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Research Paper

Fractional Sumudu Decomposition Method for Solving PDEs of Fractional Order

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Abstract. In this paper, the fractional Sumudu decomposition method (FSDM) is employed to handle the time-fractional PDEs and system of time-fractional PDEs. The fractional derivative is described in the Caputo sense. The approximate solutions are obtained by using FSDM, which is the coupling method of fractional decomposition method and Sumudu transform. The method, in general, is easy to implement and yields good results. Illustrative examples are included to demonstrate the validity and applicability of the proposed technique.

Keywords: Fokker Plank equation; Nonlinear gas dynamic equation; Sumudu transform; Adomian decomposition method.

1. Introduction

Fractional differential equations (FDEs) have gained much attention from researchers due to their ability to enhance real-world issues, used in various fields of engineering and physics. Numerous physical marvels in signal processing, chemical physics, electrochemistry of corrosion, probability, and statistics, acoustics, and electromagnetic are precisely modeled by DEs of fractional order [1]. Nonlinear partial differential equations (PDEs) can be considered the generalization of the differential equations of integer order [2]. In the modern age, it is impossible to imagine the modeling of many real-world problems without using fractional partial differential equations (FPDEs). Indeed, fractional calculus can be called this century's calculus [3] because of the diversity of applications in different areas of science and technology [4].

Many numerical and analytical techniques have been suggested for the solutions of linear and nonlinear partial differential equations of fractional order such as homotopy analysis technique [5], variational iteration method [6-8], homotopy perturbation method [9-11], Laplace homotopy perturbation method [12], Laplace decomposition method [13], Sumudu variational iteration method [14], variation iteration transform method [15], reduce differential transform method [16], series expansion method [17], and other methods [18-24]. This paper considers the efficiency of fractional Sumudu decomposition method (FSDM) to solve the time-fractional Fokker Plank equation, nonlinear time-fractional gas dynamics equation, and system of time-fractional partial differential equations. The FSDM is a graceful coupling of two powerful techniques, namely ADM and Sumudu transform methods, and gives a more refined convergent series solution.

Our aim is to extend the application of the proposed method to obtain the approximate analytical solutions to fractional partial differential equations. The remaining sections of this work are organized as follows. In Section 2, some background notations of fractional calculus are presented. In Section 3, the analysis of the fractional Sumudu decomposition method is discussed. Applications of FSDM are shown in Section 4. The conclusion of this paper is given in Section 5.

2. Preliminaries

Some fractional calculus definitions and notation needed in the course of this work are discussed in this section.

Definition 1 [10,11,14]. A real function $\varphi(\mu), \mu > 0$, is said to be in the space $C_\vartheta, \vartheta \in R$ if there exists a real number $q, (q > \vartheta)$, such that $\varphi(\mu) = \mu^q \varphi_1(\mu)$, where $\varphi_1(\mu) \in C[0,\infty)$, and it is said to be in the space C_ϑ^m if $\varphi^{(m)} \in C_\vartheta, m \in N$.

Definition 2 [10,11]. The fractional derivative of $\varphi(\mu)$ in the Caputo sense is defined as

$$D^\delta \varphi(\mu) = I^{m-\delta} D^m \varphi(\mu) = \frac{1}{\Gamma(m-\delta)} \int_0^\mu (\mu-\tau)^{m-\delta-1} \varphi^{(m)}(\tau) d\tau, \tag{1}$$

for $m-1 < \delta \le m, m \in N, \mu > 0, \varphi \in C_{-1}^m$.

The following are the basic properties of the operator D^δ:

1. $D^\delta I^\delta \varphi(\mu) = \varphi(\mu)$.
2. $I^\delta D^\delta \varphi(\mu) = \varphi(\mu) - \sum_{k=0}^{m-1} \varphi^{(k)}(0) \frac{\mu^k}{k!}$.

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Definition 3 [10,11,14]. The Mittag–Leffler function E_δ with $\delta > 0$ is defined as

$$E_\delta(\mu) = \sum_{m=0}^{\infty} \frac{\mu^\delta}{\Gamma(m\delta+1)}. \tag{2}$$

Definition 4 [14]. The Sumudu transform is defined over the set of function

$$A = \{\varphi(\tau) / \exists M, \omega_1, \omega_2 > 0, |\varphi(\tau)| < Me^{|\tau|/\omega_j}, if \ \ \tau \in (-1)^j \times [0,\infty)\}$$

by the following formula

$$\boldsymbol{S}[\varphi(\tau)](\omega) = \int_0^\infty e^{-\tau}\varphi(\omega\tau)d\tau, \ \ \omega \in (-\omega_1, \omega_2). \tag{3}$$

Lemma 1 [14]. The Sumudu transform of the Caputo fractional derivative is defined as

$$\boldsymbol{S}\left[D_\tau^{m\delta}\varphi(\mu,\tau)\right] = \omega^{-m\delta}\boldsymbol{S}[\varphi(\mu,\tau] - \sum_{k=0}^{m-1} \omega^{(-m\delta+k)}\,\varphi^{(k)}(\mu,0), m-1 < m\delta < m. \tag{4}$$

3. Fractional Sumudu Decomposition Method (FSDM).

Let us consider a general fractional nonlinear partial differential equation of the form:

$$D_\tau^\delta\varphi(\mu,\tau) + R[\varphi(\mu,\tau)] + N[\varphi(\mu,\tau)] = g(\mu,\tau), \tag{5}$$

with the initial condition

$$\varphi(\mu,0) = f(\mu) \tag{6}$$

where $D_\tau^\delta\varphi(\mu,\tau)$is the Caputo fractional derivative of the function $\varphi(\mu,\tau)$ defined as:

$$D_\tau^\delta\varphi(\mu,\tau) = \frac{\partial^\delta\varphi(\mu,\tau)}{\partial\tau^\delta} = \begin{cases} \dfrac{1}{\Gamma(m-\delta)}\displaystyle\int_0^\tau (\tau-\omega)^{m-\delta-1}\frac{\partial^m\varphi(\mu,\omega)}{\partial\tau^m}d\omega\,, & m-1<\delta<m \\ \dfrac{\partial^m\varphi(\mu,\tau)}{\partial\tau^m}, & \delta = m \in N \end{cases}$$

and R is the linear differential operator, N represents the general nonlinear differential operator and $g(\mu,\tau)$ is the source term. Taking the ST on both sides of Eq.(5), we have

$$S\left[D_\tau^\delta\varphi(\mu,\tau)\right] + S\left[R[\varphi(\mu,\tau)]\right] + S\left[N[\varphi(\mu,\tau)]\right] = S[g(\mu,\tau)]. \tag{7}$$

