J. Appl. Comput. Mech., 8(2) (2022) 528-544
DOI: 10.22055/JACM.2020.31860.1935

ISSN: 2383-4536
jacm.scu.ac.ir

Shahid Chamran
University of Ahvaz

Journal of
Applied and Computational Mechanics

Research Paper

Haar Wavelet Method for Solving High-Order Differential Equations with Multi-Point Boundary Conditions

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Received December 10 2019; Revised April 23 2020; Accepted for publication April 24 2020.
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Abstract. In This paper, the developed Haar wavelet method for solving boundary value problems is described. As known, the orthogonal Haar basis functions are applied widely for solving initial value problems, but In this study, the method for solving systems of ODEs associated with multipoint boundary conditions is generalized in separated or non-separated forms. In this technique, a system of high-order boundary value problems of ordinary differential equations is reduced to a system of algebraic equations. The experimental results confirm the computational efficiency and simplicity of the proposed method. Also, the implementation of the method for solving the systems arising in the real world for phenomena in fluid mechanics and construction engineering approves the applicability of the approach for a variety of problems.

Keywords: High-order differential equations, Separated and non-separated boundary conditions, Haar wavelets, Multi-point boundary value problems.

1. Introduction

The systems of ordinary differential equations (ODEs) with boundary value conditions, called boundary value problems (BVPs), are well known for their applications in biology, chemistry, physics, and engineering [1-14]. There are many different reliable methods that can find the solution for the simple forms of boundary conditions, but the mathematical models of many phenomena in the real world are forced by different forms of boundary conditions such as multipoint separated boundary conditions or non-separated boundary conditions. Regarding the importance, the boundary value problems have been solved several times by many different methods such as finite difference method, spline method, radial basis functions, and many other numerical and analytical methods [15-27]. We remind that more difficulty of boundary conditions implies developing of the numerical methods to find the solution of the ordinary differential systems. In this study, we follow the Haar wavelet method in [28] and describe an effective scheme that is able to solve the arbitrary order differential equations with separated or non-separated boundary conditions.

Recently, orthogonal wavelet bases have been widely discussed and applied to approximate the solutions of some difficult systems defined in engineering and science, such as differential and integro-differential equations [29], fractional integral equations [30], fractional partial differential equations [31], fractional order differential equations [32], Fredholm-Volterra fractional integro-differential equations [33], fractional optimal control problems [34], time fractional diffusion-wave equation [35], two-dimensional convection-dominated and two-dimensional near singular elliptic equations [36], regularized long wave (RLW) equation [37], two-dimensional time fractional reaction-subdiffusion equation [38], modified Burgers' equation [39], coupled KdV equation [40], etc. For more research works, we refer the interested reader to [41-53]. Among the wavelet families, the Haar wavelet, which is the simplest and easy to implement, is well known for its use in signal analysis and image processing [54]. However, the discontinuity of the Haar wavelets prevents using them directly for solving differential equations, considering the highest order derivative of the unknown function to approximate gives continuous solutions. This method widely has been applied for solving the initial value problems (IVPs), but there are a few articles that can describe it for boundary value problems. However, some particular types of BVPs are solved by the Haar wavelet method, and it is still a costly challenge [19, 43, 55].

Obviously, the approximation of a given function is more accurate when the basis functions are chosen from a larger family of functions. In this study, the chosen basis functions are not only Haar basis functions but also can be chosen among monomials. Precisely, we extend the unknown function that appeared in differential equations under study in terms of Haar basis functions and monomials. This idea helps us to simply implement wavelet method for differential equations with boundary conditions. Furthermore, this technique solves higher order of BVPs and systems with more complexity such as multipoint separated or non-separated boundary conditions.

Published online: June 15 2020

In this work, the Haar wavelet method is used to solve a system of arbitrary-order boundary value problems associated with separated or non-separated boundary conditions. Let us consider the n-th order differential equations

$$F(Y(t),t)=0,\quad t\in[a,b], \tag{1}$$

associated with the following conditions

$$\begin{aligned} &g_1(Y(t_0),Y(t_1),\dots,Y(t_k))=0\\ &g_2(Y(t_0),Y(t_1),\dots,Y(t_k))=0\\ &\vdots\\ &g_n(Y(t_0),Y(t_1),\dots,Y(t_k))=0 \end{aligned} \tag{2}$$

where

$$\mathrm{Y}(t)=(y(t),y'(t),\dots,y^{(n-1)}(t),y^{(n)}(t)), \tag{3}$$

$$Y(t)=(y(t),y'(t),\dots,y^{(n-1)}(t)), \tag{4}$$

$t\in\mathbb{R}$ and $t_0,t_1,\dots,t_k$ (not necessarily distinct) are given real finite constants. If k = 0 the problem become the initial value problem and for k = 1 the problem will be called a two-point boundary value problem. It is also called multipoint boundary value problem if k > 1. In the particular case, for distinct $t_0,t_1,\dots,t_{n-1}$, the boundary conditions in the following type

$$\begin{aligned} &g_1(Y(t_0))=0,\\ &g_2(Y(t_1))=0,\\ &\vdots\\ &g_n(Y(t_{n-1}))=0, \end{aligned}$$

may be called separated and if the conditions are not separated, so will be called non-separated type. However this type of systems arising in engineering and sciences has important applications, but it is not possible to be solved analytically for arbitrary choices of F and g_i, $i=1,\dots,n$. Therefore, the numerical methods for obtaining an approximated solution of Eq. (1) with higher accuracy still is an interest of researchers.

To make the article more readable, in Section 2 a short description on the Haar wavelets is added. In Section 3 the description of method shows the Haar wavelet how can be applied to solve the high order differential equations with different kind of boundary conditions. The numerical results are illustrated in Section 4, and expectedly confirm the convergence and applicability of the method. Section 5 includes some phenomena in science and engineering that their models may be shown by the high order differential equations with separated and non-separated boundary conditions. Finally, a brief conclusion is drawn in Section 6.

2. Haar Basis Functions

The Haar wavelet family for $t\in[0,1]$ is defined as follows [42]

$$h_i(t)=\begin{cases} 1, & t\in\left[\dfrac{k}{m},\dfrac{k+0.5}{m}\right),\\ -1, & t\in\left[\dfrac{k+0.5}{m},\dfrac{k+1}{m}\right),\\ 0, & \text{otherwise}, \end{cases} \tag{5}$$

where m indicates level of wavelet $(m=2^j\ j=0,1,\dots,J)$ and $k=0,1,\dots,m-1$ is the translation parameter. Here J is the maximum level of resolution. Also, the index i is calculated according to the formula $i=m+k$ and the integer J determines the maximal level of resolution. The index $i=0$ is corresponds to the scaling function

$$h_0(t)=\begin{cases} 1, & t\in[0,1),\\ 0, & \text{otherwise}. \end{cases} \tag{6}$$

Each Haar wavelet is composed of a couple of constant steps of opposite sign during its subinterval and is zero elsewhere. So, they have the following orthogonality relationship:

$$\int_0^1 h_i(t)h_l(t)dt=\begin{cases} 2^{-j}, & i=l=2^j+k,\\ 0, & i\neq l. \end{cases} \tag{7}$$

Any square integrable function $f(t)$ in the interval $[0,1]$ can be expanded in a Haar series with an infinite number of terms as

$$f(t)=\sum_{n=0}^{\infty}c_n h_n(t),\quad t\in[0,1], \tag{8}$$

where the Haar coefficients c_n are calculated as follows

$$c_0 = \int_0^1 f(t)h_0(t)dt,$$

$$c_n = 2^j \int_0^1 f(t)h_n(t)dt, \quad n = 2^j + k, \ j \geq 0, \ 0 \leq k < 2^j.$$

In fact, the series expansion (8) contains an infinite number of terms for smooth $f(t)$. If $f(t)$ is a piecewise constant or may be approximated as a piecewise constant, the sum in (8) will be terminated after $m = 2^J$ terms, that is

$$f(t) \simeq f_m(t) = \sum_{n=0}^{m-1} c_n h_n(t) = C_m^T h_m(t), \quad t \in [0,1], \tag{9}$$

where

$$C_m = [c_0, c_1, \dots, c_{m-1}]^T \quad \text{and} \quad h_m(t) = [h_0(t), h_1(t), \dots, h_{m-1}(t)]^T. \tag{10}$$

