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Research Paper

Axisymmetric Problem of the Elasticity Theory for the Radially Inhomogeneous Cylinder with a Fixed Lateral Surface

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Abstract. By the method of the asymptotic integration of the equations of elasticity theory, the axisymmetric problem of elasticity theory is studied for a radially inhomogeneous cylinder of small thickness. It is considered that the elasticity moduli are arbitrary positive continuous functions of the radius of the cylinder. It is also assumed that the lateral surface of the cylinder is fixed, and stresses are imposed at the end faces of the cylinder, which leave the cylinder in equilibrium. The analysis is carried out when the cylinder thickness tends to zero. It is shown that solutions corresponding to the first and second iterative processes that determine the internal stress-strain state of the radially inhomogeneous cylinder with a fixed surface do not exist. The third iterative process defines solutions that have the boundary layer character equivalent to the Saint-Venant end effect on the theory of inhomogeneous plates. The stresses determined by the third iterative process are localized at the ends of the cylinder and decrease exponentially with distance from the ends. The asymptotic integration method is used to study the problem of torsion of the radially inhomogeneous cylinder of small thickness. The nature of the stress-strain state is analyzed.

Keywords: Radially inhomogeneous cylinder, Equilibrium equations, Matrix differential operator, Asymptotic method, Boundary layer, Papkovich spectral problem, Saint-Venant end effect.

1. Introduction

The inhomogeneity of the mechanical properties of real elastic bodies is an experimentally confirmed fact. Inhomogeneous materials are widely used in various fields of technology. Various materials are developed, the characteristics of which, in particular, the elastic moduli, can vary continuously along some directions [1].

The complexity of the phenomena arising from the deformation of inhomogeneous plates and shells has given rise to a number of applied theories. Despite the number of such theories for inhomogeneous shells based on various hypotheses, the areas of their applicability have not been studied enough. The existence of various applied theories for inhomogeneous shells poses the problem of their critical analysis based on a strong mathematical approach, i.e. from the position of three-dimensional equations of the theory of elasticity. In order to develop new refined applied theories, it is important to analyze the stress-strain state of inhomogeneous shells from the perspective of three-dimensional equations of the elasticity theory.

The study of the stress-strain state of the inhomogeneous bodies basing on three-dimensional equations of the elasticity theory is associated with significant mathematical difficulties. Along with this, from the physical point of view, new qualitative and quantitative effects arise which also require development of the constructive solution methods.

The spatial problems of the elasticity theory for a cylinder have been a subject of a number of studies. Asymptotic methods made a significant contribution to these solutions [2, 3]. In [3], an asymptotic theory of a transversely isotropic cylinder of small thickness was developed. In [4], the method of homogeneous solutions was used to analyze the three-dimensional problem of the theory of elasticity for the homogeneous isotropic cylinder of small thickness and a comparison of the asymptotic solution with the solutions obtained by applied theories is given. In [5], the torsion problem was studied for a radially layered cylinder with alternating hard and soft layers. A possible break of the principle of Saint-Venant in its classical formulation is indicated. In [6], the axisymmetric problem of elasticity theory for the radially layered cylinder with alternating hard and soft layers was considered. The existence of weakly damped boundary layer solutions is shown.

In [7], the bending deformation of a multilayer cylinder with the most general form of cylindrical anisotropy was studied. In [8], the Almansi-Michell problem for an inhomogeneous anisotropic cylinder was investigated by the numerical-analytical method. In [9], an analog of the classical Lame problem for the isotropic hollow cylinder with a Young's modulus depending on the radial coordinate and with a constant Poisson's ratio was considered. In [10], the effect of the material inhomogeneity on the stress-strain state of the cylinder was investigated. In [11], the stress-strain state of an inhomogeneous orthotropic cylinder with a given inhomogeneity is studied.

In [12, 13], based on the spline collocation method and the finite element method, the three-dimensional stress-strain state of the radially inhomogeneous cylinder was studied and the comparison of the obtained numerical results were given in [14], the analysis of the stress-strain state of the radially inhomogeneous cylinder subjected to uniform internal pressure was considered.

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In [15], the elasticity theory problem was analyzed for the radially inhomogeneous hollow cylinder with constant Poisson's ratio and Young's modulus, which is an exponential function of radius.

In [16-19], the thermomechanical behavior of the hollow radially inhomogeneous cylinders was studied. In [20], assuming that the thermoelastic parameters are power functions of the radial coordinate, the thermomechanical state of the radially inhomogeneous anisotropic cylinder was studied.