Using the property of the ST, we obtain

$$S[\varphi(\mu,\tau)] = \varphi(\mu,0) + \omega^\delta S[g(\mu,\tau)] - \omega^\delta S\left[R[\varphi(\mu,\tau)] + N[\varphi(\mu,\tau)]\right]. \tag{8}$$

Operating with the ST on both sides of Eq. (8) gives

$$\varphi(\mu,\tau) = f(\mu) + S^{-1}\left(\omega^\delta S[g(\mu,\tau)]\right) - S^{-1}\left(\omega^\delta S\left[R[\varphi(\mu,\tau)] + N[\varphi(\mu,\tau)]\right]\right). \tag{9}$$

Now, we represent the solution as an infinite series given below

$$\varphi(\mu,\tau) = \sum_{n=0}^{\infty}\varphi_n(\mu,\tau), \tag{10}$$

and the nonlinear term can be decomposed as

$$N[\varphi(\mu,\tau)] = \sum_{n=0}^{\infty} A_n(\varphi_0,\varphi_1,\dots,\varphi_n), \tag{11}$$

where

$$A_n(\varphi_0,\varphi_1,\dots,\varphi_n) = \frac{1}{n!}\frac{\partial^n}{\partial\lambda^n}\left[N\left(\sum_{i=0}^{\infty}\lambda^i\varphi_i\right)\right]_{\lambda=0}.$$

Substituting Eqs.(10) and (11) in Eq.(9), we get

$$\sum_{n=0}^{\infty}\varphi_n(\mu,\tau) = f(\mu) + S^{-1}\left(\omega^\delta S[g(\mu,\tau)]\right) - S^{-1}\left(\omega^\delta S\left[R\left[\sum_{n=0}^{\infty}\varphi_n(\mu,\tau)\right] + \sum_{n=0}^{\infty}A_n\right]\right). \tag{12}$$

On comparing both sides of the Eq. (12), we get

$$\begin{aligned} &\varphi_0(\mu,\tau) = f(\mu) + S^{-1}\left(\omega^\delta S[g(\mu,\tau)]\right), \\ &\varphi_1(\mu,\tau) = -S^{-1}\left(\omega^\delta S[R[\varphi_0(\mu,\tau)] + A_0]\right), \\ &\varphi_2(\mu,\tau) = -S^{-1}\left(\omega^\delta S[R[\varphi_1(\mu,\tau)] + A_1]\right), \end{aligned} \tag{13}$$

$$\vdots$$

$$\varphi_n(\mu,\tau) = -S^{-1}\left(\omega^\delta S[R[\varphi_{n-1}(\mu,\tau)] + A_{n-1}]\right),\ \ n \geq 1.$$

Finally, we approximate the analytical solution $\varphi(\mu,\tau)$ by truncated series:

$$\varphi(\mu,\tau) = \sum_{n=0}^{\infty} \varphi_n(\mu,\tau). \tag{14}$$

4. Applications

In this section, we will implement the proposed method FSDM for solving time-fractional differential equations.

Example 1. First, we consider the time-fractional Fokker-Plank equation

$$D_\tau^\delta \varphi(\mu,\tau) = -\frac{\partial}{\partial\mu}\left(\frac{4}{\mu}\varphi(\mu,\tau) - \frac{\mu}{3}\right)\varphi(\mu,\tau) + \frac{\partial^2}{\partial\mu^2}\varphi^2(\mu,\tau), \tag{15}$$

where $\tau > 0,\ \mu \in R,\ 0 < \delta \leq 1$, subject to the initial and boundary conditions

$$\varphi(\mu,0) = \mu^2, \qquad \varphi(0,\tau) = 0, \varphi(1,\tau) = E_\delta(\mu^\delta). \tag{16}$$

If $\delta = 1$, then the exact solution of Eq.(15) is $\varphi(\mu,\tau) = \mu^2 e^\tau$.

From Eqs. (13) and (4.1), the successive approximations are

$$\varphi_0(\mu,\tau) = \varphi(\mu,0),$$
$$\varphi_{n+1}(\mu,\tau) = \boldsymbol{S}^{-1}\left(\omega^\delta \boldsymbol{S}\left[-\frac{\partial}{\partial\mu}\left(\frac{4}{\mu}\varphi_n(\mu,\tau) - \frac{\mu}{3}\right)\varphi_n(\mu,\tau) + \frac{\partial^2}{\partial\mu^2}A_n\right]\right), n \geq 0 \tag{17}$$

where

$$A_0 = \varphi_0^2$$
$$A_1 = 2\varphi_0\varphi_1$$
$$A_2 = 2\varphi_0\varphi_2 + \varphi_1^2$$
$$\vdots$$

Then, we have

$$\varphi_0(\mu,\tau) = \mu^2,$$
$$\varphi_1(\mu,\tau) = \boldsymbol{S}^{-1}\left(\omega^\delta \boldsymbol{S}\left[-\frac{\partial}{\partial\mu}\left(\frac{4}{\mu}\varphi_0(\mu,\tau) - \frac{\mu}{3}\right)\varphi_0(\mu,\tau) + \frac{\partial^2}{\partial\mu^2}A_0\right]\right) = \frac{\mu^2\tau^\delta}{\Gamma(\delta+1)},$$
$$\varphi_2(\mu,\tau) = \boldsymbol{S}^{-1}\left(\omega^\delta \boldsymbol{S}\left[-\frac{\partial}{\partial\mu}\left(\frac{4}{\mu}\varphi_1(\mu,\tau) - \frac{\mu}{3}\right)\varphi_1(\mu,\tau) + \frac{\partial^2}{\partial\mu^2}A_1\right]\right) = \frac{\mu^2\tau^{2\delta}}{\Gamma(2\delta+1)},$$
$$\varphi_3(\mu,\tau) = \boldsymbol{S}^{-1}\left(\omega^\delta \boldsymbol{S}\left[-\frac{\partial}{\partial\mu}\left(\frac{4}{\mu}\varphi_2(\mu,\tau) - \frac{\mu}{3}\right)\varphi_2(\mu,\tau) + \frac{\partial^2}{\partial\mu^2}A_2\right]\right) = \frac{\mu^2\tau^{3\delta}}{\Gamma(3\delta+1)},$$
$$\vdots$$
$$\varphi_n(\mu,\tau) = \frac{\mu^2\tau^{n\delta}}{\Gamma(n\delta+1)}.$$

