



Journal of Applied and Computational Mechanics



Research Paper

Numerical Scheme based on Non-polynomial Spline Functions for the System of Second Order Boundary Value Problems arising in Various Engineering Applications

Anju Chaurasia¹, Yogesh Gupta², Prakash C. Srivastava³

¹ Department of Mathematics, Birla Institute of Technology, Allahabad-211010 (U.P), India, Email: anjuchaurasiya@rediffmail.com

² Department of Mathematics, Jaypee Institute of Information Technology, Noida-201309 (U.P), India, Email: yogesh4july@gmail.com

³ Department of Mathematics, Birla Institute of Technology, Patna-800014 (Bihar), India, Email: prakash_bit123@rediffmail.com

Received January 29 2020; Revised March 30 2020; Accepted for publication April 26 2020.

Corresponding author: Y. Gupta (yogesh4july@gmail.com)

© 2022 Published by Shahid Chamran University of Ahvaz

Abstract. Several applications of computational science and engineering, including population dynamics, optimal control, and physics, reduce to the study of a system of second-order boundary value problems. To achieve the improved solution of these problems, an efficient numerical method is developed by using spline functions. A non-polynomial cubic spline-based method is proposed for the first time to solve a linear system of second-order differential equations. Convergence and stability of the proposed method are also investigated. A mathematical procedure is described in detail, and several examples are solved with numerical and graphical illustrations. It is shown that our method yields improved results when compared to the results available in the literature.

Keywords: Linear System, Second-order boundary-value problems, Numerical approximation, Cubic non-polynomial spline, Convergence analysis, Error analysis.

1. Introduction

Many problems in science, engineering, and mathematics can be reduced to solving systems of equations with notable examples in modeling and simulation of physical systems, and the verification and validation of engineering designs [1]. These systems of boundary value problems (BVPs) with different types of boundary conditions (BCs) are powerful tools to define many realism matters and so, constitute a very interesting topic for researchers. Modeling of real world problems into various orders of system of BVPs can be often seen in the field of population dynamics, brine tank cascade, compartment analysis, pond pollution, chemo stats, irregular heartbeats and lidocaine, flow of nutrient in an aquarium, forecasting prices, boxcars, electrical network, coupled spring-mass systems, logging timber by helicopter, earthquake effects on buildings, etc. [2].

In recent times, second order system of BVPs has been considered by several researchers. They have put much effort to solve these problems numerically and developed some efficient and accurate methods. For instance, second-order linear and non-linear systems were studied by Geng and Cui [3] in the reproducing kernel space, whereas Dehghan *et al.* [4] and Gamel [5] presented a sinc-collocation method to solve them. Lu in [6] introduced an analytical method based on the variational iterations to obtain the approximate solution of this system of second order BVPs. Over again, Laplace homotopy analysis was instituted by Bataineh and his team [7] as well as Ogunlaran *et al.* [8] to solve the system of non-linear differential equations.

Due to ever widening range of applications and mathematical consequences of spline functions, solution of BVPs using them has been an active field for researcher [9]. They have sufficiently closely been approximated to systems of BVPs by spline functions based approaches as well. For example, Dehghan *et al.* [10] used B-spline scaling functions and obtain the solution of the non-linear system of second-order BVPs. Caglar *et al.* applied collocation method with B-splines to solve the second order system of BVPs [11]. Khuri and Sayfy [12] used same approach and acquired the result for a generalized system of BVPs. Heilat *et al.* [13] employed extended cubic B-spline based scheme and solve the linear case of above discussed problem. A singular boundary value problem was solved by Goh and his group [14] with extended cubic uniform B-spline functions based method.



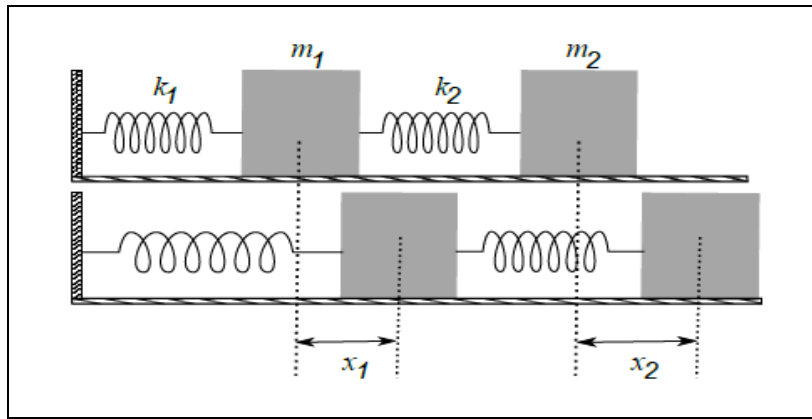


Fig.1. System of two masses and two springs

In this work, we study the solution of the following linear system of second-order BVPs:

$$\begin{aligned} u''(x) + a_1(x) u'(x) + a_2(x) u(x) + a_3(x) v''(x) + a_4(x) v'(x) + a_5(x) v(x) &= f_1(x), \\ v''(x) + b_1(x) v'(x) + b_2(x) v(x) + b_3(x) u''(x) + b_4(x) u'(x) + b_5(x) u(x) &= f_2(x), \end{aligned} \tag{1}$$

subject to

$$u(0) = u(1) = 0, \quad v(0) = v(1) = 0, \tag{2}$$

where $0 \leq x \leq 1$, $f_1(x)$ and $f_2(x)$ are given functions and $a_j(x)$ and $b_j(x)$, for $j = 1, 2, 3, 4, 5$ are continuous. Here, we emphasize that the coupled system of second-order BVPs is just a special case of our anticipated problem (1)-(2). A coupled spring-mass system is one of the very simple and prevalent applications involving above considered system. A schematic diagram is shown in Figure 1, illustrates the coupling of simple oscillators where two blocks of weight m_1 and m_2 are attached with two springs. x_1 and x_2 are the elongations of each spring from equilibrium and k_1, k_2 are the spring constants. [15]. The detailed explanation of the existence of solution for these systems can be easily found in [16]-[19]. It is quite considered that on the specified interval, the proposed system (1)-(2) have unique solutions.

From above we can perceive that a number of different techniques like reproducing kernel, sinc-collocation, variational iteration, laplace homotopy analysis along with B-spline collocation method have been projected to obtain the solution of the system of second-order BVPs, for linear and non-linear cases both. Here, we propose an efficient numerical method generated by non-polynomial cubic splines to attain the solution of the proposed system (1)-(2). Our solution methodology is based on the development of an algorithm that is formed by exponential and trigonometric cubic spline functions, which solves the deliberated problem efficiently. Present method is developed by means of following function space:

$$\begin{aligned} T_3(x) &= \text{Span} \{1, x, e^{kx}, \sin(kx)\}, \\ &= \text{Span} \left\{ 1, x, \frac{2}{k^2} (e^{kx} - kx), \left(\frac{6}{k^3}\right) (\sin(kx) - kx) \right\}, \end{aligned} \tag{3}$$

where k is the frequency of the non-polynomial functions. It follows that if $k \rightarrow 0$, T_3 reduces to $\text{Span}\{1, x, x^2, x^3\}$ [20]. The present combination of function space was also considered by Chaurasia et al. [21] to solve the fourth order system of BVPs, but with quintic non-polynomial splines.

Here, we have organized our work in this way. We have deliberated the development of a non-polynomial spline method in section 2. In section 3, detailed solution methodology is discussed for the solution of linear second-order system of BVPs. Convergence and stability analysis of the proposed method are given in section 4 and 5, respectively. In section 6, three examples are solved to verify the practicality of our developed scheme with graphical depictions. Section 7 concludes the study with remarks.

2. Development of Non-polynomial Spline Method

Set a framework of an equally spaced partition of an interval $[a, b]$, dividing into N equal sections as $a = x_0 < x_1 < x_2 < \dots < x_N = b$, $h = b - a / N$. Our spline functions $P_{1j}(x)$ and $P_{2j}(x)$ hold the following structures in every section of the interval:

$$P_{1j}(x) = c_{1j} \sin k(x - x_j) + c_{2j} e^{k(x-x_j)} + c_{3j}(x - x_j) + c_{4j}, \quad j = 0, 1, 2, \dots, N; \tag{4}$$

$$P_{2j}(x) = d_{1j} \sin k(x - x_j) + d_{2j} e^{k(x-x_j)} + d_{3j}(x - x_j) + d_{4j}, \quad j = 0, 1, 2, \dots, N, \tag{5}$$

where, c_{ij} , and d_{ij} ; $i=1,2,3,4$ are constants and k is free parameter, which can be real or purely imaginary. Functions $P_{1j}(x)$ and $P_{2j}(x)$, which interpolate $S(x)$ and $s(x)$, respectively at x_j and reduce to cubic splines as $k \rightarrow 0$. Let $u(x)$ and $v(x)$ be the exact solutions of (1). S_j and s_j be approximation to $u_j = u(x_j)$ and $v_j = v(x_j)$ respectively, obtained by the segment $P_{1j}(x)$ and $P_{2j}(x)$ of the spline functions passing through the points (x_j, S_j) , (x_{j+1}, S_{j+1}) and (x_j, s_j) , (x_{j+1}, s_{j+1}) . Then our proposed mixed splines are defined by the functions $S(x) = P_{1j}(x)$ and $s(x) = P_{2j}(x)$; $j = 0, 1, 2, \dots, N$. Now, we assume

$$\begin{aligned} P_{1j}(x_j) &= S_j, & P_{1j}(x_{j+1}) &= S_{j+1}, \\ P_{1j}^{(2)}(x_j) &= M_j, & P_{1j}^{(2)}(x_{j+1}) &= M_{j+1}, \end{aligned} \tag{6}$$