Commonly, the collocation points are defined as the following form

$$\tau_l = \frac{2l-1}{2m}, \quad l = 1,2,\dots,m, \tag{11}$$

which also can be chosen non-uniformly [56]. Now substituting the collocation points leads to

$$f(\tau_l) \simeq \sum_{n=0}^{m-1} c_n h_n(\tau_l) = C_m^T h_m(\tau_l), \quad l = 1,2,\dots,m. \tag{12}$$

The matrix form of the above linear system is

$$F^T = C_m^T H_m, \tag{13}$$

where $F = [f(\tau_1), f(\tau_2), \dots, f(\tau_m)]^T$, H_m is the Haar wavelet matrix of order m defined by

$$H_m = \begin{bmatrix} h_0 \\ h_1 \\ \vdots \\ h_{m-1} \end{bmatrix} = \begin{bmatrix} h_{0,0} & h_{0,1} & \cdots & h_{0,m-1} \\ h_{1,0} & h_{1,1} & \cdots & h_{1,m-1} \\ \vdots & \vdots & \ddots & \vdots \\ h_{m-1,0} & h_{m-1,1} & \cdots & h_{m-1,m-1} \end{bmatrix}, \tag{14}$$

and $h_0, h_1, \dots, h_{m-1}$ are the discrete form of the Haar wavelet bases.

Theorem 2.1 [29] Let $f(t) = u^{(n)}(t) \in L_2[0,1]$ is a continuous function on $[0,1]$ and its first derivative is bounded that is

$$\exists L > 0 : |\frac{df}{dt}| \leq L, \ n \geq 2.$$

Then the Haar wavelet method, based on the proposed in [57, 58], will be convergent i.e. $e_J(t) = f(t) - f_M(t), M = 2^J$ vanishes as J goes to infinity. The convergence is of order two

$$\| e_J(t) \|_2 \leq \mathcal{O}(\frac{1}{2^{J+1}})^2.$$

Remark 2.2 [43] The Haar wavelets belong to the group of piecewise constant functions. It is known that if the function is sufficiently smooth, then the convergence rate for the piecewise constant function is $O(M^{-2})$ where $M = 2^J$; this result can be transferred also to the Haar wavelet approach. So it could be expected that by doubling the number of collocation points the error roughly decreases four times. Considering the system mentioned in (1) which is the n-th order boundary-value problem, to be able approximate the highest order derivative of the unknown function and prevent the discontinuity seen in (9), we first should define

$$P_1(t) = \left[\int_0^t h_0(s)ds, \int_0^t h_1(s)ds, \dots, \int_0^t h_{m-1}(s)ds \right]^T, \tag{15}$$

$$P_2(t) = \left[\int_0^t \int_0^t h_0(s)ds^2, \int_0^t \int_0^t h_1(s)ds^2, \dots, \int_0^t \int_0^t h_{m-1}(s)ds^2 \right]^T, \tag{16}$$

$$\vdots$$

$$P_n(t) = \left[P_{0n}(t), P_{1n}(t), \dots, P_{m-1,n}(t) \right]^T, \tag{17}$$

where

$$P_{in}(t) = \underbrace{\int_0^t \int_0^t \cdots \int_0^t}_{\text{n-times}} h_i(s)ds^n = \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} h_i(s)ds, \quad i = 0,1,\dots,m-1. \tag{18}$$

These integrals can be evaluated using the definition of Haar wavelet for $i=1,2,\ldots,m-1$ and are given as follows:

$$P_{in}(t)=\begin{cases} 0, & t\in\left[0,\dfrac{k}{m}\right), \\ \dfrac{1}{n!}\left(t-\dfrac{k}{m}\right)^n, & t\in\left[\dfrac{k}{m},\dfrac{k+0.5}{m}\right), \\ \dfrac{1}{n!}\left[\left(t-\dfrac{k}{m}\right)^n-2\left(t-\dfrac{k+0.5}{m}\right)^n\right], & t\in\left[\dfrac{k+0.5}{m},\dfrac{k+1}{m}\right), \\ \dfrac{1}{n!}\left[\left(t-\dfrac{k}{m}\right)^n-2\left(t-\dfrac{k+0.5}{m}\right)^n+\left(t-\dfrac{k+1}{m}\right)^n\right], & t\in\left[\dfrac{k+1}{m},1\right), \end{cases} \tag{19}$$

and for $i=0$ we can get

$$P_{0n}(t)=\begin{cases} \dfrac{t^n}{n!}, & t\in[0,1), \\ 0, & \text{otherwise.} \end{cases} \tag{20}$$

Remark 2.3 More generally, we define the Haar wavelet family over the interval $[a,b)$ as

$$\tilde{h}_0(t)=\begin{cases} 1 & t\in[a,b), \\ 0 & \text{otherwise,} \end{cases} \tag{21}$$

$$\tilde{h}_i(t)=\begin{cases} 1 & t\in[\alpha,\beta), \\ -1 & t\in[\beta,\theta), \\ 0 & \text{otherwise,} \end{cases} \tag{22}$$

where

$$\alpha=a+(b-a)\left(\frac{k}{m}\right),\ \beta=a+(b-a)\left(\frac{k+0.5}{m}\right),\ \theta=a+(b-a)\left(\frac{k+1}{m}\right), \tag{23}$$

$m=2^j,\ j=0,1,\ldots,J$ and $k=0,1,\ldots,m-1$. Also, the integrals of Haar function $\tilde{h}_i(t)$ can be evaluated as:

$$\tilde{P}_{0n}(t)=\begin{cases} \dfrac{1}{n!}(t-a)^n, & t\in[a,b), \\ 0, & \text{otherwise,} \end{cases} \tag{24}$$

and for $i=1,2,\ldots,m-1$

$$\tilde{P}_{in}(t)=\begin{cases} 0, & t\in[a,\alpha), \\ \dfrac{1}{n!}(t-\alpha)^n, & t\in[\alpha,\beta), \\ \dfrac{1}{n!}\left[(t-\alpha)^n-2(t-\beta)^n\right], & t\in[\beta,\theta), \\ \dfrac{1}{n!}\left[(t-\alpha)^n-2(t-\beta)^n+(t-\theta)^n\right], & t\in[\theta,b). \end{cases} \tag{25}$$

Now $y^{(n)}(t)$ may be considered as follows

$$y^{(n)}(t)\simeq\sum_{i=0}^{m-1}c_ih_i(t)=C_m^Th_m(t),\ \ t\in[0,1], \tag{26}$$

which gives

$$y(t)\simeq\sum_{i=0}^{m-1}c_iP_{in}(t)+\frac{y^{(n-1)}(0)}{(n-1)!}t^{n-1}+\ldots+\frac{y''(0)}{2!}t^2+y'(0)t+y(0),\ \ t\in[0,1]. \tag{27}$$

Similarly, by using the collocations points (11) we have

$$y(\tau_l)\simeq\sum_{i=0}^{m-1}c_iP_{in}(\tau_l)+\sum_{k=0}^{n-1}\frac{y^{(k)}(0)}{k!}\tau_l^k\quad l=1,2,\ldots,m, \tag{28}$$

and the matrix form of the above linear system is

$$Y_m^T = C_m^T Q_m + \sum_{k=0}^{n-1} \frac{y^{(k)}(0)}{k!} T_k^T, \tag{29}$$

where

$$Q_m = \begin{bmatrix} q_{0,0} & q_{0,1} & \cdots & q_{0,m-1} \\ q_{1,0} & q_{1,1} & \cdots & q_{1,m-1} \\ \vdots & \vdots & & \vdots \\ q_{m-1,0} & q_{m-1,1} & \cdots & q_{m-1,m-1} \end{bmatrix}, \quad q_{i,k} = P_{in}(\tau_k), \quad i,k = 0,1,\ldots,m-1, \tag{30}$$

$$Y_m = [y(\tau_1), y(\tau_2), \ldots, y(\tau_m)]^T, T_k = [\tau_1^k, \tau_2^k, \ldots, \tau_m^k]^T, \quad k = 0,1,\ldots,n-1. \tag{31}$$