Therefore, the solution of Eq.(15) is given by

$$\varphi(\mu,\tau) = \mu^2\left[1 + \frac{\tau^\delta}{\Gamma(\delta+1)} + \frac{\tau^{2\delta}}{\Gamma(2\delta+1)} + \cdots + \frac{\tau^{n\delta}}{\Gamma(n\delta+1)}\right] = \mu^2 E_\delta(\tau^\delta). \tag{18}$$

Setting $\delta = 1$ in Eq.(18), we reproduce the solution of the problem as follows:

$$\varphi(\mu,\tau) = \mu^2\left[1 + \tau + \frac{\tau^2}{2!} + \cdots + \frac{\tau^n}{n!}\right]. \tag{19}$$

This solution is equivalent to the exact solution in closed form:

$$\varphi(\mu,\tau) = \mu^2 e^\tau. \tag{20}$$

In Figure 1, we plot the graph of the approximate solution for Eqs. (15) and (16) when $\delta = 0.7, 0.8, 0.9, 1$. In Figure 2, 3D surface solution for Eqs. (15) and (16) when $\delta = 0.5, 0.7, 0.9, 1$. In Table 1, we evaluated the numerical values of the approximate solution $\varphi(\mu,\tau)$ of the problem (15) obtained by the FSDM and the exact solution φ_{EX} given by Eq. (20) through different values of μ,τ when $\delta = 0.7, 0.8, 0.9$ to make a numerical comparison between $\varphi(\mu,\tau)$ and φ_{EX}.

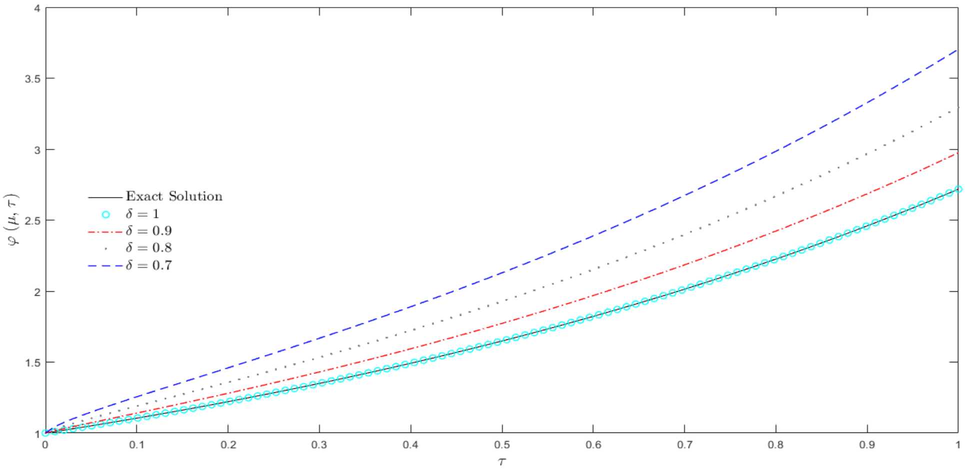

Fig. 1. Plots of approximate solution $\varphi(\mu,\tau)$ for different values of δ with fixed value $\mu = 1$.

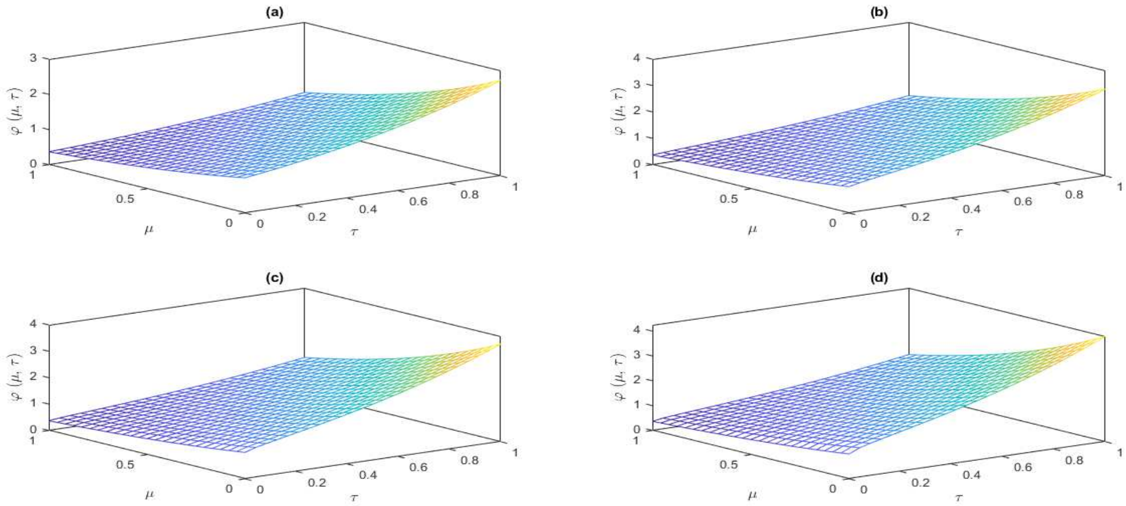

Fig. 2. The surface graph of the approximate solution $\varphi(\mu,\tau)$ of Eqs. (15) and (16): (a) $\varphi(\mu,\tau)$ when $\delta = 0.5$, (b) $\varphi(\mu,\tau)$ when $\delta = 0.7$, (c) $\varphi(\mu,\tau)$ when $\delta = 0.9$, (d) $\varphi(\mu,\tau)$ when $\delta = 1$.

Table 1. Numerical values of the approximate and exact solutions among different values of μ and τ when $\delta = 0.7, 0.8, 0.9$ for Example 1.