and

$$\begin{aligned} P_{2j}(x_j) &= s_j, & P_{2j}(x_{j+1}) &= s_{j+1}, \\ P_{2j}^{(2)}(x) &= m_j, & P_{2j}^{(2)}(x_{j+1}) &= m_{j+1}, \end{aligned} \tag{7}$$

to get the following value of coefficients

$$\begin{aligned} c_{1j} &= \frac{1}{k^2 \sin \theta} [e^\theta M_j - M_{j+1}], & c_{2j} &= \frac{1}{k^2} [M_j], \\ c_{3j}(x_j) &= \frac{S_{j+1} - S_j}{h} + \frac{M_{j+1} + M_j}{k^2 h} - \frac{2e^\theta M_j}{k^2 h}, & c_{4j} &= S_j - \frac{1}{k^2} [M_j], \end{aligned}$$

and

$$\begin{aligned} d_{1j} &= \frac{1}{k^2 \sin \theta} [e^\theta m_j - m_{j+1}], & d_{2j} &= \frac{1}{k^2} [m_j], \\ d_{3j}(x_j) &= \frac{s_{j+1} - s_j}{h} + \frac{m_{j+1} + m_j}{k^2 h} - \frac{2e^\theta m_j}{k^2 h}, & d_{4j} &= s_j - \frac{1}{k^2} [m_j], \end{aligned}$$

where, $j=0,1,\dots,N$ and $\vartheta=kh$. At the knots, apply the first derivative continuity of spline functions to acquire the following relations:

$$S_{j-1} - 2S_j + S_{j+1} = h^2 [\alpha M_{j-1} + \beta M_j + \gamma M_{j+1}], \quad j = 1, 2, \dots, N - 1 \tag{8}$$

and

$$s_{j-1} - 2s_j + s_{j+1} = h^2 [\alpha m_{j-1} + \beta m_j + \gamma m_{j+1}], \quad j = 1, 2, \dots, N - 1, \tag{9}$$

where

$$\begin{aligned} \alpha &= [\theta e^\theta \{\sin(\theta) + \cos(\theta)\} + \sin(\theta)(1 - 2e^\theta)] / \theta^2 \sin(\theta), \\ \beta &= [2e^\theta \sin(\theta) - \theta e^\theta - \theta \{\sin(\theta) + \cos(\theta)\}] / \theta^2 \sin(\theta), \\ \gamma &= [\theta - \sin(\theta)] / \theta^2 \sin(\theta), \end{aligned}$$

Equations (8)-(9) provides us a system of $2N-2$ linear algebraic equations in the $2N-2$ unknowns S_j and s_j , for $j = 1, 2, \dots, N-1$. As $k \rightarrow 0$, $\alpha = 1/6$, $\beta = 4/6$ and $\gamma = 1/6$, then our schemes (8)-(9) reduce to the cubic spline scheme [22].

The local truncation errors T_{1j} and T_{2j} , for $j=1, 2, \dots, N-1$ can be written as:

$$T_{1j} = \{1 - (\alpha + \beta + \gamma)\} h^2 u_j^{(2)} + (\alpha - \gamma) h^3 u_j^{(3)} + \left\{ \frac{1}{12} - \frac{1}{2}(\alpha + \gamma) \right\} h^4 u_j^{(4)} + O(h^5). \tag{10}$$

and

$$T_{2j} = \{1 - (\alpha + \beta + \gamma)\} h^2 v_j^{(2)} + (\alpha - \gamma) h^3 v_j^{(3)} + \left\{ \frac{1}{12} - \frac{1}{2}(\alpha + \gamma) \right\} h^4 v_j^{(4)} + O(h^5). \tag{11}$$

Equation (10) designs the second order methods for various choices of parameters α , β and γ , where $\alpha+\beta+\gamma=1$ and $\alpha=\gamma$ in the following way:

Case (i): For $\alpha=1/6$, $\beta=4/6$ and $\gamma=1-\alpha-\beta$, the truncation error

$$T_{1j} = \left(-\frac{1}{12} \right) h^4 u_j^{(4)} + O(h^5).$$

Case (ii): For $\alpha=1/8$, $\beta=6/8$, the truncation error

$$T_{1j} = \left(-\frac{1}{24} \right) h^4 u_j^{(4)} + O(h^5).$$

Case (iii): $\alpha=1/18$, $\beta=16/18$, the truncation error

$$T_{1j} = \left(\frac{1}{36} \right) h^4 u_j^{(4)} + O(h^5).$$

In the same way, we can obtain the truncation errors for scheme (9) by using Eq. (11).