In [21], on the basis of the asymptotic integration method for equations of the elasticity theory, the axisymmetric problem was studied for the radially inhomogeneous cylinder of small thickness, when the elastic moduli vary linearly in radius. In [22], the behavior of the solution of the three-dimensional problem of elasticity theory for a radially inhomogeneous transversely isotropic hollow cylinder of small thickness was studied by the asymptotic method. Inhomogeneous and homogeneous solutions are constructed and an asymptotic analysis of the stress-strain state, determined by the homogeneous solutions is carried out. In [23] the survey is given on the study of the stress-stain state of the inhomogeneous elastic body and solution methods are described.

The aim of the work is to reveal the features and develop effective methods for calculating the stress-strain state of the radially inhomogeneous cylinder of small thickness with a fixed lateral surface.

To achieve this goal, the following problems are set:

- Constructing a solution by the method of asymptotic integration of equations of the elasticity theory and study the nature of the constructed solutions;
- Studying the behavior of the constructed solutions both in the inner part of the cylinder and near the border;
- Construction of asymptotic formulas for the displacements and stresses, allowing one to calculate the three-dimensional stress-strain state of the radially inhomogeneous cylinder;
- Carrying out the numerical analysis.

2. Statement of the Boundary Value Problems for the Radially Inhomogeneous Cylinder

The axisymmetric problem of the elasticity theory for a radially inhomogeneous isotropic hollow cylinder of small thickness (Fig. 1) is considered. Suppose the volume of the cylinder in the cylindrical coordinate system is:

$$\Gamma = \{r_1 \le r \le r_2, 0 \le \phi \le 2\pi, -l \le z \le l\}.$$

The system of the equilibrium equations in the absence of mass forces in the cylindrical coordinate system r,ϕ,z has a form [24]:

$$\begin{cases} \dfrac{\partial \sigma_{rr}}{\partial r} + \dfrac{\partial \sigma_{rz}}{\partial z} + \dfrac{\sigma_{rr} - \sigma_{\phi\phi}}{r} = 0, \\ \dfrac{\partial \sigma_{rz}}{\partial r} + \dfrac{\partial \sigma_{zz}}{\partial z} + \dfrac{\sigma_{rz}}{r} = 0, \end{cases} \tag{1}$$

$$\frac{\partial \sigma_{r\phi}}{\partial r} + \frac{\partial \sigma_{\phi z}}{\partial z} + \frac{2\sigma_{r\phi}}{r} = 0\,, \tag{2}$$

where $\sigma_{rr},\sigma_{rz},\sigma_{\phi\phi},\sigma_{zz},\sigma_{r\phi},\sigma_{\phi z}$ are the components of the stress tensor, which are expressed through the components of the displacement vector $u_r(r,z)$, $u_z(r,z)$, $u_\phi(r,z)$ by the following way [24]:

$$\begin{cases} \sigma_{rr} = \left(2\tilde{G} + \tilde{\lambda}\right)\dfrac{\partial u_r}{\partial r} + \tilde{\lambda}\left(\dfrac{u_r}{r} + \dfrac{\partial u_z}{\partial z}\right), \\ \sigma_{rz} = \tilde{G}\left(\dfrac{\partial u_r}{\partial z} + \dfrac{\partial u_z}{\partial r}\right), \end{cases} \tag{3}$$

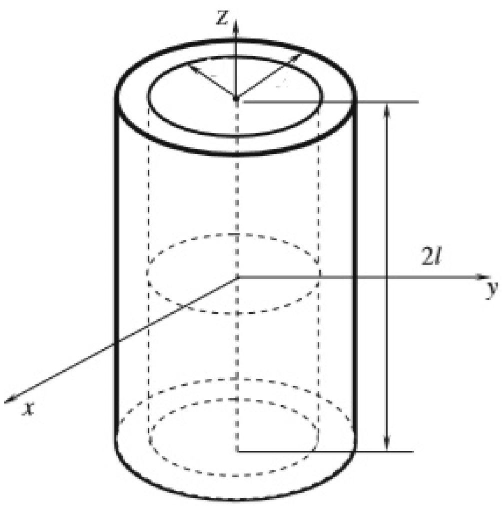

Figure 1. Radially inhomogeneous isotropic hollow cylinder

$$\begin{cases} \sigma_{zz} = \left(2\tilde{G}+\tilde{\lambda}\right)\dfrac{\partial u_z}{\partial z} + \tilde{\lambda}\left(\dfrac{\partial u_r}{\partial r} + \dfrac{u_r}{r}\right), \\ \sigma_{\phi\phi} = \left(2\tilde{G}+\tilde{\lambda}\right)\dfrac{u_r}{r} + \tilde{\lambda}\left(\dfrac{\partial u_r}{\partial r} + \dfrac{\partial u_z}{\partial z}\right), \end{cases} \tag{3-cont.}$$

$$\sigma_{r\phi} = \tilde{G}\left(\frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r}\right), \quad \sigma_{\phi z} = \tilde{G}\frac{\partial u_\phi}{\partial z}. \tag{4}$$