3. Description of the Method

Regarding to the previous section, if the initial values are given in (27), the function can be approximated by Haar basis functions and consequently we can expand the solution of initial value problems. Now we want to propose the technique for solving boundary value problems in the general form of (1) on the interval $[a,b]$ when some initial values might be un-given. We assume that $y^{(n)}(t)$ can be expanded in terms of Haar wavelets as

$$y^{(n)}(t) = \sum_{i=0}^{m-1} c_i h_i(t), \qquad t \in [a,b]. \tag{32}$$

which gives

$$y^{(n-1)}(t) = \sum_{i=0}^{m-1} c_i \tilde{P}_{i1}(t) + y^{(n-1)}(a),$$

$$y^{(n-2)}(t) = \sum_{i=0}^{m-1} c_i \tilde{P}_{i2}(t) + t\, y^{(n-1)}(a) + y^{(n-2)}(a),$$

$$y^{(n-3)}(t) = \sum_{i=0}^{m-1} c_i \tilde{P}_{i3}(t) + \frac{t^2}{2} y^{(n-1)}(a) + t\, y^{(n-2)}(a) + y^{(n-3)}(a),$$

$$\vdots$$

$$y(t) = \sum_{i=0}^{m-1} c_i \tilde{P}_{in}(t) + \sum_{k=0}^{n-1} \frac{t^k}{k!} y^{(k)}(a).$$

Without loss of generality, we may assume $y^{(j)}(a), j = 0,1,\ldots,n-1$ are unknown. Thus the above equations can be rewritten as follow

$$y^{(n-1)}(t) = \sum_{i=0}^{m-1} c_i \tilde{P}_{i1}(t) + c_m,$$

$$y^{(n-2)}(t) = \sum_{i=0}^{m-1} c_i \tilde{P}_{i2}(t) + t c_m + c_{m+1},$$

$$y^{(n-3)}(t) = \sum_{i=0}^{m-1} c_i \tilde{P}_{i3}(t) + \frac{t^2}{2} c_m + t c_{m+1} + c_{m+2},$$

$$\vdots$$

$$y(t) = \sum_{i=0}^{m-1} c_i \tilde{P}_{in}(t) + \sum_{k=0}^{n-1} \frac{t^k}{k!} c_{m+k},$$

where $c_{m+k} = y^{(n-k-1)}(a),\ k = 0,1,\ldots,n-1$. Now replacing the expansion of $y^{(j)}(t)$, $j = 0,1,\ldots,n$ into the system of (1) and (2) and discretizing the results at the collocation points (11), we get following equations

$$F(Y(\tau_l), \tau_l) = 0, \quad l = 1,2,\ldots,m, \tag{33}$$

$$g_i(Y(t_0), Y(t_1), \ldots, Y(t_k)) = 0, \quad i = 1,2,\ldots,n. \tag{34}$$

Solving the obtained system of (33) and (34) including $(m+n)$ equations gives the value of unknown coefficients $c_i, i = 0,1,\ldots,(m+n-1)$ and therefore the functions $y^{(j)}(t), j = 0,1,\ldots,n$ are approximately identified.

Remark 3.1 The point is worth to mention that we can simply do some small modifications when some of $y^{(i)}(a), i = 0,1,\ldots,n-1$, are given. Particularly if $y^{(i)}(a)$, $i=0,1,\dots,n-1$ all are given, the system become an initial value problem and no need to define any $c_i, i=m,\dots,m+n-1$. Also we can keep the structure of the algorithm and input the given initial value into the described scheme. Obviously the first state considers the value of $y^{(i)}(a)$, $1\le i\le n$, precisely and the second state find them approximately such that there are good agreement between precise and approximated values. In this paper, the reported results are based on the second assumption.

Remark 3.2 Definitely we need $(n+m)$ equations which are linear independent to find a unique solution including $(n+m)$ unknown coefficients. Note that the intersection of $t_0,t_1,\dots,t_k$ defined in (2), and the collocation points defined (11) should be an empty set. If there is any common point, we can simply change the collocation points non-uniformly. Obviously, there are many set of points can be appropriate candidate which lead to the independent equations [56].

4. Numerical Experiments

In this section, we solve some examples to describe more details of the implementation of rationalized Haar basis functions. The obtained numerical results confirm the validity and efficiency of the proposed method. The associated computations with the examples were performed using MAPLE 17 with 64 digits precision on a Personal Computer. To see the convergence of the proposed method, we employ the following error formulas

$$e_r = \| y^{(r)}_{\text{Exact}}(t) - y^{(r)}_{\text{Haarwavelet}}(t)\|_\infty, \quad r=0,1,2,\dots,$$

$$ER_r = \text{Log}_{10}(| y^{(r)}_{\text{Exact}}(t) - y^{(r)}_{\text{Haarwavelet}}(t)|), \quad r=0,1,2,\dots.$$

Also, we obtained the computational order (C-Order) of the proposed method by using the following formula [59]

$$\text{C}-\text{Order} = \frac{\text{Log}(\frac{\| E_J(t)\|_\infty}{\| E_{J+1}(t)\|_\infty})}{\text{Log}(2)},$$

where $E_J(t) = y_{\text{Exact}}(t) - y_M(t)$.

Example 1. Consider the following ordinary differential equation [24, 25]

$$y^{(6)}(t) + y(t) = 6(2t\cos(t) + 5\sin(t)), \qquad t\in[-1,1],$$

with the separated boundary conditions

$$\begin{gathered} y(-1)=y(1)=0,\\ y''(-1) = -4\cos(-1)+2\sin(-1),\\ y''(1) = 4\cos(1)+2\sin(1),\\ y^{(4)}(-1) = 8\cos(-1) - 12\sin(-1),\\ y^{(4)}(1) = -8\cos(1)-12\sin(1), \end{gathered}$$

and the exact solution

$$y(t) = (t^2-1)\sin(t).$$

According to the algorithm, we firstly approximate $y^{(6)}(t)$ as follows

$$y^{(6)}(t) \simeq \sum_{i=0}^{m-1} c_i \tilde{h}_i(t), \qquad t\in[-1,1],$$

where $\tilde{h}_i(t)$ are the Haar basis functions defined in (22) on $[a,b]=[-1,1]$. Let assume $J=2$ which means $m=4$, by integration one can find $y^{(j)}(t), j=0,1,\dots,5,$ as follow

$$y^{(5)}(t) = \sum_{i=0}^{3} c_i \tilde{P}_{i1}(t) + c_4,$$

$$y^{(4)}(t) = \sum_{i=0}^{3} c_i \tilde{P}_{i2}(t) + tc_4 + c_5,$$

$$\vdots$$

$$y(t) = \sum_{i=0}^{3} c_i \tilde{P}_{i6}(t) + \frac{t^5}{120}c_4 + \frac{t^4}{24}c_5 + \frac{t^3}{6}c_6 + \frac{t^2}{2}c_7 + tc_8 + c_9.$$