3. Composite Non-Polynomial Spline Solution for Linear System of Second Order BVPs

To illustrate the development of an approximation for Eq. (1), first we discretize it and get the following form:



$$F_{kl} = \begin{bmatrix} f_{kl}(x_1) & f_{kl}(x_2) & 0 & 0 & & & & \\ f_{kl}(x_1) & f_{kl}(x_2) & f_{kl}(x_3) & 0 & & & & \\ 0 & f_{kl}(x_2) & f_{kl}(x_3) & f_{kl}(x_4) & & & & \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \\ & & & & & & f_{kl}(x_{N-2}) & f_{kl}(x_{N-1}) \end{bmatrix},$$

where the terms used in the above matrices are defined as

$$\begin{aligned} f_{11}(x_j) &= \frac{(a_2(x_j) - a_3(x_j)b_5(x_j))}{\varphi(x_j)}; & f_{12}(x_j) &= \frac{(a_1(x_j) - a_3(x_j)b_4(x_j))}{\varphi(x_j)}; \\ f_{13}(x_j) &= \frac{(a_4(x_j) - a_3(x_j)b_1(x_j))}{\varphi(x_j)}; & f_{14}(x_j) &= \frac{(a_5(x_j) - a_3(x_j)b_2(x_j))}{\varphi(x_j)}; \\ f_{15}(x_j) &= \frac{(f_1(x_j) - a_3(x_j)f_2(x_j))}{\varphi(x_j)}; & \text{where } \varphi(x_j) &= (1 - a_3(x_j)b_3(x_j)); \quad j = 1, 2, \dots, N - 1. \end{aligned}$$

Similarly, matrices F_{kl} , for $k= 2$ and $l=1, 2,3,4,5$ can be defined by swapping the functions a and b . For example,

$$f_{21}(x_j) = \frac{(b_2(x_j) - b_3(x_j)a_5(x_j))}{\varphi(x_j)},$$

and so on. By substituting the value of A and C , simplify the system (15) and we get the tri-diagonal system of linear equations. On solving this system by using any suitable method, the solution matrix is attained. Thus, solution matrix S_l gives the approximations S_j and s_j to the solution $u(x)$ and $v(x)$ at the points x_1, x_2, \dots, x_{N-1} .

4. Convergence Analysis

In this section, we examine our developed scheme in terms of convergence. We can write the error equation of the method (15) as

$$AE=T, \tag{19}$$

where, $E = (E_1 \ E_2)^t$, the error of discretization with $E_{1j} = u_j - S_j, E_{2j} = v_j - s_j$, where $E= U-S_1, U = (u_j, v_j)$ and matrix A is according Eq.(17) and $T= (T_{1j} \ T_{2j})^t$, for $j = 1, 2, \dots, N-1$, the local truncation errors described in eqs. (10)- (11). Let us assume that B_1F_{ij} and $B_2F_{ij} \geq 0$ in Eq. (18), for all i, j . So, the diagonal blocks A_{11} and A_{22} are invertible [24] and hold the following condition

$$(1 + d_2^*)(1 + d_{1^*}) < (1 + d_2^*) + (1 + d_{1^*}),$$

where,

$$d_2^* = \|A_{12}A_{22}^{-1}\|_\infty \text{ and } d_{1^*} = \|A_{21}A_{11}^{-1}\|_\infty.$$

Then matrix A , defined in Eq. (17) is invertible and so A^{-1} exists. From Eq. (19) and norm inequalities, we have

$$\|E\| \leq \|A^{-1}\| \|T\|, \tag{20}$$

where matrix $\|A^{-1}\|$ satisfies the following condition, according to Gil [24]

$$\|A^{-1}\| \leq \frac{\max(\|A_{11}^{-1}\|, \|A_{22}^{-1}\|) \cdot (1 + d_2^*)(1 + d_{1^*})}{(1 + d_2^*) + (1 + d_{1^*}) - (1 + d_2^*)(1 + d_{1^*})}. \tag{21}$$

As $\|T\| \leq O(h^4)$ and from the classifications of the matrices $A_{kl}; k, l=1, 2$ defined in Eq. (18), we can have

$$\|E\| \leq O(h^2). \tag{22}$$

From above, it follows that $\|E\| \rightarrow 0$ as $h \rightarrow 0$. So we can conclude that method (15) is second order convergent.

5. Stability Analysis

Here, we will check the stability of our scheme (15) when applied to system (1)–(2). Introducing the separate perturbations $\delta u(x)$ and $\delta v(x)$ in the given system suppose the errors e_j and e_k have been occurred in the calculation of A and C respectively specified in Eq. (16). Let S_1^* be the solution of the perturbed system, then the above system (16) can be written as

$$(A + e_j)S_1^* = C + e_k \tag{23}$$

Let A is non-singular and also assume that



$$\|e_j\| < \left(\frac{1}{2\|A^{-1}\|} \right), \tag{24}$$

Then, $A + e_j$ is also non-singular and

$$\|(A + e_j)^{-1}\| \leq 2\|A^{-1}\|. \tag{25}$$

From (16) and (23), it follows that

$$S_1 - S_1^* = (A + e_j)^{-1} (e_j S_1 - e_k) \tag{26}$$

Also, from the definition of norm

$$\|A^{-1}\|_\infty \leq \left[\min_{0 \leq j \leq N} \left(|a_{j,k}| - \sum_{j \neq k} |a_{j,k}| \right) \right]^{-1} = w \text{ (say)} < \infty \tag{27}$$

So, from (25), (26) and (27), we can have

$$\|S_1 - S_1^*\|_\infty < 2w \{ \|e_j\|_\infty \|S_1\|_\infty + \|e_k\|_\infty \}, \tag{28}$$

which shows that our scheme is stable.