		$\delta = 0.7$	$\delta = 0.8$	$\delta = 0.9$		Absolute Error
μ	τ	φ	φ	φ	φ_{Ex}	$\lvert\varphi - \varphi_{Ex}\rvert$
	0.25	0.0977	0.0903	0.0846	0.0803	0.0044
0.25	0.50	0.1331	0.1205	0.1108	0.1030	0.0077
	0.75	0.1766	0.1582	0.1438	0.1323	0.0115
	0.25	0.3906	0.3610	0.3385	0.3210	0.0175
0.5	0.50	0.5322	0.4820	0.4431	0.4122	0.0309
	0.75	0.7064	0.6327	0.5753	0.5293	0.0460
	0.25	1.5625	1.4440	1.3540	1.2840	0.0700
1	0.50	2.1289	1.9281	1.7725	1.6487	0.1238
	0.75	2.8256	2.5307	2.3010	2.1170	0.1840

Example 2. Consider the following nonlinear time-fractional gas dynamics equation:

$$D_\tau^\delta \varphi(\mu,\tau) + \frac{1}{2}\left(\varphi^2(\mu,\tau)\right)_\mu - \varphi(\mu,\tau) + \varphi^2(\mu,\tau) = 0, \tag{21}$$

where $0 < \delta \le 1$, subject to the initial and boundary conditions

$$\varphi(\mu,0) = e^{-\mu}, \qquad \varphi(0,\tau) = E_\delta(\tau^\delta), \qquad \varphi(1,\tau) = e^{-1}E_\delta(\tau^\delta). \tag{22}$$

If $\delta = 1$, then the exact solution of Eq. (21) is $\varphi(\mu,\tau) = e^{\tau-\mu}$.
From Eqs. (13) and (21), the successive approximations are

$$\varphi_0(\mu,\tau) = \varphi(\mu,0), \tag{23}$$

$$\varphi_{n+1}(\mu,\tau) = -\boldsymbol{S}^{-1}\left(\omega^{\delta}\boldsymbol{S}\left[\frac{1}{2}A_n - \varphi_n(\mu,\tau) + B_n\right]\right), n \geq 0$$

where

$$A_0 = (\varphi_0^2)_\mu$$
$$A_1 = (2\varphi_0\varphi_1)_\mu$$
$$A_2 = (2\varphi_0\varphi_2 + \varphi_1^2)_\mu$$
$$\vdots$$
$$B_0 = \varphi_0^2$$
$$B_1 = 2\varphi_0\varphi_1$$
$$B_2 = 2\varphi_0\varphi_2 + \varphi_1^2$$

Then, we have

$$\varphi_0(\mu,\tau) = e^{-\mu},$$
$$\varphi_1(\mu,\tau) = -\boldsymbol{S}^{-1}\left(\omega^{\delta}\boldsymbol{S}\left[\frac{1}{2}A_0 - \varphi_0(\mu,\tau) + B_0\right]\right) = \frac{\tau^{\delta}}{\Gamma(\delta+1)}e^{-\mu},$$
$$\varphi_2(\mu,\tau) = -\boldsymbol{S}^{-1}\left(\omega^{\delta}\boldsymbol{S}\left[\frac{1}{2}A_1 - \varphi_1(\mu,\tau) + B_1\right]\right) = \frac{\tau^{2\delta}}{\Gamma(2\delta+1)}e^{-\mu},$$
$$\varphi_3(\mu,\tau) = -\boldsymbol{S}^{-1}\left(\omega^{\delta}\boldsymbol{S}\left[\frac{1}{2}A_2 - \varphi_2(\mu,\tau) + B_2\right]\right) = \frac{\tau^{3\delta}}{\Gamma(3\delta+1)}e^{-\mu},$$
$$\vdots$$
$$\varphi_n(\mu,\tau) = \frac{\tau^{n\delta}}{\Gamma(n\delta+1)}e^{-\mu}.$$

Therefore, the solution of Eq.(21) is given by

$$\varphi(\mu,\tau) = e^{-\mu}\left[1 + \frac{\tau^{\delta}}{\Gamma(\delta+1)} + \frac{\tau^{2\delta}}{\Gamma(2\delta+1)} + \cdots + \frac{\tau^{n\delta}}{\Gamma(n\delta+1)}\right] = e^{-\mu}E_{\delta}(\tau^{\delta}). \tag{24}$$

Setting $\delta = 1$ in Eq.(24), we reproduce the solution of the problem as follows:

$$\varphi(\mu,\tau) = e^{-\mu}\left[1 + \tau + \frac{\tau^2}{2!} + \cdots + \frac{\tau^n}{n!}\right]. \tag{25}$$

This solution is equivalent to the exact solution in closed form:

$$\varphi(\mu,\tau) = e^{\tau-\mu}. \tag{26}$$

In Figure 3, we plot the graph of the approximate solution for (21) and (22) when $\delta = 0.7, 0.8, 0.9, 1$. In Figure 4, 3D surface solution fo r (21) and (22) when $\delta = 0.5, 0.7, 0.9, 1$. In Table 2, we evaluated the numerical values of the approximate solution $\varphi(\mu,\tau)$ of the prob lem (21) obtained by the FSDM and the exact solution φ_{EX} given by equation (26) through different values of μ,τ when $\delta = 0.7, 0.8, 0$ $.9, 1$ to make a numerical comparison between $\varphi(\mu,\tau)$ and φ_{EX}.

Example 3. Consider the following system of nonlinear time-fractional PDEs:

$$\begin{aligned} D_\tau^{\delta}\varphi(\mu,\tau) - \psi_\mu + \psi + \varphi = 0, \quad 0 < \delta \leq 1 \\ D_\tau^{\gamma}\psi(\mu,\tau) - \varphi_\mu + \psi + \varphi = 0, \quad 0 < \gamma \leq 1 \end{aligned} \tag{27}$$

with the initial and boundary conditions

$$\begin{aligned} &\varphi(\mu,0) = \sinh(\mu) \\ &\psi(\mu,0) = \cosh(\mu), \\ &\varphi(0,\tau) = -E_{\delta}(\tau^{\delta}), \varphi(1,\tau) = \sinh(1)\,E_{\delta}(\tau^{\delta}) - \cosh(1)\,E_{\delta}(\tau^{\delta}) \\ &\psi(0,\tau) = E_{\delta}(\tau^{\delta}), \psi(1,\tau) = \cosh(1)E_{\delta}(\tau^{\delta}) - \sinh(1)E_{\delta}(\tau^{\delta}) \end{aligned} \tag{28}$$