Introduce the new dimensionless variables ρ and ξ:

$$\rho = \frac{1}{\varepsilon}\ln\left(\frac{r}{r_0}\right), \quad \xi = \frac{z}{r_0}, \tag{5}$$

where $\varepsilon = \ln(r_2/r_1)/2$ is a small parameter characterizing the thickness of the cylinder, $r_0 = \sqrt{r_1 r_2}$, $\rho \in [-1;1]$, $\xi \in [-l_0;l_0]$, $l_0 = l/r_0$. Eqs. (3), (4) in new dimensionless variables ρ,ξ take the form:

$$\begin{aligned} \sigma_{\rho\rho} &= H\frac{e^{-\varepsilon\rho}}{\varepsilon}\frac{\partial u_\rho}{\partial \rho} + \lambda\left(e^{-\varepsilon\rho}u_\rho + \frac{\partial u_\xi}{\partial \xi}\right), \\ \sigma_{\xi\xi} &= H\frac{\partial u_\xi}{\partial \xi} + \lambda\left(\frac{e^{-\varepsilon\rho}}{\varepsilon}\frac{\partial u_\rho}{\partial \rho} + e^{-\varepsilon\rho}u_\rho\right), \\ \sigma_{\phi\phi} &= He^{-\varepsilon\rho}u_\rho + \lambda\left(\frac{e^{-\varepsilon\rho}}{\varepsilon}\frac{\partial u_\rho}{\partial \rho} + \frac{\partial u_\xi}{\partial \xi}\right), \\ \sigma_{\rho\xi} &= G\left(\frac{\partial u_\rho}{\partial \xi} + \frac{e^{-\varepsilon\rho}}{\varepsilon}\frac{\partial u_\xi}{\partial \rho}\right), \end{aligned} \tag{6}$$

$$\sigma_{\rho\phi} = Ge^{-\varepsilon\rho}\left(\frac{1}{\varepsilon}\frac{\partial u_\phi}{\partial \rho} - u_\phi\right), \quad \sigma_{\phi\xi} = G\frac{\partial u_\phi}{\partial \xi}, \tag{7}$$

where $u_\rho = u_r/r_0$, $u_\xi = u_z/r_0$, $u_\phi = u_\phi/r_0$, $\sigma_{\rho\rho} = \sigma_{rr}/G_*$, $\sigma_{\xi\xi} = \sigma_{zz}/G_*$, $\sigma_{\phi\phi} = \sigma_{\phi\phi}/G_*$, $\sigma_{\rho\xi} = \sigma_{rz}/G_*$, $\sigma_{\rho\phi} = \sigma_{r\phi}/G_*$, $\sigma_{\phi\xi} = \sigma_{\phi z}/G_*$, $G = \tilde{G}/G_*$, $\lambda = \tilde{\lambda}/G_*$ are dimensionless quantities, G_* is some characteristic parameter, having shear modulus dimension $H = 2G + \lambda$.

By substituting (6) into (1) and (7) into (2), we obtain the equations of equilibrium in displacements:

$$\left(L_0 + \frac{\partial}{\partial \xi}(L_1 + L_2) + \frac{\partial^2}{\partial \xi^2}L_3\right)\bar{u} = \bar{0}, \tag{8}$$

$$\frac{\partial}{\partial \rho}\left[2G\left(\frac{\partial u_\phi}{\partial \rho} - \varepsilon u_\phi\right)\right] + 2\varepsilon G\left(\frac{\partial u_\phi}{\partial \rho} - \varepsilon u_\phi\right) + 2\varepsilon^2 G e^{2\varepsilon\rho}\frac{\partial^2 u_\phi}{\partial \xi^2} = 0. \tag{9}$$