6. Results and Discussions

To show the usefulness of our developed spline based scheme, we have solved three problems of linear system of second-order BVPs. All computations, for these problems were carried out using Matlab software.

6.1 Problem 1

$$\begin{aligned} u^{(2)}(x) + xu(x) + xv(x) &= f_1(x), \\ v^{(2)}(x) + 2xv(x) + xu(x) &= f_2(x), \end{aligned} \tag{29}$$

subject to BCs

$$u(0) = u(1) = 0, v(0) = v(1) = 0, \text{ where } 0 < x < 1, f_1(x) = 2 \text{ and } f_2(x) = -2.$$

The theoretical solution for $u(x)$ and $v(x)$ are given by

$$u(x) = x^2 - x; v(x) = x - x^2. \tag{30}$$

Numerical solution acquired by our method for two different choices of α, β and γ are given in Table 1 for $N=21$. Table 1 confirms that our method gives better accuracy than the methods obtained in [11] and [13].

Table 1. Maximum Absolute Errors for Problem 1 ($N=21$).

	$u(x)$	$v(x)$
Our Method ($\alpha=\gamma=1/6, \beta=4/6$)	3.3×10^{-16}	6.6×10^{-16}
Our Method ($\alpha=\gamma=1/8, \beta=6/8$)	3.0×10^{-16}	1.1×10^{-16}
B-spline method[11]	3.4×10^{-15}	1.8×10^{-15}
Extended cubic B-spline method [13]	1.7×10^{-13}	1.6×10^{-13}

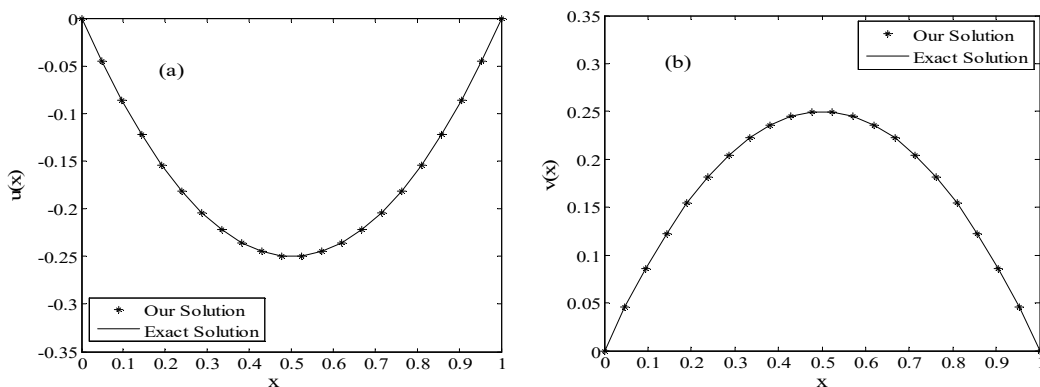


Fig.2 (a) & (b). Comparison of approximate values and exact values for Problem 1 at $N=21$.



Table 2. Maximum absolute errors, Problem 2.

Our Method	N=50		N=100		N=200	
	u(x)	v(x)	u(x)	v(x)	u(x)	v(x)
$\alpha=\gamma=1/6, \theta=4/6$	1.20×10^{-3}	3.07×10^{-4}	7.77×10^{-5}	7.77×10^{-5}	3.07×10^{-4}	8.94×10^{-6}
$\alpha=\gamma=1/8, \theta=6/8$	1.20×10^{-3}	2.78×10^{-5}	3.07×10^{-4}	7.09×10^{-6}	7.77×10^{-5}	1.78×10^{-6}
$\alpha=\gamma=1/18, \theta=16/18$	1.22×10^{-3}	1.68×10^{-5}	3.09×10^{-4}	4.31×10^{-6}	7.80×10^{-5}	1.09×10^{-6}
Our Method	N=21		N=41		N=61	
	u(x)	v(x)	u(x)	v(x)	u(x)	v(x)
$\alpha=\gamma=1/18, \theta=16/18$	6.58×10^{-3}	8.77×10^{-5}	1.80×10^{-3}	2.47×10^{-5}	8.25×10^{-4}	1.14×10^{-5}
B-spline method [11]	1.89×10^{-3}	9.60×10^{-5}	4.74×10^{-4}	2.40×10^{-5}	2.10×10^{-4}	1.07×10^{-5}