If $\delta = \gamma = 1$ then the exact solution of Eq.(27) is $\varphi(\mu,\tau) = \sinh(\mu-\tau)$, $\psi(\mu,\tau) = \cosh(\mu-\tau)$.
Taking the Sumudu transform on both sides of Eq.(27), we have

$$\boldsymbol{S}[D_\tau^{\delta}\varphi(\mu,\tau)] = \boldsymbol{S}[\psi_\mu - \psi - \varphi]$$
$$\boldsymbol{S}[D_\tau^{\gamma}\psi(\mu,\tau)] = \boldsymbol{S}[\varphi_\mu - \psi - \varphi],$$

Using the property of the Sumudu transform and the initial condition in (28), we obtain

$$\begin{aligned} \boldsymbol{S}[\varphi(\mu,\tau)] = \sinh(\mu) + \omega^{\delta}\boldsymbol{S}[\psi_\mu - \psi - \varphi], \\ \boldsymbol{S}[\psi(\mu,\tau)] = \cosh(\mu) + \omega^{\gamma}\boldsymbol{S}[\varphi_\mu - \psi - \varphi], \end{aligned} \tag{29}$$

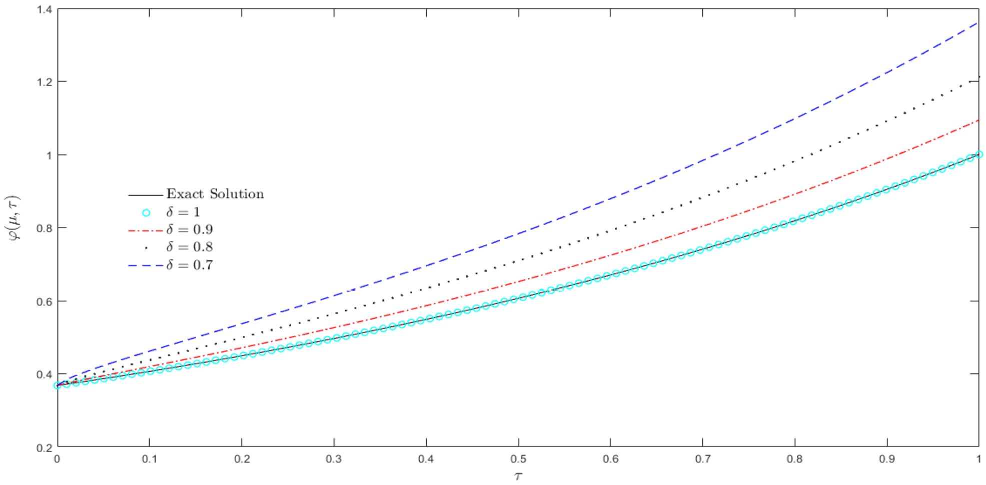

Fig. 3. Plots of approximate solution $\varphi(\mu,\tau)$ for different values of δ with fixed value $\mu = 1$.

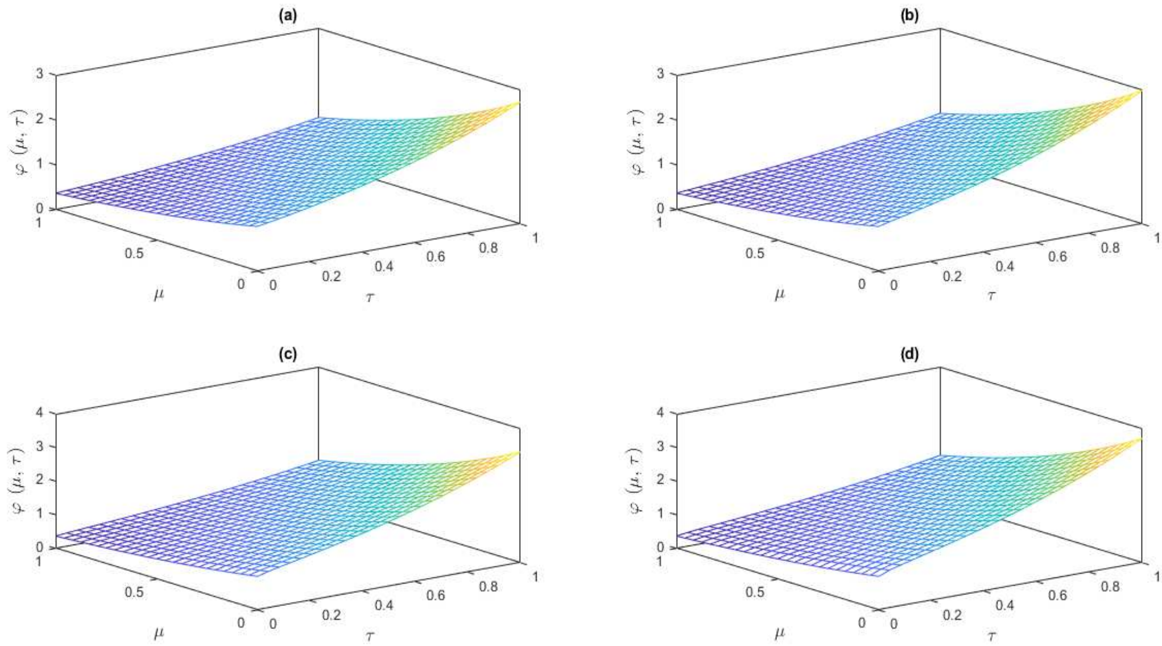

Fig. 4. The surface graph of the approximate solution $\varphi(\mu,\tau)$ of Eqs. (21) and (22): (a) $\varphi(\mu,\tau)$ when $\delta = 0.5$, (b) $\varphi(\mu,\tau)$ when $\delta = 0.7$, (c) $\varphi(\mu,\tau)$ when $\delta = 0.9$, (d) $\varphi(\mu,\tau)$ when $\delta = 1$.

Table 2. Numerical values of the approximate and exact solutions among different values of μ and τ when $\delta = 0.7, 0.8, 0.9, 1$ for Example 2.

