7770