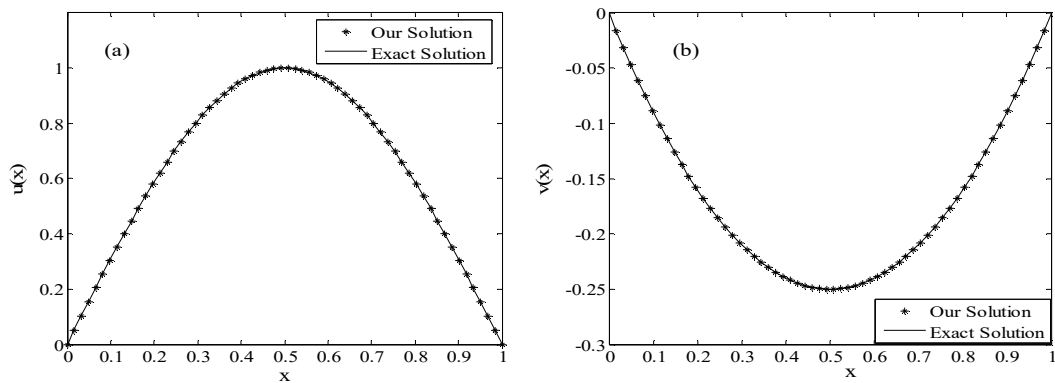


Fig. 3 (a) & (b). Comparison of approximate values and exact values for Problem 2 at N=61.

6.2 Problem 2

$$\begin{aligned}
 u^{(2)}(x) + (2x - 1)u^{(1)}(x) + \cos(\pi x)v^{(1)}(x) &= f_1(x), \\
 v^{(2)}(x) + xu(x) &= f_2(x),
 \end{aligned}
 \tag{31}$$

subject to BCs

$$\begin{aligned}
 u(0) = u(1) = 0, v(0) = v(1) = 0, \text{ where } 0 < x < 1, \\
 f_1(x) = -\pi^2 \sin(\pi x) + (2x - 1)(\pi + 1)\cos(\pi x) \text{ and } f_2(x) = 2 + x\sin(\pi x).
 \end{aligned}$$

The exact solution for u(x) and v(x) are

$$u(x) = \sin(\pi x), v(x) = x^2 - x.
 \tag{32}$$

The maximum absolute error and its comparison with B-spline method [11] for problem 2 are summarized in Table 2, at different values of N.

6.3 Problem 3

$$\begin{aligned}
 u^{(2)}(x) + u^{(1)}(x) + xu(x) + v^{(1)}(x) + 2xv(x) &= f_1(x), \\
 u^{(2)}(x) + v(x) + 2u^{(1)}(x) + x^2u(x) &= f_2(x),
 \end{aligned}
 \tag{33}$$

subject to BCs

$$\begin{aligned}
 u(0) = u(1) = 0, v(0) = v(1) = 0, \text{ where } 0 < x < 1, \\
 f_1(x) = -2(x + 1) \cos(x) + \pi \cos(\pi x) + 2x\sin(\pi x) + (4x - 2x^2 - 4)\sin(x) \text{ and} \\
 f_2(x) = -4(x - 1) \cos(x) - 2(2 - x^2 + x^3)\sin(x) - (\pi^2 - 1)\sin(\pi x).
 \end{aligned}$$

The exact solution for u(x) and v(x) are

$$u(x) = 2(1 - x)\sin(\pi x), v(x) = \sin(\pi x).
 \tag{34}$$

The maximum absolute errors for problem 3 are summarized in Table 3-4, at different values of N with comparison of results given in [3] and [4]. Thus, if we look at the results given in Tables 1-4, it is clear that our non-polynomial spline based method is convergent as accuracy improves, when we increase the number of mesh points. Our method is quite comparable with the methods suggested in [3], [4] and [11]. Even, for problem 1 our method gives pretty much better results than B-spline method proposed in [11] and [13]. Figures 2-4 have been depicted to demonstrate the numerical results for problems 1-3, respectively.



Table 3. Maximum absolute errors, Problem 3 (N=25).

x	Reproducing kernel space [3]		Sinc-collocation method [4]		Our method ($\alpha = \gamma = 1/6, \beta = 4/6$)	
	u(x)	v(x)	u(x)	v(x)	u(x)	v(x)
0.08	3.3×10^{-3}	7.7×10^{-3}	3.2×10^{-3}	1.5×10^{-3}	5.4×10^{-3}	3.9×10^{-3}
0.24	7.7×10^{-3}	2.2×10^{-2}	9.4×10^{-4}	7.0×10^{-3}	1.3×10^{-3}	8.5×10^{-4}
0.40	9.7×10^{-3}	2.7×10^{-2}	2.0×10^{-3}	7.4×10^{-3}	1.7×10^{-3}	1.3×10^{-3}
0.56	9.5×10^{-3}	2.7×10^{-2}	2.2×10^{-4}	1.0×10^{-2}	4.1×10^{-3}	2.9×10^{-3}
0.72	7.3×10^{-3}	2.0×10^{-2}	4.1×10^{-3}	4.4×10^{-3}	5.9×10^{-3}	4.0×10^{-3}
0.88	3.4×10^{-3}	9.4×10^{-3}	1.0×10^{-2}	2.1×10^{-2}	7.0×10^{-3}	4.7×10^{-3}
0.96	1.1×10^{-3}	3.1×10^{-3}	2.1×10^{-3}	6.9×10^{-3}	7.3×10^{-3}	5.0×10^{-3}

Table 4. Maximum absolute errors, Problem 3.