Here, $\bar{u} = \bar{u}(\rho,\xi) = (u_\rho(\rho;\xi); u_\xi(\rho;\xi))^T$, L_k are matrix differential operators of the form:

$$L_0 = \left\| \begin{matrix} \partial\left((H\partial + \varepsilon\lambda)e^{-\varepsilon\rho}\right) + \\ +2G\varepsilon(\partial - \varepsilon)e^{-\varepsilon\rho} \end{matrix} \quad \begin{matrix} 0 \\ \partial\left(Ge^{-\varepsilon\rho}\partial\right) + \varepsilon Ge^{-\varepsilon\rho}\partial \end{matrix} \right\|$$

$$L_1 = \begin{Vmatrix} 0 & \varepsilon(G\partial + \partial(\lambda)) \\ 0 & 0 \end{Vmatrix}, \quad L_2 = \begin{Vmatrix} 0 & 0 \\ \varepsilon(\partial(G) + \lambda\partial) + (G+\lambda)\varepsilon^2 & 0 \end{Vmatrix},$$

$$L_3 = \begin{Vmatrix} Ge^{\varepsilon\rho}\varepsilon^2 & 0 \\ 0 & \varepsilon^2 He^{\varepsilon\rho} \end{Vmatrix}, \quad \partial = \frac{\partial}{\partial \rho}.$$

Assume that $G = G(\rho)$, $\lambda = \lambda(\rho)$ are arbitrary positive continuous functions of the variable ρ. Suppose that the lateral surface of the cylinder is fixed, i.e.

$$u_\rho = 0;\ u_\xi = 0 \quad \text{at} \quad \rho = \pm 1, \tag{10}$$

$$u_\phi = 0 \quad \text{at} \quad \rho = \pm 1, \tag{11}$$

and the end faces are subjected to the following boundary conditions

$$\sigma_{\rho\xi}\big|_{\xi=\pm l_0} = f_{1s}(\rho),\ \ \sigma_{\xi\xi}\big|_{\xi=\pm l_0} = f_{2s}(\rho), \tag{12}$$

$$\sigma_{\phi\xi}\big|_{\xi=\pm l_0} = f_{3s}(\rho). \tag{13}$$

Here $f_{1s}(\rho)$, $f_{2s}(\rho)$, $f_{3s}(\rho)$ $(s=1,2)$ are smooth enough functions, satisfying equilibrium conditions. Problem (9), (11) and (13) describes the torsion of the radially inhomogeneous cylinder.

3. Construction of Solutions for the Radially Inhomogeneous Cylinder of Small Thickness

A solution was sought to (8) and (10) in the form

$$u_\rho(\rho,\xi) = u(\rho)m'(\xi); \qquad u_\xi(\rho,\xi) = w(\rho)m(\xi), \tag{14}$$

where $m''(\xi) - \mu^2 m(\xi) = 0$.

Substituting (14) into (8), (10) yields:

$$(L_0 + L_1 + \mu^2(L_2 + L_3))\bar{a} = \bar{0} \tag{15}$$

$$E\bar{a} = \bar{0} \ at \ \ \rho = \pm 1 \tag{16}$$

where $\bar{a}(\rho) = (u(\rho)\ ,\ \ w(\rho))^T$, $E = [1, 0; 0, 1]$.

To solve (15), (16), we use the asymptotic method based on three iterative processes [25-28]. The first iteration process corresponds to trivial solutions. Solutions having the nature of the edge effect corresponding to the second iterative process for the radially inhomogeneous cylinder with a fixed side surface do not exist.

According to the third iterative process, we seek solution of (15), (16) in the form:

$$\begin{aligned} &u^{(3)} = \varepsilon^2\beta_0^{-1}\left(c_0 + \varepsilon c_1 + \ldots\right), \qquad w^{(3)} = \varepsilon\left(b_0 + \varepsilon b_1 + \ldots\right),\\ &\mu = \varepsilon^{-1}\left(\beta_0 + \varepsilon b_1 + \ldots\right). \end{aligned} \tag{17}$$

By substituting (17) into (15) and (16), we get

$$A(\beta_0)\bar{f}_0 = \bar{0} \tag{18}$$

for the first terms of the expansion, where

$$A(\beta_0)\bar{f}_0 = \{\left(A_0 + \beta_0 A_1 + \beta_0^2 A_2\right)\bar{f}_0,\quad \bar{f}_0\big|_{\rho=\pm1} = \bar{0}\},$$

$$A_0 = \begin{pmatrix} \partial(H\partial) & 0 \\ 0 & \partial(G\partial) \end{pmatrix},\ A_1 = \begin{pmatrix} 0 & \partial(\lambda) + G\partial \\ \partial(G) + \lambda\partial & 0 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} G & 0 \\ 0 & H \end{pmatrix},\ \bar{f}_0 = (c_0, b_0)^T.$$

Problem (18) coincides with the problem on the potential solution for a plate non-uniform thickness [2, 29].