The chosen uniform collocation points, $-3/4$, $-1/4$, $1/4$ and $3/4$, should be substituted into the following equation

$$\sum_{i=0}^{3} c_i \tilde{h}_i(t) + \sum_{i=0}^{3} c_i \tilde{P}_{i6}(t) + \frac{t^5}{120}c_4 + \frac{t^4}{24}c_5 + \frac{t^3}{6}c_6 + \frac{t^2}{2}c_7 + tc_8 + c_9 - 6(2t\cos(t)+5\sin(t)) = 0,$$

which gives four algebraic linear equations. Also, we will get other six linear equations as follow

$$\sum_{i=0}^{3} c_i \widetilde{P}_{i6}(-1) - \frac{1}{120}c_4 + \frac{1}{24}c_5 - \frac{1}{6}c_6 + \frac{1}{2}c_7 - c_8 + c_9 = 0,$$

$$\sum_{i=0}^{3} c_i \widetilde{P}_{i6}(1) + \frac{1}{120}c_4 + \frac{1}{24}c_5 + \frac{1}{6}c_6 + \frac{1}{2}c_7 + c_8 + c_9 = 0,$$

$$\sum_{i=0}^{3} c_i \widetilde{P}_{i4}(-1) - \frac{1}{6}c_4 + \frac{1}{2}c_5 - c_6 + c_7 + 4\cos(-1) - 2\sin(-1) = 0,$$

$$\sum_{i=0}^{3} c_i \widetilde{P}_{i4}(1) + \frac{1}{6}c_4 + \frac{1}{2}c_5 + c_6 + c_7 - 4\cos(1) + 2\sin(1) = 0,$$

$$\sum_{i=0}^{3} c_i \widetilde{P}_{i2}(-1) - c_4 + c_5 - 8\cos(-1) + 12\sin(-1) = 0,$$

$$\sum_{i=0}^{3} c_i \widetilde{P}_{i2}(1) + c_4 + c_5 + 8\cos(1) + 12\sin(1) = 0,$$

which are included ten unknown coefficients. Solving the obtained system gives

$$c_0 = 0, \qquad c_5 = 11.56970,$$

$$c_1 = -18.94644, \quad c_6 = 11.19923,$$

$$c_2 = -8.38589, \quad c_7 = 1.09517,$$

$$c_3 = -8.38589, \quad c_8 = -0.77553,$$

$$c_4 = -2.85037, \quad c_9 = 0.03760,$$

to evaluate $y^{(r)}, r = 0,1,\dots,6$ and Table 1 includes the observed absolute error by these values. Table 2 shows the computational orders obtained for this example. Also, the absolute error function when $J = 6$ is illustrated in Figure 1 which shows high accuracy of the method. It is interesting to note that following the Remark 3.1 regarding to

$$c_5 = y^{(4)}(-1), \qquad c_7 = y''(-1), \qquad c_9 = y(-1),$$

which are given, the scheme also can be reduced to 7 equations including 7 unknown coefficients. Furthermore, this example with other boundary conditions discussed in [24, 25] also straightforwardly can be reduced to the system of linear equations as described.

Example 2. Consider the following two-point boundary value problem [20, 24]

$$y^{(6)}(t) + t\,y(t) + \left(t^3 + 11t + 24\right)e^t = 0, \qquad t \in [0,1],$$

with the separated boundary conditions

$$y(0) = y(1) = 0,$$

$$y''(0) = 0, \qquad y''(1) = -4e,$$

$$y^{(4)}(0) = -8, \qquad y^{(4)}(1) = -16e,$$

Table 1. The observed maximum absolute error for different values of J for Example 1.

e_r	$J=2$	$J=3$	$J=4$	$J=5$	$J=6$
e_0	3.5×10^{-4}	1.1×10^{-4}	2.9×10^{-5}	7.1×10^{-6}	1.7×10^{-6}
e_1	1.1×10^{-3}	3.4×10^{-4}	9.0×10^{-5}	2.3×10^{-5}	5.5×10^{-6}
e_2	3.6×10^{-3}	1.1×10^{-3}	2.8×10^{-4}	6.9×10^{-5}	1.7×10^{-5}
e_3	1.2×10^{-2}	3.5×10^{-3}	9.1×10^{-4}	2.4×10^{-4}	5.9×10^{-4}
e_4	7.1×10^{-2}	1.5×10^{-2}	3.1×10^{-3}	8.0×10^{-4}	1.8×10^{-4}
e_5	8.0×10^{-1}	1.9×10^{-1}	5.0×10^{-2}	1.2×10^{-2}	3.0×10^{-3}

Table 2. Computational orders obtained for Examples 1-3.

J	Example 1	Example 2	Example 3
3	1.6699	1.9542	2.2439
4	1.9234	1.9296	2.0780
5	1.9320	2.0418	2.0324
6	2.0623	2.0875	2.0265

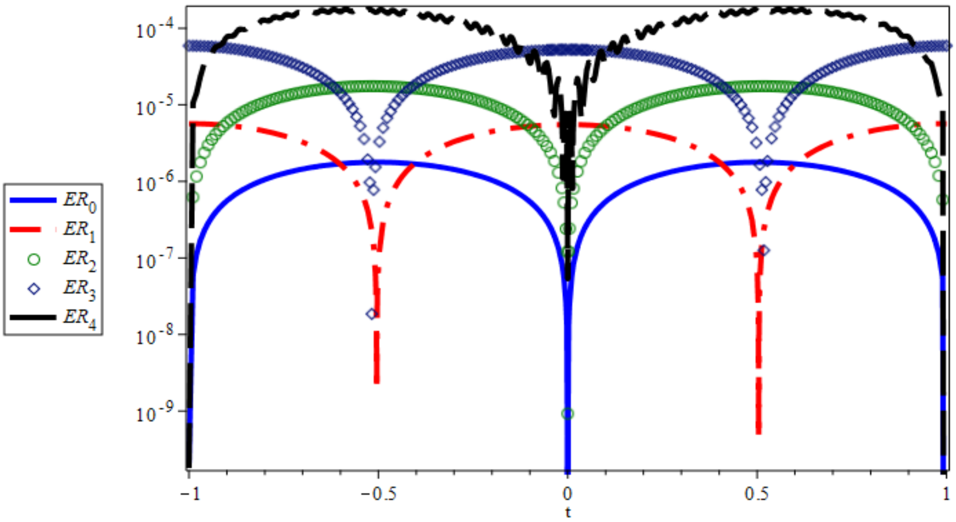

Fig. 1. The absolute error functions of $y, y', y'', y^{(3)}, y^{(4)}$ when the level of resolutions is $J = 6$ for Example 1.

and the exact solution

$$y(t) = t(1-t)e^{t}.$$

Similarly, we approximate $y^{(6)}(t)$ using the Haar basis functions defined in (22) on $[a,b]=[0,1]$ and then successively integration to prepare needed terms. As before, assume $m = 4$

$$\sum_{i=0}^{3} c_i h_i(t) + \sum_{i=0}^{3} c_i t \tilde{P}_{i6}(t) + \frac{t^6}{120}c_4 + \frac{t^5}{24}c_5 + \frac{t^4}{6}c_6 + \frac{t^3}{2}c_7 + t^2 c_8 + t c_9 + \left(t^3 + 11t + 24\right)e^{t} = 0,$$

and uniform collocation points $1/8$, $3/8$, $5/8$ and $7/8$ which gives four algebraic linear equations. Also, we will get other 6 linear equations as follow

$$\sum_{i=0}^{3} c_i \tilde{P}_{i6}(0) + c_9 = 0,$$

$$\sum_{i=0}^{3} c_i \tilde{P}_{i6}(1) + \frac{1}{120}c_4 + \frac{1}{24}c_5 + \frac{1}{6}c_6 + \frac{1}{2}c_7 + c_8 + c_9 = 0,$$

$$\sum_{i=0}^{3} c_i \tilde{P}_{i4}(0) + c_7 = 0,$$

$$\sum_{i=0}^{3} c_i \tilde{P}_{i4}(1) + \frac{1}{6}c_4 + \frac{1}{2}c_5 + c_6 + c_7 + 4e = 0,$$

$$\sum_{i=0}^{3} c_i \tilde{P}_{i2}(0) + c_5 + 8 = 0,$$

$$\sum_{i=0}^{3} c_i \tilde{P}_{i2}(1) + c_4 + c_5 + 16e = 0,$$

Solving the obtained system gives

$$c_0 = -52.70217, \quad c_5 = -8,$$

$$c_1 = 17.75331, \quad c_6 = 3.00940,$$

$$c_2 = 6.17749, \quad c_7 = 0,$$

$$c_3 = 12.04343, \quad c_8 = 1.0009,$$

$$c_4 = -14.71856, \quad c_9 = 0,$$

to evaluate $y^{(r)}, r = 0,1,\ldots,6$ and Table 3 includes the absolute error by these values. Table 2 shows the computational orders obtained for Example 2. Also, the absolute error function when $J = 6$ is illustrated in Figure 2 which shows computational efficiency of the method.