		$\delta = 0.7$	$\delta = 0.8$	$\delta = 0.9$		Absolute Error
μ	τ	φ	φ	φ	φ_{Ex}	$\lvert\varphi - \varphi_{Ex}\rvert$
	0.25	1.2169	1.1246	1.0545	1.0000	0.0545
0.25	0.50	1.6580	1.5016	1.3804	1.2840	0.0964
	0.75	2.2006	1.9709	1.7920	1.6487	0.1433
	0.25	0.9477	0.8759	0.8212	0.7788	0.0424
0.50	0.50	1.2913	1.1694	1.0751	1.0000	0.0751
	0.75	1.7138	1.5350	1.3956	1.2840	0.1116
	0.25	0.5748	0.5312	0.4981	0.4724	0.0257
1.00	0.50	0.7832	0.7093	0.6521	0.6065	0.0455
	0.75	1.0395	0.9310	0.8465	0.7788	0.0677

Operating with the Sumudu inverse on both sides of Eq.(29), we have

$$\begin{aligned} \varphi(\mu,\tau) &= \sinh(\mu) + \boldsymbol{S}^{-1}\big(\,\omega^{\delta}\,\boldsymbol{S}[\psi_{\mu} - \psi - \varphi]\big) \\ \psi(\mu,\tau) &= \cosh(\mu) + \boldsymbol{S}^{-1}\big(\,\omega^{\gamma}\,\boldsymbol{S}[\varphi_{\mu} - \psi - \varphi]\big). \end{aligned} \tag{30}$$

Suppose that

$$\varphi(\mu,\tau) = \sum_{n=0}^{\infty} \varphi_n(\mu,\tau), \tag{31}$$

$$\psi(\mu,\tau) = \sum_{n=0}^{\infty} \psi_n(\mu,\tau), \tag{32}$$

Substituting Eqs.(31) and (32) in Eq.(30), we get

$$\begin{aligned}\sum_{n=0}^{\infty}\varphi_n &= \sinh(\mu) + \boldsymbol{S}^{-1}\left(\omega^{\delta}\,\boldsymbol{S}\left[\frac{\partial}{\partial\mu}\left(\sum_{n=0}^{\infty}\psi_n\right) - \sum_{n=0}^{\infty}\psi_n - \sum_{n=0}^{\infty}\varphi_n\right]\right)\\ \sum_{n=0}^{\infty}\psi_n &= \cosh(\mu) + \boldsymbol{S}^{-1}\left(\omega^{\gamma}\,\boldsymbol{S}\left[\frac{\partial}{\partial\mu}\left(\sum_{n=0}^{\infty}\varphi_n\right) - \sum_{n=0}^{\infty}\psi_n - \sum_{n=0}^{\infty}\varphi_n\right]\right).\end{aligned} \tag{33}$$

On comparing both sides of Eq.(33), we obtain

$$\varphi_0(\mu,\tau) = \sinh(\mu)$$
$$\psi_0(\mu,\tau) = \cosh(\mu)$$

$$\varphi_1(\mu,\tau) = \boldsymbol{S}^{-1}\left(\omega^{\delta}\,\boldsymbol{S}\left[\frac{\partial}{\partial\mu}\psi_0 - \psi_0 - \varphi_0\right]\right)$$

$$\psi_1(\mu,\tau) = \boldsymbol{S}^{-1}\left(\omega^{\gamma}\,\boldsymbol{S}\left[\frac{\partial}{\partial\mu}\varphi_0 - \psi_0 - \varphi_0\right]\right),$$

$$= -\cosh(\mu)\frac{\tau^{\delta}}{\Gamma(\delta+1)},$$

$$= -\sinh(\mu)\frac{\tau^{\gamma}}{\Gamma(\gamma+1)},$$

$$\varphi_2(\mu,\tau) = \boldsymbol{S}^{-1}\left(\omega^{\delta}\,\boldsymbol{S}\left[\frac{\partial}{\partial\mu}\psi_1 - \psi_1 - \varphi_1\right]\right)$$

$$\psi_2(\mu,\tau) = \boldsymbol{S}^{-1}\left(\omega^{\gamma}\,\boldsymbol{S}\left[\frac{\partial}{\partial\mu}\varphi_1 - \psi_1 - \varphi_1\right]\right),$$

$$= -\cosh(\mu)\frac{\tau^{\delta+\gamma}}{\Gamma(\delta+\gamma+1)} + \sinh(\mu)\frac{\tau^{\delta+\gamma}}{\Gamma(\delta+\gamma+1)} + \cosh(\mu)\frac{\tau^{2\delta}}{\Gamma(2\delta+1)},$$

$$= -\sinh(\mu)\frac{\tau^{\delta+\gamma}}{\Gamma(\delta+\gamma+1)} + \cosh(\mu)\frac{\tau^{\delta+\gamma}}{\Gamma(\delta+\gamma+1)} + \sinh(\mu)\frac{\tau^{2\gamma}}{\Gamma(2\gamma+1)}.$$

$$\vdots$$

Therefore, the solution of Eq.(27) is given by

$$\begin{aligned}\varphi(\mu,\tau) &= \sinh(\mu)\left[1 + \frac{\boldsymbol{\tau}^{\delta+\gamma}}{\Gamma(\delta+\gamma+1)} + \cdots\right] - \cosh(\mu)\left[\frac{\boldsymbol{\tau}^{\delta}}{\Gamma(\delta+1)} + \frac{\boldsymbol{\tau}^{\delta+\gamma}}{\Gamma(\delta+\gamma+1)} - \frac{\boldsymbol{\tau}^{2\delta}}{\Gamma(2\delta+1)} + \cdots\right]\\ \psi(\mu,\tau) &= \cosh(\mu)\left[1 + \frac{\boldsymbol{\tau}^{\delta+\gamma}}{\Gamma(\delta+\gamma+1)} + \cdots\right] + \sinh(\mu)\left[\frac{\boldsymbol{\tau}^{\gamma}}{\Gamma(\gamma+1)} + \frac{\boldsymbol{\tau}^{\delta+\gamma}}{\Gamma(\delta+\gamma+1)} - \frac{\tau^{2\gamma}}{\Gamma(2\gamma+1)} + \cdots\right].\end{aligned} \tag{34}$$