At the next stage, the boundary value problem is obtained for determining $\bar{f}_1 = (c_1, b_1)^T$ and β_1 :

$$\begin{cases} t(\beta_0)\bar{f}_1 = \left(\rho A_0 + A_3 - \beta_0 A_4 - \beta_0^2 \rho\cdot A_2 - 2\beta_1(\beta_0 A_2 + A_5)\right)\bar{f}_0, \\ \bar{f}_1\big|_{\rho=\pm1} = \bar{0}, \end{cases} \tag{19}$$

where

$$t(\beta_0)\bar{f}_1 = (A_0 + \beta_0 A_1 + \beta_0^2 A_2)\bar{f}_1,\quad A_3 = \begin{pmatrix} \lambda\partial - \partial(\lambda) & 0 \\ 0 & 0 \end{pmatrix},\quad A_4 = \begin{pmatrix} 0 & 0 \\ (G+\lambda) & 0 \end{pmatrix},\quad A_5 = \begin{pmatrix} 0 & 0 \\ \partial(G) + \lambda\partial & 0 \end{pmatrix},\quad \bar{f}_1 = (c_1, b_1)^T.$$

The condition for the solvability of (19) is orthogonality of the right hand side of the solution of the adjoint problem

$$A^*(\beta_0)\bar{f}_0^* = A(-\bar{\beta}_0)\bar{f}_0^* = \bar{0},$$

with $\bar{f}_0^* = (\bar{c}_0^*, \bar{b}_0^*)^T$. From the solvability condition for β_1

$$\beta_1 = \frac{E_1}{E_2},$$

$$E_1 = \int_{-1}^{1} \left[\lambda c_0' \bar{c}_0^* - \beta_0^2 G \rho c_0 \bar{c}_0^* + \rho (G b_0')' \cdot \bar{b}_0^* - \beta_0 (G+\lambda) c_0 \bar{b}_0^* - \beta_0^2 \rho H b_0 \bar{b}_0^* - \rho H c_0' \left(\bar{c}_0^*\right)' - H c_0' \bar{c}_0^* + \lambda c_0 \left(\bar{c}_0^*\right)' \right] d\rho,$$

$$E_2 = 2\int_{-1}^{1} [\beta_0 (G c_0 \bar{c}_0^* + H b_0 \bar{b}_0^*) + \lambda c_0' \bar{b}_0^* - G c_0 (\bar{b}_0^*)'] d\rho$$

Note that among $\{\beta_{0k}\}$ there are not pure imaginary quantities [2, 29].

The expression of functions is used $c_{0k}(\rho), b_{0k}(\rho)$ in terms of the Papkovich functions for the plate with inhomogeneous thickness [2, 28, 29]

$$c_{0k} = -\beta_{0k}^{-3} \left(p_0 \psi_k''\right)' - 2\beta_{0k}^{-1} p_1 \psi_k' + \beta_{0k}^{-1} \left(p_2 \psi_k\right)',$$

$$b_{0k} = p_0 \beta_{0k}^{-2} \psi_k'' - p_2 \psi_k,$$

$$p_0 = \frac{H}{4G(G+\lambda)},\ p_1 = \frac{1}{2G},\ p_2 = \frac{\lambda}{4G(G+\lambda)}.$$

Here, $\psi_k(\rho)$ is a solution to the generalized Papkovich spectral problem for the inhomogeneous case [2, 28, 29]:

$$\begin{cases} (p_0 \psi_k'')'' + \beta_{0k}^2 \left[2(p_1 \psi_k')' - (p_2 \psi_k)'' - p_2 \psi_k'' \right] + \beta_{0k}^4 p_0 \psi_k = 0, \\ (p_0 \psi_k'' - \beta_{0k}^2 p_2 \psi_k)\big|_{\rho=\pm 1} = 0, \\ \left[(p_0 \psi_k'')' + 2\beta_{0k}^2 p_1 \psi_k' - \beta_{0k}^2 (p_2 \psi_k)' \right]\big|_{\rho=\pm 1} = 0. \end{cases} \tag{20}$$