Example 3. Consider the following nonlinear differential equation [21]

$$y^{(4)}(t) + y'(t)y'''(t) - 4(y'(t))^2 y''(t) = 0, \qquad t \in [0,1],$$

Table 3. The observed maximum absolute error for different values of J for Example 2.

e_r	$J=2$	$J=3$	$J=4$	$J=5$	$J=6$
e_0	3.1×10^{-4}	8.0×10^{-5}	2.1×10^{-5}	5.1×10^{-6}	1.2×10^{-6}
e_1	9.6×10^{-4}	2.5×10^{-4}	6.3×10^{-5}	1.6×10^{-6}	4.0×10^{-6}
e_2	3.0×10^{-3}	7.9×10^{-4}	2.0×10^{-4}	5.0×10^{-5}	1.3×10^{-5}
e_3	9.6×10^{-3}	2.6×10^{-3}	6.8×10^{-4}	1.7×10^{-4}	4.1×10^{-5}
e_4	3.9×10^{-2}	8.9×10^{-3}	2.1×10^{-3}	5.0×10^{-4}	1.2×10^{-4}
e_5	5.8×10^{-1}	1.3×10^{-1}	3.4×10^{-2}	8.2×10^{-3}	2.1×10^{-3}

Table 4. The observed maximum absolute error for different values of J for Example 3.

e_r	$J=2$	$J=3$	$J=4$	$J=5$	$J=6$
e_0	9.0×10^{-5}	1.9×10^{-5}	4.5×10^{-6}	1.1×10^{-6}	2.7×10^{-7}
e_1	6.1×10^{-4}	1.2×10^{-4}	3.0×10^{-5}	7.1×10^{-6}	1.8×10^{-6}
e_2	5.5×10^{-3}	1.6×10^{-3}	4.2×10^{-4}	1.1×10^{-4}	2.7×10^{-5}
e_3	6.2×10^{-2}	2.4×10^{-2}	6.9×10^{-3}	1.9×10^{-3}	4.9×10^{-4}
e_4	2.2	1.3	0.7	3.0×10^{-1}	1.8×10^{-1}

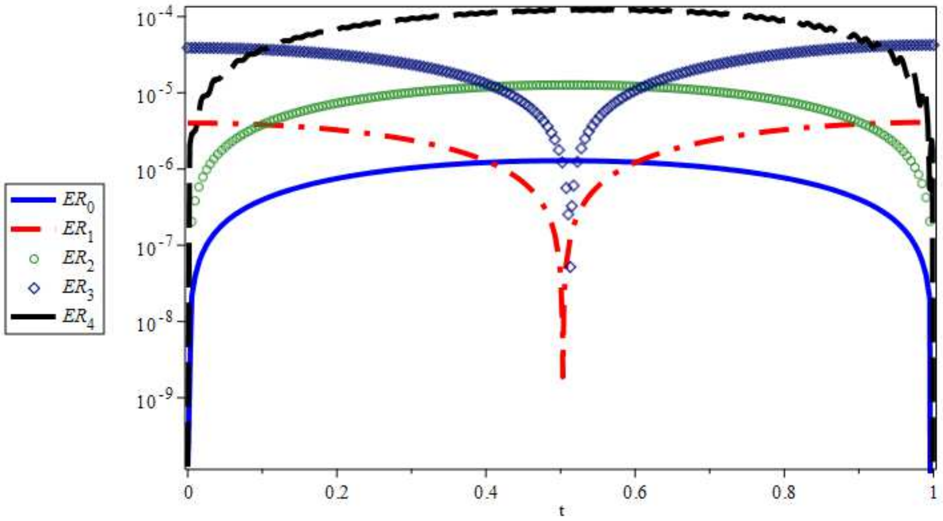

Fig. 2. The absolute error functions of $y, y', y'', y^{(3)}, y^{(4)}$ when the level of resolutions is $J=6$ for Example 2.

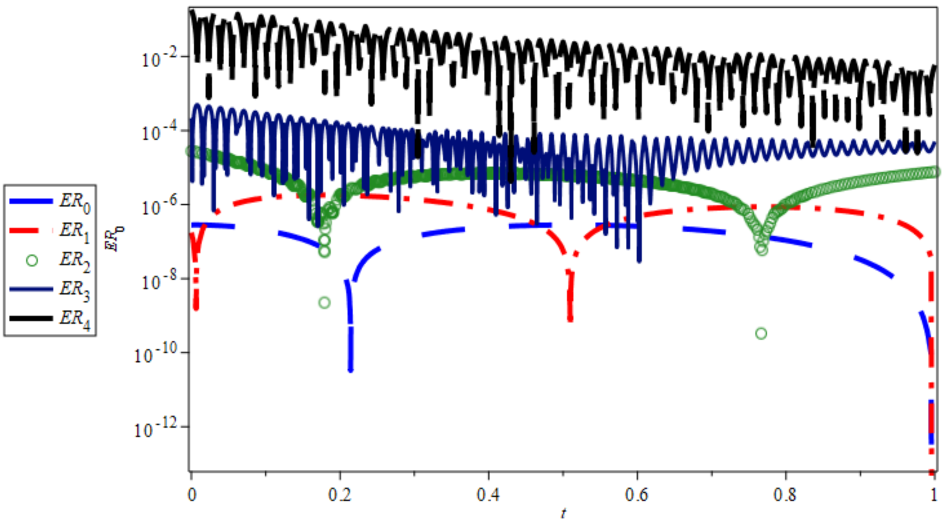

Fig. 3. The absolute error functions of $y, y', y'', y^{(3)}, y^{(4)}$ when the level of resolutions is $J=6$ for Example 3.

with the non-separated boundary conditions

$$y(0)+y(\frac{1}{2})+y(1)=\ln(3), \quad y(1)=\ln(2),$$

$$y'(0)+3y'(\tfrac{1}{2})-2y'(1)=2,\quad y'(1)=\frac{1}{2},$$

and the exact solution

$$y(t)=\ln(1+t).$$

The observed maximum absolute error for different values of J to evaluate $y^{(r)}, r=0,1,\ldots,4$ are reported in Table 4 and the absolute error functions of $y, y', y'', y^{(3)}, y^{(4)}$ are demonstrated in Figure 3. Also, Table 2 shows the computational orders obtained for this example.

Example 4. Consider the fifth-order nonlinear boundary value problems [17, 26, 27]

$$y^{(5)}(t)-e^{-t}y^{2}(t)=0,\qquad t\in[0,1],$$

which is associated with one of the following cases:

Case I: The two-point separated boundary conditions

$$y(0)=y'(0)=y''(0)=1,\qquad y(1)=y'(1)=e,$$

Case II: The two-point non-separated boundary conditions

$$y^{2}(0)-2y(0)y(1)+y''(1)+y^{(4)}(1)-1=0,$$

$$y^{(3)}(1)+y''(0)-y'(1)+y^{(3)}(0)-2y^{(4)}(0)=0,$$

$$y(0)y''(0)+(y'(1))^{2}-y''(1)y'(1)-2y^{(3)}+1=0,$$

$$y''(0)+y^{(3)}(1)-2y^{(4)}(1)-y^{(4)}(0)+e=0,$$

$$y^{(3)}(0)-y(1)+y''(1)+y(0)-2=0,$$

Case III: The multipoint non-separated boundary conditions

$$y(0)+y'(\tfrac{2}{5})+y''(\ln(2))+y^{(4)}(\tfrac{5}{6})+y(1)=3+e+e^{\frac{2}{5}}+e^{\frac{5}{6}},$$

$$y'(0)-y^{(4)}(\tfrac{1}{8})+y(\tfrac{1}{3})+y''(\tfrac{5}{6})+y^{(3)}(1)=1+e-e^{\frac{1}{8}}+e^{\frac{1}{3}}+e^{\frac{5}{6}},$$

$$y(0)+y'(\tfrac{2}{5})+y(\ln(2))+y^{(3)}(\tfrac{4}{9})+y''(1)=3+e+e^{\frac{2}{5}}+e^{\frac{4}{9}},$$

$$y''(0)+y(\tfrac{1}{3})+y'(\ln(2))-y^{(4)}(\tfrac{5}{6})+y(1)=3+e+e^{\frac{1}{3}}-e^{\frac{5}{6}},$$

$$2y^{(3)}(0)+y''(\tfrac{1}{3})+y^{(3)}(\ln(\sqrt{3}))+y'(\tfrac{8}{9})+y^{(4)}(1)=2+\sqrt{3}+e+e^{\frac{1}{3}}+e^{\frac{8}{9}},$$

and the exact solution $y=e^{t}$. The observed maximum absolute error for different cases of boundary conditions to evaluate $y^{(r)}, r=0,1,\ldots,5$ are shown in Table 5. Also, the absolute error function for the second case when $J=7$ are demonstrated in Figure 4.