Setting $\delta = \gamma$ in Eq.(34), we obtain

$$\begin{aligned}\varphi(\mu,\tau) &= \sinh(\mu)\left[1 + \frac{\boldsymbol{\tau}^{2\delta}}{\Gamma(2\delta+1)} + \cdots\right] - \cosh(\mu)\left[\frac{\boldsymbol{\tau}^{\delta}}{\Gamma(\delta+1)} + \frac{\boldsymbol{\tau}^{3\delta}}{\Gamma(3\delta+1)} + \cdots\right].\\ \psi(\mu,\tau) &= \cosh(\mu)\left[1 + \frac{\boldsymbol{\tau}^{2\delta}}{\Gamma(2\delta+1)} + \cdots\right] - \sinh(\mu)\left[\frac{\boldsymbol{\tau}^{\delta}}{\Gamma(\delta+1)} + \frac{\tau^{3\delta}}{\Gamma(3\delta+1)} + \cdots\right].\end{aligned} \tag{35}$$

Setting $\delta = 1$ in Eq.(35), we reproduce the solution of the problem as follows:

$$\begin{aligned}\varphi(\mu,\tau) &= \sinh(\mu)\left[1 + \frac{\tau^2}{2!} + \cdots\right] - \cosh(\mu)\left[\boldsymbol{\tau} + \frac{\tau^3}{3!} + \cdots\right].\\ \psi(\mu,\tau) &= \cosh(\mu)\left[1 + \frac{\tau^2}{2!} + \cdots\right] - \sinh(\mu)\left[\boldsymbol{\tau} + \frac{\tau^3}{3!} + \cdots\right].\end{aligned} \tag{36}$$

This solution is equivalent to the exact solution in closed form:

$$\begin{aligned}\varphi(\mu,\tau) &= \sinh(\mu-\tau).\\ \psi(\mu,\tau) &= \cosh(\mu-\tau).\end{aligned} \tag{37}$$

In Figure 5, we plot the graph of the approximate solution for (27) and (28) when $\delta = 0.7, 0.8, 0.9, 1$. In Figure 6 and Figure 7, 3D surface solutions for (27) and (28) when $\delta = 0.5, 0.7, 0.9, 1$. In Table 3, we evaluated the numerical values of the approximate solution $\varphi(\mu,\tau)$ of problem (27) obtained by the FSDM and the exact solution φ_{EX} given by equation (37) through different values of μ,τ when $\delta = 0.7, 0.8, 0.9$ to make a numerical comparison between $\varphi(\mu,\tau)$ and φ_{EX}. In Table 4, we evaluated the numerical values of the approximate solution $\psi(\mu,\tau)$ of problem (27) obtained by the FSDM and the exact solution ψ_{EX} given by equation (37) through different values of μ,τ when $\delta = 0.7, 0.8, 0.9$ to make a numerical comparison between $\psi(\mu,\tau)$ and ψ_{EX}.

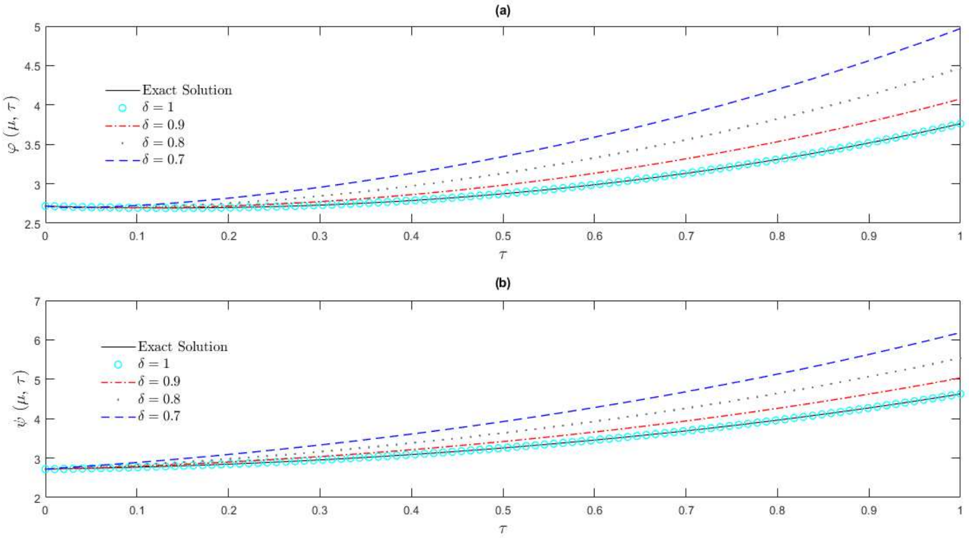

Fig. 5. (a) Plots of approximate solution $\varphi(\mu,\tau)$ for different values of δ and (b) Plots of approximate solution $\psi(\mu,\tau)$ for different values of δ with fixed value $\mu = 1$.

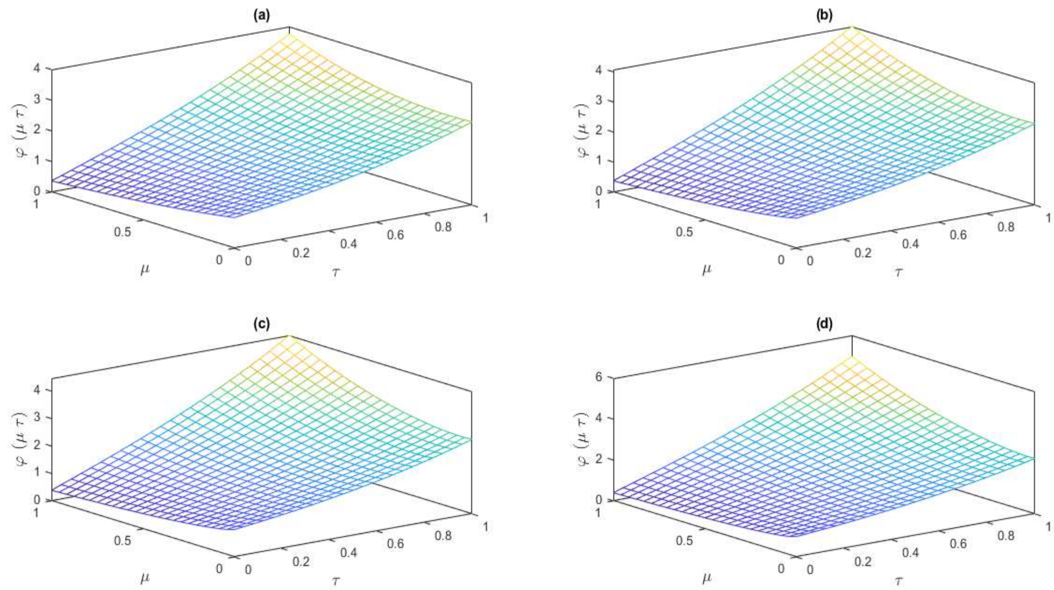

Fig. 6. The surface graph of the approximate solution $\varphi(\mu,\tau)$ of Eqs. (27) and (28): (a) $\varphi(\mu,\tau)$ when $\delta = 0.5$, (b) $\varphi(\mu,\tau)$ when $\delta = 0.7$, (c) $\varphi(\mu,\tau)$ when $\delta = 0.9$, (d) $\varphi(\mu,\tau)$ when $\delta = 1$.