The solutions corresponding to the third iteration process have the form:

$$u_\rho(\rho,\xi) = \sum_{k=1}^{\infty} u_k(\rho) m_k'(\xi),\ \ u_\xi(\rho,\xi) = \sum_{k=1}^{\infty} w_k(\rho) m_k(\xi), \tag{21}$$

$$\begin{aligned} \sigma_{\rho\xi} &= \sum_{k=1}^{\infty} \sigma_{1k}(\rho) m_k(\xi), \quad \sigma_{\xi\xi} = \sum_{k=1}^{\infty} \sigma_{2k}(\rho) m_k'(\xi), \\ \sigma_{\rho\rho} &= \sum_{k=1}^{\infty} \sigma_{3k}(\rho) m_k'(\xi), \quad \sigma_{\phi\phi} = \sum_{k=1}^{\infty} \sigma_{4k}(\rho) m_k'(\xi), \end{aligned} \tag{22}$$

where

$$u_k(\rho) = \varepsilon^2 \left[-\beta_{0k}^{-4} \left(p_0 \psi_k''\right)' - 2\beta_{0k}^{-2} p_1 \psi_k' + \beta_{0k}^{-2} \left(p_2 \psi_k\right)' + O(\varepsilon) \right],$$

$$w_k(\rho) = \varepsilon \left(p_0 \beta_{0k}^{-2} \psi_k'' - p_2 \psi_k + O(\varepsilon) \right),$$

$$\sigma_{1k}(\rho) = -\psi_k' + O(\varepsilon), \qquad \sigma_{2k}(\rho) = \varepsilon \left(\beta_{0k}^{-2} \psi_k'' + O(\varepsilon) \right),$$

$$\sigma_{3k}(\rho) = \varepsilon \left(\psi_k + O(\varepsilon) \right), \qquad \sigma_{4k}(\rho) = \varepsilon \lambda (p_0 - p_2) \left(\psi_k + \beta_{0k}^{-2} \psi_k'' + O(\varepsilon) \right),$$

$$m_k(\xi) = C_k ch(\mu_k \xi) + D_k sh(\mu_k \xi),$$

C_k and D_k are arbitrary constants. Solution of (21), (22) have the character of a boundary layer equivalent to the Saint-Venant boundary effect on the theory of inhomogeneous plates [2, 29]. The stresses determined by the third iterative process are localized at the ends of the cylinder and decrease exponentially with distance from the ends. The stress damping coefficient is of order $O(\varepsilon^{-1})$ with respect to ε.

To determine the unknown constants C_k, D_k $(k = 1,2,...)$ we use the Lagrange variational principle. In the considered case this principle takes the following form [3, 4]:

$$\sum_{s=1}^{2}\int_{-1}^{1}\left[(\sigma_{\rho\xi}-f_{1s}(\rho))\delta u_{\rho}+(\sigma_{\xi\xi}-f_{2s}(\rho))\delta u_{\xi}\right]\Big|_{\xi=\pm l_0}\cdot e^{2\varepsilon\rho}d\rho=0. \tag{23}$$

Substituting (21) and (22) into (23) and assuming $\delta C_k,\delta D_k$ variable variations, from (23) we obtain the following infinite system of the linear algebraic equation

$$\sum_{k=1}^{\infty}g'_{jk}C_{k0}=\tau'_j,\quad (j=1,2,...) \tag{24}$$

$$\sum_{k=1}^{\infty}g''_{jk}D_{k0}=\tau''_j,\quad (j=1,2,...) \tag{25}$$

where

$$g'_{jk}=\beta_{0k}^{-1}sh\left(\left(\frac{\beta_{0k}}{\varepsilon}+\beta_{1k}\right)l_0\right)\cdot sh\left(\left(\frac{\beta_{0j}}{\varepsilon}+\beta_{1j}\right)l_0\right)\int_{-1}^{1}\psi''_k\cdot\left(p_0\beta_{0j}^{-2}\psi''_j-p_2\psi_j\right)d\rho+$$
$$+\beta_{0j}\cdot ch\left(\left(\frac{\beta_{0j}}{\varepsilon}+\beta_{1j}\right)l_0\right)\cdot ch\left(\left(\frac{\beta_{0k}}{\varepsilon}+\beta_{1k}\right)l_0\right)\cdot\int_{-1}^{1}\left[\beta_{0j}^{-4}\left(p_0\psi''_j\right)'+2\beta_{0j}^{-2}p_1\psi'_j-\beta_{0j}^{-2}\left(p_2\psi_j\right)'\right]\psi'_k d\rho,$$