Example 5. Consider the nonlinear singular two-point boundary value problem [63, 61]

$$(ty'(t))'+te^{y(t)}=0,\qquad t\in[0,1],$$

with the boundary conditions

$$y'(0)=0,\quad y(1)=0.$$

Table 5. The observed maximum absolute error for different cases of boundary conditions for Example 4.

	Case I		Case II		Case III	
e_r	$J=5$	$J=6$	$J=5$	$J=6$	$J=5$	$J=6$
e_0	1.8×10^{-8}	2.5×10^{-5}	1.5×10^{-6}	6.1×10^{-6}	1.0×10^{-4}	9.2×10^{-6}
e_1	7.3×10^{-8}	7.1×10^{-5}	4.2×10^{-6}	1.7×10^{-5}	1.5×10^{-4}	6.8×10^{-6}
e_2	1.1×10^{-6}	1.4×10^{-4}	8.1×10^{-6}	3.5×10^{-5}	2.6×10^{-4}	1.7×10^{-5}
e_3	1.2×10^{-5}	1.9×10^{-4}	1.2×10^{-5}	5.1×10^{-5}	4.5×10^{-4}	3.0×10^{-5}
e_4	1.7×10^{-4}	3.2×10^{-4}	2.0×10^{-5}	8.0×10^{-5}	5.0×10^{-4}	3.9×10^{-5}
e_5	4.1×10^{-2}	4.1×10^{-2}	1.1×10^{-2}	2.2×10^{-2}	4.2×10^{-2}	1.3×10^{-2}

Table 6. The observed maximum absolute error for Example 5.

$M=2^J$ (Mesh points)	Present method	n (No. of terms)	HPM [63]	n (Mesh points)	B-spline method [61]
8	1.1617×10^{-5}	10	1.3395×10^{-5}	20	3.1607×10^{-5}
16	2.2876×10^{-6}	14	3.0159×10^{-6}	40	7.8742×10^{-6}

Table 7. The observed maximum absolute error for Example 6.

$M=2^J$ (Mesh points)	Present method	n (No. of terms)	HPM [63]	n (Mesh points)	FDM solution [60]
8	2.3666×10^{-4}	12	2.6371×10^{-4}	16	3.6400×10^{-4}
16	1.3922×10^{-5}	22	1.6482×10^{-5}	32	2.4900×10^{-5}

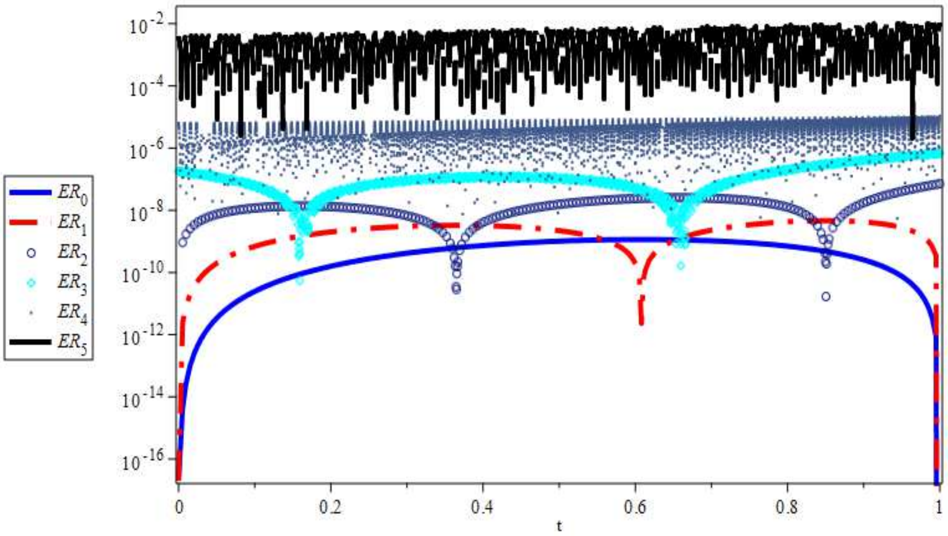

Fig. 4. The absolute error functions of $y, y', y'', y^{(3)}, y^{(4)}, y^{(5)}$ when the level of resolutions is $J=7$ for Example 4 with boundary defined in Case II.

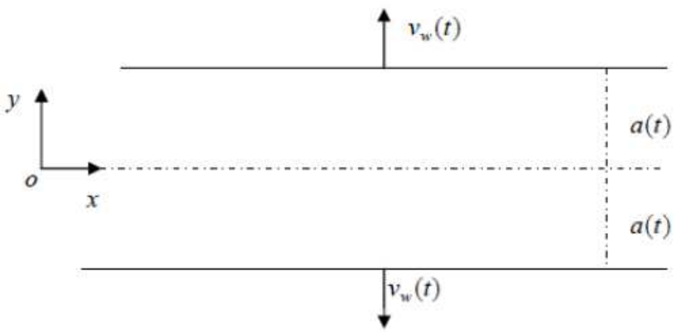

Fig. 5. Geometry of the problem [62].

The exact solution is given by $y(t)=2\ln((A+1)/(At^2+1))$ with $A=3-2\sqrt{2}$. Table 6 shows a comparison between the maximum absolute error obtained by presented method, the homotopy perturbation method [63] and the B-spline method [61].

Example 6. Consider the nonlinear singular boundary value problem describing the equilibrium of the isothermal gas sphere [63, 60]

$$(t^2y'(t))'+t^2y^5(t)=0, \qquad t\in[0,1],$$

with the boundary conditions

$$y'(0)=0, \quad y(1)=\frac{\sqrt{3}}{2}.$$

The exact solution is $y(t)=\sqrt{3/(3+t^2)}$. Table 7 shows a comparison between the maximum absolute error obtained by presented method, the homotopy perturbation method [63] and the three-point fourth-order finite-difference method (FDM) [60].