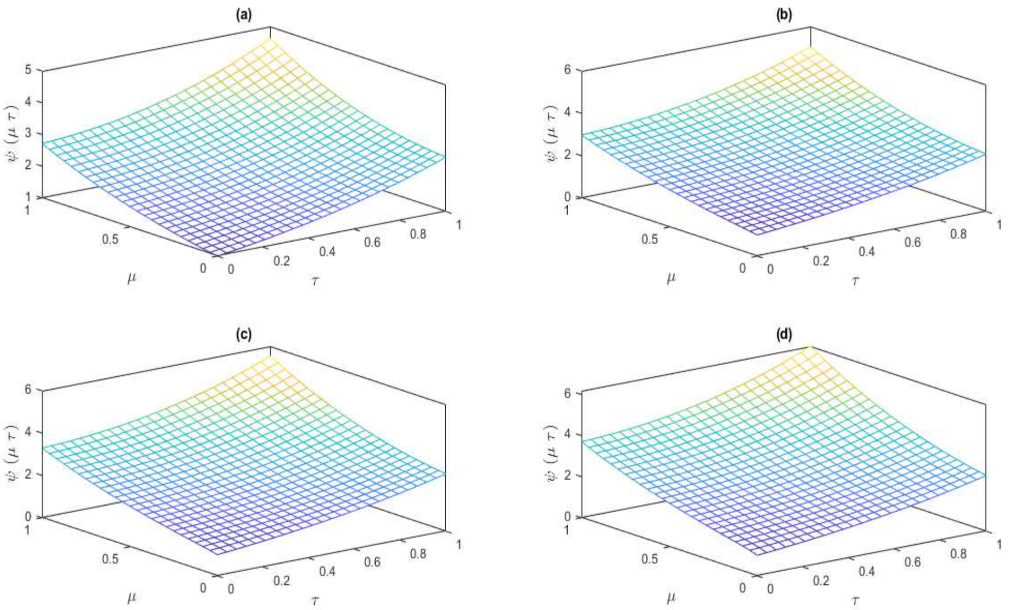

Fig. 7. The surface graph of the approximate solution $\psi(\mu,\tau)$ of Eqs. (27) and (28): (a) $\psi(\mu,\tau)$ when $\delta = 0.5$, (b) $\psi(\mu,\tau)$ when $\delta = 0.7$, (c) $\psi(\mu,\tau)$ when $\delta = 0.9$, (d) $\psi(\mu,\tau)$ when $\delta = 1$.

Table 3. Numerical values of the approximate and exact solutions $\varphi(\mu,\tau)$ among different values of μ and τ when $\delta = 0.7, 0.8, 0.9$ for Example 3.

		$\delta = 0.7$	$\delta = 0.8$	$\delta = 0.9$		Absolute Error
μ	τ	φ	φ	φ	φ_{Ex}	$\lvert\varphi - \varphi_{Ex}\rvert$
	0.25	1.0936	1.0984	1.1108	1.1276	0.0168
0.25	0.50	1.1008	1.0670	1.0487	1.0421	0.0066
	0.75	1.1889	1.1147	1.0605	1.0220	0.0385
	0.25	1.5783	1.5545	1.5461	1.5473	0.0012
0.50	0.50	1.7249	1.6388	1.5806	1.5431	0.0375
	0.75	1.9918	1.8385	1.7230	1.6358	0.0871
	0.25	2.8819	2.7946	2.7414	2.7107	0.0307
1.00	0.50	3.3442	3.1336	2.9821	2.8735	0.1086
	0.75	4.0315	3.6854	3.4211	3.2168	0.2043

Table 4. Numerical values of the approximate and exact solutions $\psi(\mu,\tau)$ among different values of μ and τ when $\delta = 0.7, 0.8, 0.9$ for Example 3.

		$\delta = 0.7$	$\delta = 0.8$	$\delta = 0.9$		Absolute Error
μ	τ	ψ	ψ	ψ	ψ_{Ex}	$\lvert\psi - \psi_{Ex}\rvert$
	0.25	1.7828	1.6691	1.5848	1.5211	0.0637
0.25	0.50	2.3337	2.1307	1.9754	1.8537	0.1216
	0.75	3.0307	2.7267	2.4907	2.3028	0.1878
	0.25	2.1151	1.9990	1.9152	1.8537	0.0615
0.50	0.50	2.6851	2.4672	2.3023	2.1752	0.1271
	0.75	3.4263	3.0939	2.8368	2.6333	0.2035
	0.25	3.2075	3.0642	2.9653	2.8966	0.0687
1.00	0.50	3.9266	3.6360	3.4198	3.2569	0.1629
	0.75	4.9015	4.4468	4.0967	3.8218	0.2749

5. Conclusion

The coupling of the Adomian decomposition method (ADM) and the Sumudu transform method (STM) in the sense of Caputo fractional derivative, was proved very effective in solving time-fractional partial differential equations and system of time-fractional partial differential equations. The solution was provided by the proposed algorithm in a series form that converges rapidly to the exact solution if it exists. From the obtained results, it is clear that the FSDM yields very accurate solutions using only a few iterates. As a result, the conclusion that comes through this work is that FSDM can be applied to other fractional partial differential equations of higher-order due to the efficiency and flexibility in the application, as can be seen in the proposed examples.

Author Contributions

H.K. Jassim wrote some sections of the manuscript; H.A. Al-Rkhais prepared some other sections of the paper and analyzed. All authors have read and approved the final version of the manuscript.

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Conflict of Interest

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Nomenclature

ADM	Adomian decomposition method	PDEs	Partial differential equations
ST	Sumudu transform		
FSDM	Fractional Sumudu decomposition method		

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