$$\tau'_j=\frac{1}{2}sh\left(\left(\frac{\beta_{0j}}{\varepsilon}+\beta_{1j}\right)l_0\right)\cdot\int_{-1}^{1}(f_{22}-f_{21})\left(p_0\beta_{0j}^{-2}\psi''_j-p_2\psi_j\right)d\rho+\frac{1}{2}\beta_{0j}ch\left(\left(\frac{\beta_{0j}}{\varepsilon}+\beta_{1j}\right)l_0\right)\times\int_{-1}^{1}(f_{11}+f_{12})\left[-\beta_{0j}^{-4}\left(p_0\psi''_j\right)'-2\beta_{0j}^{-2}p_1\psi'_j+\beta_{0j}^{-2}\left(p_2\psi_j\right)'\right]d\rho,$$

$$g''_{jk}=\beta_{0k}^{-1}ch\left(\left(\frac{\beta_{0k}}{\varepsilon}+\beta_{1k}\right)l_0\right)\cdot ch\left(\left(\frac{\beta_{0j}}{\varepsilon}+\beta_{1j}\right)l_0\right)\cdot\int_{-1}^{1}\psi''_k\left(p_0\beta_{0j}^{-2}\psi''_j-p_2\psi_j\right)d\rho+$$
$$+\beta_{0j}\cdot sh\left(\left(\frac{\beta_{0j}}{\varepsilon}+\beta_{1j}\right)l_0\right)\cdot sh\left(\left(\frac{\beta_{0k}}{\varepsilon}+\beta_{1k}\right)l_0\right)\cdot\int_{-1}^{1}\left[\beta_{0j}^{-4}\left(p_0\psi''_j\right)'+2\beta_{0j}^{-2}p_1\psi'_j-\beta_{0j}^{-2}\left(p_2\psi_j\right)'\right]\psi'_k d\rho,$$

$$\tau''_j=\frac{1}{2}\cdot ch\left(\left(\frac{\beta_{0j}}{\varepsilon}+\beta_{1j}\right)l_0\right)\cdot\int_{-1}^{1}(f_{21}+f_{22})\left(p_0\beta_{0j}^{-2}\psi''_j-p_2\psi_j\right)d\rho+\frac{1}{2}\beta_{0j}sh\left(\left(\frac{\beta_{0j}}{\varepsilon}+\beta_{1j}\right)l_0\right)\times\int_{-1}^{1}(f_{12}-f_{11})\left[-\beta_{0j}^{-4}\left(p_0\psi''_j\right)'-2\beta_{0j}^{-2}p_1\psi'_j+\beta_{0j}^{-2}\left(p_2\psi_j\right)'\right]d\rho,$$

$$C_k=C_{k0}+\varepsilon C_{k1}+...\quad,\quad D_k=D_{k0}+\varepsilon D_{k1}+...$$

The definition of the constants C_{kp},D_{kp} $(p=1,2,...)$ is invariably reduced to the systems whose matrices coincide with the matrices of systems (24), (25).

The system of the infinite linear algebraic equations (24), (25) is always solvable under physically well-posed conditions imposed on the right-hand side [3, 4]. The solvability and convergence of the reduction method for (24), (25) was proved in [30].

4. Torsion Problem Analysis

Consider the torsion problem for the radially inhomogeneous cylinder of small thickness, i.e. problem was studied (9), (11) and (13).

The solution of (9) is sought in the form:

$$u_{\phi}(\rho,\xi)=\upsilon(\rho)m(\xi), \tag{26}$$

where $m''(\xi)-\alpha^2m(\xi)=0$.

Substituting (26) into (9), (11),

$$[G(\rho)(\upsilon'(\rho)-\varepsilon\upsilon(\rho))]'+\varepsilon G(\rho)(\upsilon'(\rho)-\varepsilon\upsilon(\rho))+\alpha^2\varepsilon^2G(\rho)e^{2\varepsilon\rho}\upsilon(\rho)=0, \tag{27}$$

$$\upsilon(\rho)=0\ \text{ at }\ \rho=\pm1 \tag{28}$$

Eqs. (27) and (28) in the form are presented as

$$A\upsilon=\alpha^2\upsilon, \tag{29}$$

where

$$Av = \left\{ -\frac{1}{\varepsilon^2 G(\rho) e^{2\varepsilon\rho}} [G(\rho)(v'(\rho) - \varepsilon v(\rho))]' - \frac{1}{\varepsilon e^{2\varepsilon\rho}} (v'(\rho) - \varepsilon v(\rho)); v(\rho)\big|_{\rho=\pm 1} = 0 \right\}.$$