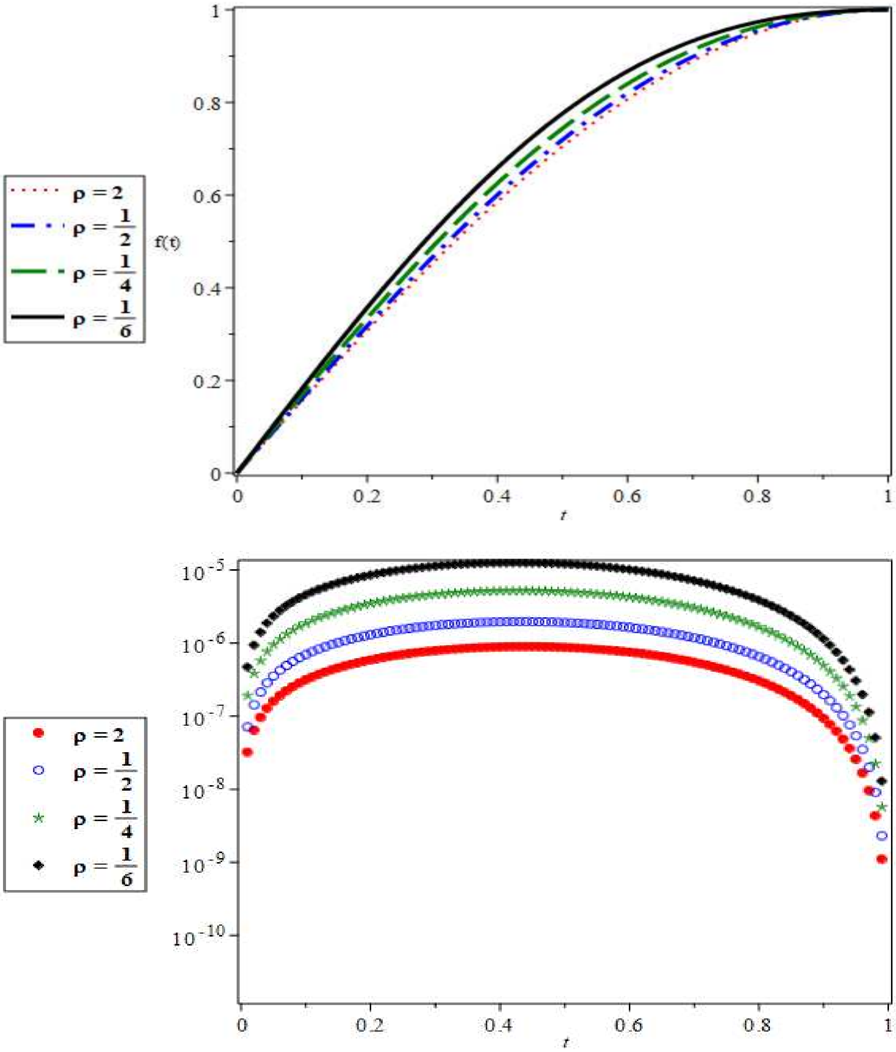

Fig. 6. Velocity Profiles $f(t)$ and absolute error function for $M = R = 1$ and various values of ρ.

5. Applications

The high order differential equations with separated and non-separated boundary are applied to model many problems in different areas of science and engineering such as mechanics, physics, electronics, etc. In this section we present two applications of these problems studied in this paper. These materials are taken from [62, 18].

5.1 The unsteady MHD flow of a viscous fluid between moving parallel plates

Because of its many industrial and practical applications, the unsteady MHD flow of a viscous fluid between moving parallel plates is an important area of study. Such a flow problem lends itself to applications in bearings with liquid-metal lubrications [62].

We consider the unsteady hydromagnetic squeezing flow of an incompressible two-dimensional viscous fluid between two infinite plates.

Figure 5 which is taken from [62], shows the flow geometry of the problem where $a(t)$ denotes the distance between each plate and the x-axis at any time t. The unsteady conservation of mass and momentum equations describing the flow are

$$u_x + v_y = 0,$$

$$\rho(u_t + uu_x + vu_y) = -p_x + \nu(u_{xx} + u_{yy}) - \sigma B_0^2 u,$$

$$\rho(v_t + uv_x + vv_y) = -p_y + \nu(v_{xx} + v_{yy}) - \sigma B_0^2 u,$$

where u and v are the velocity components along the x- and y-directions, respectively, ν denotes the kinematic viscosity, ρ denotes the density of the fluid, σ is the electrical conductivity of the fluid and B_0 is the magnitude of the uniform magnetic field acting along the y-axis. We impose the boundary conditions

$$u_y = 0, \;\; v = 0, \;\; \text{at} \;\; y = 0,$$

$$u = 0, \;\; v = v_\omega(t) \;\; \text{at} \;\; y = a(t) > 0,$$

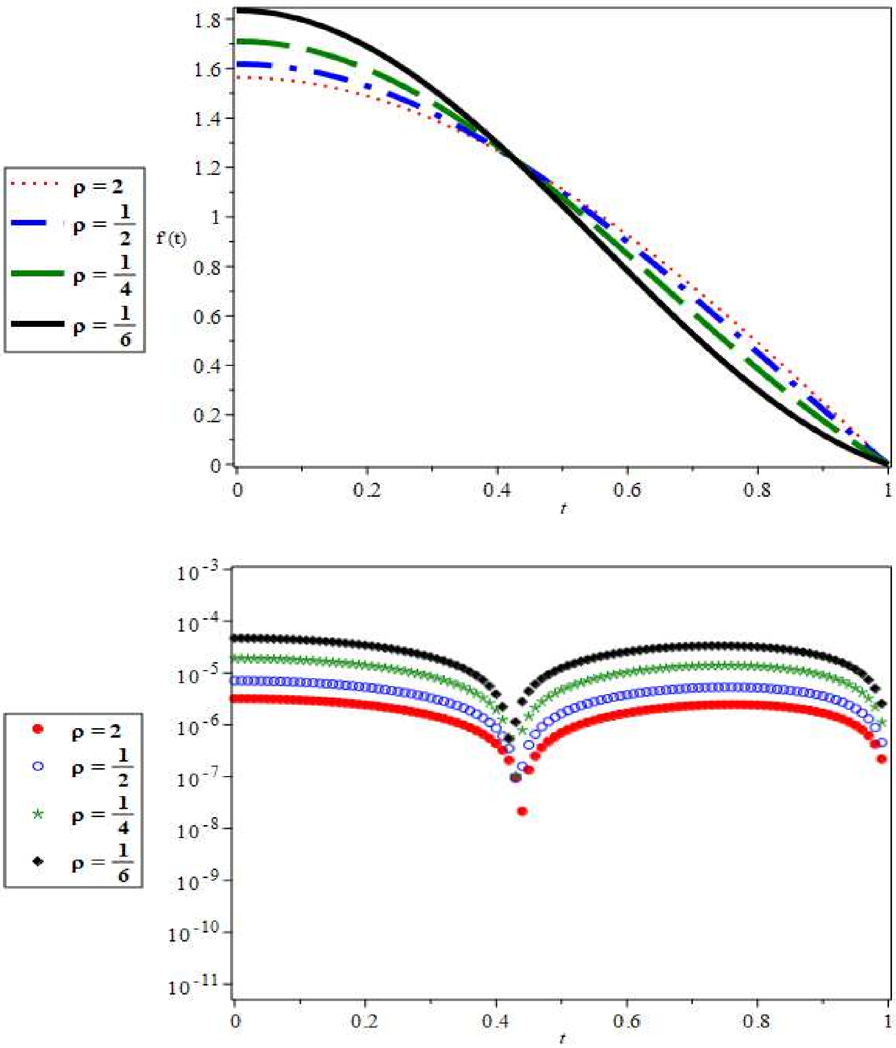

Fig. 7. Velocity Profiles $f'(t)$ and absolute error function for $M = R = 1$ and various values of ρ.

where $v_{\omega}(t) = da(t)/dt$ is the velocity of the plate. As is said in [62], the governing equations can be converted to the following nonlinear differential equation with separated boundary conditions

$$f^{(4)}(t) - M^2 f''(t) = R\left(f(t)f'''(t) - f'(t)f''(t) - tf'''(t) - \left(2 + \frac{1}{\rho}\right)f''(t)\right),$$

$$f(0) = 0 \;\; f(1) = 1, \;\; f''(0) = 0, \;\; f'(1) = 0,$$

where R is the parameter describing the movement of the plates, $R > 0$ corresponds to the plate moving apart, while $R < 0$ corresponds to the plates moving together, and M^2 is the magnetic parameter. In order to assess the validity and accuracy of the obtained results, we compare the approximate results given by presented method for $J = 6$ with numerical solution obtained by fourth order Runge-Kutta method and shooting method for $M = R = 1$ and $\rho = 2, \frac{1}{2}, \frac{1}{4}, \frac{1}{6}$ in Figures 6 and 7. It can be seen from Figures 6 and 7 that the solutions obtained by the proposed procedure are in good agreement with the RK4-based solutions.

5.2 Multipoint boundary value problem for optimal bridge design

The second application is about the role of ordinary differential equation with non-separated boundary conditions in designing bridges [18]. Consider a second order ordinary differential equation in bridge design

$$y''(t) + f(t) + g(t, y(t)) = 0,$$

where $y(t)$ denotes the displacement of the bridge from the unloaded position. As said in [18], small size bridges are often designed with two supported points (See Figure 8(left)) and large size of them are sometimes contrived with multipoint supports (See Figure 8(right)), which leads standard two-point boundary value conditions

$$y(0) = 0, \quad y(1) = 1,$$

and multipoint boundary value conditions, respectively.