Our Method	N=50		N=100		N=200	
	u(x)	v(x)	u(x)	v(x)	u(x)	v(x)
$\alpha = \gamma = 1/6, \beta = 4/6$	1.8×10^{-3}	1.4×10^{-3}	4.8×10^{-4}	3.7×10^{-4}	1.2×10^{-4}	9.6×10^{-5}

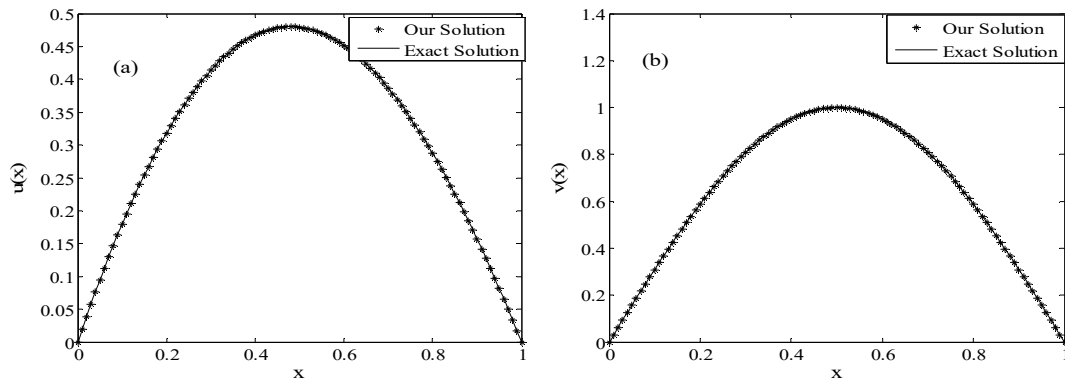


Fig. 4 (a) & (b): Comparison of approximate values and exact values for Problem 3 at N=100

7. Conclusions

In this paper, a non-polynomial cubic spline-based method was proposed to find the approximation to the solution of a linear system of second-order BVPs. Convergence and stability analysis of the method were also discussed. It was verified by the numerical results that the problems were solved by the proposed approach effectively. This method is simple and produced commendable results when compared with the methods given in ([11], [13], and others) available in the literature. Our paper could contribute remarkably to this field as it leads to the possibility to apply non-polynomial spline functions as a robust tool to approximate the system of BVPs. This work can be extended to improve the existing computational error in the field of approximation of systems of different order of BVPs and, of course, to solve non-linear problems too.

Author Contributions

Y. Gupta planned and initiated the project and recommended to solve present system of boundary value problem using given spline method. A. Chaurasia performed the convergence and stability analysis and developed the MATLAB codes. A. Chaurasia and P. C. Srivastava conducted the numerical experiments and examined the validation of result. The manuscript was written through the contribution of all authors. All authors discussed the results, reviewed, and approved the final version of the manuscript.

Acknowledgments

Not applicable.

Conflict of Interest

The authors confirm that there is no conflict of interest to declare for this publication.

Funding

The authors received no financial support for the research, authorship and publication of this article.




Data Availability Statements


The datasets generated and/or analyzed during the current study are available from the corresponding author on reasonable request.