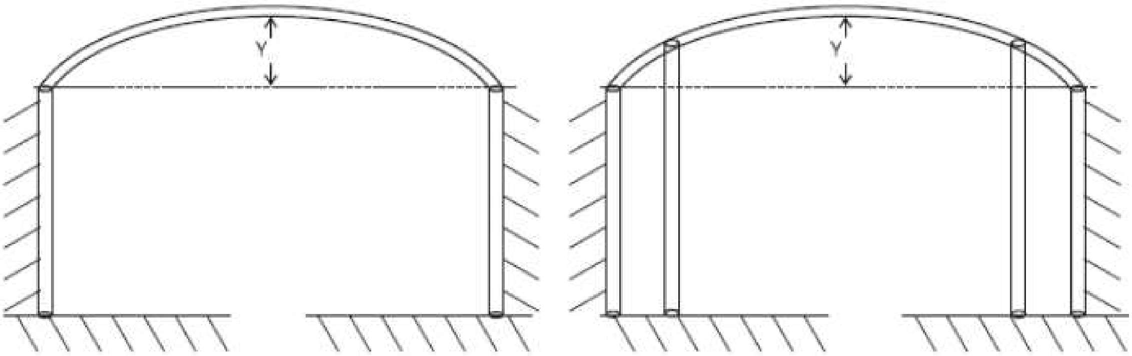

Fig. 8. Simplified models of bridge, two-point supported (left) and multipoint supported (right).

For large size, near each endpoint of the bridge, we can set up two different type of boundary conditions. If we emphasize the position of the bridge at supporting points near $t = 0$, we propose the following boundary value condition

$$y(0) = \sum_{i=1}^{p-2} \alpha_i y(\xi_i) + \lambda_1,$$

and if we are interested in controlling the angles of the bridge at supporting points near $t = 0$, we submit the following boundary value condition

$$y'(0) = \sum_{i=1}^{p-2} \alpha_i y'(\xi_i) + \lambda_1,$$

where $\xi_i \in (0,1)$, $i = 1,2\ldots,p-1$ and λ_1 is a parameter. Similar situation holds near $t = 1$ and the multipoint boundary value conditions can be formulated as

$$y(1) = \sum_{i=1}^{p-2} \beta_i y(\xi_i) + \lambda_2, \qquad \text{or} \qquad y'(1) = \sum_{i=1}^{p-2} \beta_i y'(\xi_i) + \lambda_2.$$

To solve the bridge design problems, we only need to solve the following multipoint boundary value problem [18]:

$$\begin{cases} y''(t) + f(t) + g(t, y(t)) = 0, \quad t \in [0,1), \\ y(0) = \sum_{i=1}^{p-2} \alpha_i y(\xi_i) + \lambda_1, \\ y(1) = \sum_{i=1}^{p-2} \beta_i y(\xi_i) + \lambda_2. \end{cases}$$

For example, we consider the following four-point boundary value problem of the second order [22, 23, 18]

$$y''(t) + (t^3 + t + 1)y^2(t) = f(t), \qquad t \in [0,1], \tag{35}$$

with the multipoint boundary conditions

$$y(0) - \frac{1}{6} y(\frac{2}{9}) - \frac{1}{3} y(\frac{7}{9}) = d_1,$$

$$y(1) - \frac{1}{5} y(\frac{2}{9}) - \frac{1}{2} y(\frac{7}{9}) = d_2,$$

and the exact solution

$$y(t) = \frac{1}{3}\sin(t - t^2),$$

where d_1, d_2 and $f(t)$ can be computed from the exact solution. The absolute error function for y, y' and y'' when $J = 7$ are illustrated in Figure 9. Also, the observed maximum absolute error for different J are reported in Table 8.

Table 8. The observed maximum absolute error for different values of J for differential equation (35).

e_r	$J = 2$	$J = 3$	$J = 4$	$J = 5$	$J = 6$	$J = 7$
e_0	2.2×10^{-4}	2.0×10^{-5}	4.1×10^{-6}	1.3×10^{-6}	3.1×10^{-7}	7.9×10^{-8}
e_1	8.1×10^{-4}	2.1×10^{-4}	7.0×10^{-5}	2.2×10^{-5}	6.0×10^{-6}	1.6×10^{-6}
e_2	1.7×10^{-2}	1.4×10^{-2}	9.0×10^{-3}	4.6×10^{-3}	2.5×10^{-3}	1.3×10^{-3}

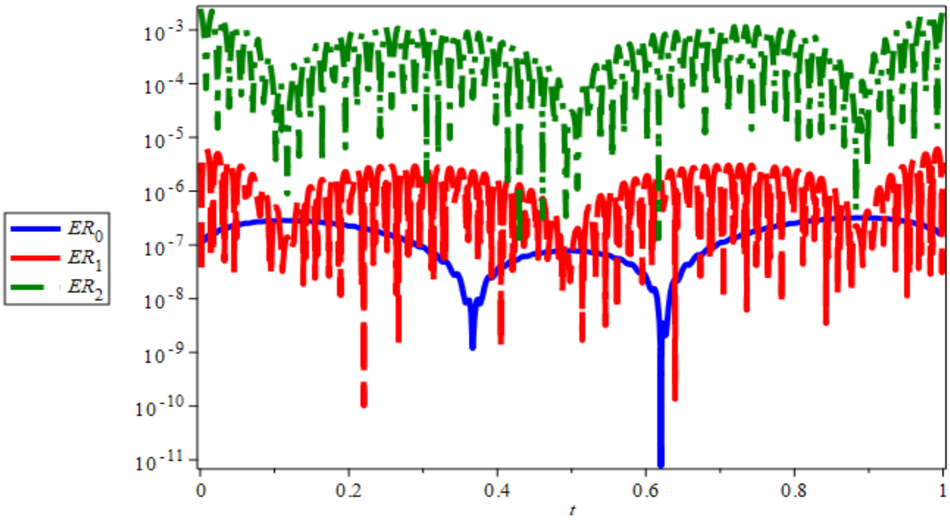

Fig. 9. The absolute error functions of y, y', y'' when the level of resolutions is $J = 7$ for differential equation (35).

6. Conclusion

According to the proposed method, the Haar basis functions were provided an efficient method to solve high order ODEs associated with the general type of initial or boundary conditions, including multipoint separated or non-separated boundary conditions, which is a large class of differential equations. Using this method, the system of PDE, which was reduced to the system of ODEs, was solved, as described in Section 5. In this paper, a technique was established, which can reduce the difficult boundary conditions to the algebraic equations. However, regarding the generality, flexibility, and importance of the technique, more theoretical points about the reliability of the solution should be discussed in the future. Even though the validity and reliability of the Haar wavelet method were approved in many articles, but the augmentation of the Haar basis functions with polynomials may cause finding a degenerate solution in particular cases. So we hope this case study attracts more researchers to clarify more details of the method. Also, interested readers can follow the next works of these authors.

Author Contributions

M. Heydari planned the scheme, initiated the project, and suggested the experiments; Z. Avazzadeh conducted the experiments and analyzed the empirical results; N. Hosseinzadeh developed the mathematical modeling and examined the theory validation. The manuscript was written through the contribution of all authors. All authors discussed the results, reviewed, and approved the final version of the manuscript.

Acknowledgments

The authors wish to express their cordial thanks to the Editors and Reviewers for their valuable suggestions and constructive comments which have served to improve the quality of this paper.

Conflict of Interest

The authors declared no potential conflicts of interest with respect to the research, authorship and publication of this article.

Funding

This work is partially supported by the AI University Research Centre (AI-URC) through XJTLU Key Programme Special Fund (KSF-P-02).

Data Availability Statements

The datasets generated and/or analyzed during the current study are available from the corresponding author on reasonable request.

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How to cite this article: Heydari M., Avazzadeh Z., Hosseinzadeh N. Haar Wavelet Method for Solving High-Order Differential Equations with Multi-Point Boundary Conditions, *J. Appl. Comput. Mech.*, 8(2), 2022, 528–544. https://doi.org/10.22055/JACM.2020.31860.1935