References

- [1] Chang, C.C., Gambhir, A., Humble, T.S., Sota, S., Quantum Annealing for Systems of Polynomial Equations, *Scientific Reports*, 9(1), 2019, 10258.
- [2] Gupta, Y., Numerical Solution of System of Boundary Value Problems using B-Spline with Free Parameter, *AIP Conference Proceedings*, 1802(1), 2017, 020006.
- [3] Geng, F., Cui, M., Solving a Non-linear System of Second Order Boundary Value Problems, *Journal of Mathematical Analysis and Applications*, 327, 2007, 1167-1181.
- [4] Dehghan, M., Saadatmandi, A., The Numerical Solution of a Nonlinear System of Second Order Boundary Value Problems Using Sinc-Collocation Method, *Mathematical and Computer Modelling*, 46, 2007, 1434-1441.
- [5] Gamel, E. M., Sinc-collocation Method for Solving Linear and Non-Linear System of Second-Order Boundary Value Problems, *Applied Mathematics*, 3(11), 2012, 1627-1633.
- [6] Lu, J. F., Variational Iteration Method for Solving a Nonlinear System of Second-Order Boundary Value Problems, *Computers and Mathematics with Applications*, 54, 2007, 1133-1138.
- [7] Bataineh, M., Noorani, S. M., Hashim, I., Modified Homotopy Analysis Method for solving system of second-order BVPs, *Communications in Nonlinear Science and Numerical Simulation*, 14(2), 2009, 430- 442.
- [8] Ogunlaran, O.M., Ademola, A.T., On the Laplace Homotopy Analysis Method for a Nonlinear System of Second-Order Boundary Value Problems, *General Mathematics Notes*, 26(2), 2015, 11-22.
- [9] Chaurasia, A., Srivastava, P.C., Gupta, Y., Solution of Higher Order Boundary Value Problems by Spline Methods, *AIP Conference Proceedings*, 1897, 2017, 020018.
- [10] Dehghan, M., Lakestani, M., Numerical Solution of Non-linear System of Second-Order Boundary Value Problems using Cubic B-Spline Scaling Functions, *International Journal of Computer Mathematics*, 85(9), 2008, 1455-1461.
- [11] Caglar, N., Caglar, H., B-spline Method for Solving Linear System of Second Order Boundary Value Problems, *Computers and Mathematics with Applications*, 57, 2009, 757-762.
- [12] Khuri, S. A., Sayfy, A., Collocation Approach for the Numerical Solution of a Generalized System of Second-Order Boundary-Value Problems, *Applied Mathematical Sciences*, 3(45), 2009, 2227-2239.
- [13] Heilat, A., Hamid, N., Ismail, A.I.M., Extended Cubic B-spline Method for Solving a Linear System of Second-Order Boundary Value Problems, *Springer Plus*, 5(1), 2016, 1-18.
- [14] Goh, J., Majid, A.A., Ismail, A.I.M., Extended Cubic Uniform B-spline for a Class of Singular Boundary Value Problems, *Science Asia*, 37(1), 2011, 79-82.
- [15] Fay, T. H., Coupled Spring Equations, *International Journal of Mathematical Education in Science and Technology*, 34(1), 2003, 65-79.
- [16] Chen, S., Hu, J., Chen, L., Wang, C., Existence Results for N-Point Boundary Value Problem of Second Order Ordinary Differential Equations, *Journal of Computational and Applied Mathematics*, 180(2), 2005, 425-432.
- [17] Cheng, X., Zhong, C., Existence of Positive Solutions for a Second-Order Ordinary Differential System, *Journal of Mathematical Analysis and Applications*, 312(1), 2005, 14-23.
- [18] Thompson, H.B., Tisdell, C., Boundary Value Problems for Systems of Difference Equations Associated with Systems of Second-Order Ordinary Differential Equations, *Applied Mathematics Letters*, 15(6), 2002, 761-766.
- [19] Thompson, H.B., Tisdell, C., The Non-existence of Spurious Solutions to Discrete Two-Point Boundary Value Problems, *Applied Mathematics Letters*, 16(1), 2003, 79-84.
- [20] Chaurasia, A., Srivastava, P.C., Gupta, Y., *Exponential Spline Approximation for the Solution of Third-Order Boundary Value Problems*, V. E. Balas et al. (eds.), Data Management, Analytics and Innovation, Advances in Intelligent Systems and Computing 808, Springer Nature Singapore Pte Ltd. 2019.
- [21] Chaurasia, A., Srivastava, P. C., Gupta, Y., Bhardwaj, A., Composite Non-polynomial Spline Solution of Boundary Value Problems in Plate Deflection Theory, *International Journal for Computational Methods in Engineering Science and Mechanics*, 20(5), 2019, 372-379.
- [22] Albasiny, E.L., Hoskins, W.D., Cubic Spline Solutions to Two-Point Boundary Value Problems, *Computer Journal*, 12, 1969, 151-153.
- [23] Jain, M.K., *Numerical Solution of Differential Equations*, Wiley Eastern, New Delhi, 1984.
- [24] Gil, M.I., Invertibility Conditions for Block Matrices and Estimates for Norms of Inverse Matrices, *Journal of Mathematics*, 33(4), 2003, 1323-1335.

ORCID ID

Anju Chaurasia  <https://orcid.org/0000-0002-4706-3792>

Yogesh Gupta  <https://orcid.org/0000-0002-9285-4526>

Prakash C. Srivastava  <https://orcid.org/0000-0002-5916-5067>



© 2022 Shahid Chamran University of Ahvaz, Ahvaz, Iran. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution-Non Commercial 4.0 International (CC BY-NC 4.0 license) (<http://creativecommons.org/licenses/by-nc/4.0/>).

How to cite this article: Chaurasia A., Gupta Y., Srivastava P.C. Numerical Scheme based on Non-polynomial Spline Functions for the System of Second Order Boundary Value Problems arising in Various Engineering Applications, *J. Appl. Comput. Mech.*, 8(1), 2022, 144-152. <https://doi.org/10.22055/JACM.2020.32435.2012>

Publisher's Note Shahid Chamran University of Ahvaz remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