The Hilbert space H with the scalar product was introduced as

$$(v,t) = \int_{-1}^{1} G(\rho) v(\rho) t(\rho) e^{2\varepsilon\rho} d\rho .$$

Let $A: H \to H$ be a symmetrical operator. For the arbitrary function $v(x) \in D_A$, $t(x) \in D_A$, we have

$$(Av,t) - (v,At) = \int_{-1}^{1} G(\rho) e^{2\varepsilon\rho} (tAv - vAt) d\rho = \frac{1}{\varepsilon^2} \int_{-1}^{1} [(G(\rho)(t'(\rho) - \varepsilon t(\rho))' v(\rho) - (G(\rho)(v'(\rho) - \varepsilon v(\rho)))' t(\rho)] d\rho$$
$$+ \frac{1}{\varepsilon} \int_{-1}^{1} G(\rho)(v(\rho) t'(\rho) - t(\rho) v'(\rho)) d\rho. \tag{30}$$

Integrating by parts and considering boundary conditions (28), from (30)

$$(Av,t) - (v,At) = 0 \quad \text{i.e.} \quad (Av,t) = (v,At) .$$

Note that

$$(Av,v)_H = \int_{-1}^{1} G(\rho) e^{2\varepsilon\rho} Av \cdot v(\rho) d\rho$$
$$= \int_{-1}^{1} G(\rho) e^{2\varepsilon\rho} \left[-\frac{1}{\varepsilon^2 G(\rho) e^{2\varepsilon\rho}} (G(v' - \varepsilon v))' - \frac{1}{\varepsilon e^{2\varepsilon\rho}} (v' - \varepsilon v) \right] v d\rho$$
$$= -\frac{1}{\varepsilon^2} \int_{-1}^{1} [(G(v' - \varepsilon v))' + \varepsilon G(\rho)(v' - \varepsilon v)] v(\rho) d\rho$$
$$= -\frac{1}{\varepsilon^2} \left[\int_{-1}^{1} (G(v' - \varepsilon v))' v(\rho) d\rho + \varepsilon \int_{-1}^{1} G(v' - \varepsilon v) v(\rho) d\rho \right] \tag{31}$$
$$= -\frac{1}{\varepsilon^2} \left[v(\rho) G(v' - \varepsilon v) \Big|_{-1}^{1} - \int_{-1}^{1} G(v' - \varepsilon v) v'(\rho) d\rho + \varepsilon \int_{-1}^{1} G(v' - \varepsilon v) v d\rho \right]$$
$$= \frac{1}{\varepsilon^2} \int_{-1}^{1} G(v' - \varepsilon v)^2 d\rho \geq 0 .$$

Let $(Av,v)_H = 0$. Then

$$\int_{-1}^{1} G(\rho)(v'(\rho) - \varepsilon v(\rho))^2 d\rho = 0 . \tag{32}$$

In (32), the integrand is nonnegative. From the vanishing of the integral, we have

$$G(\rho)(v'(\rho) - \varepsilon v(\rho))^2 = 0 \quad \text{i.e.} \quad v(\rho) = C_0 e^{\varepsilon\rho} .$$

By virtue of boundary conditions (28), it was obtained that $v(\rho) = 0$.

Thus the identity $(Av,v)_H = 0$ is satisfied only if $v(\rho) = 0$, i.e. $A: H \to H$ is a positive operator. Eigenvalues $\lambda_k(A) = \alpha_k^2$ are positive and $\lambda_k(A) \to \infty$ at $k \to \infty$. The set of the eigenfunctions $\{v_k(\rho)\}$ forms an orthogonal basis in the space H [31]:

$$(v_s, v_k) = d_s \delta_{sk} ,$$
$$d_s = (v_s, v_s)_H = \int_{-1}^{1} G(\rho) v_s^2(\rho) e^{2\varepsilon\rho} d\rho . \tag{33}$$

To solve (27), (28) the method of asymptotic integration is used. The first iterative process corresponds to the trivial solutions. The second iterative process is missing. According to the third iteration process, the solution for (27), (28) is in the form

$$v = v_0 + \varepsilon v_1 + ..., \quad \alpha = \varepsilon^{-1}(\alpha_0 + \varepsilon\alpha_1 + ...) . \tag{34}$$

After substituting (34) into (27), (28) for the first terms of expansion (34), it was obtained that

$$Bv_0 = \alpha_0^2 v_0 , \tag{35}$$

where

$$Bv_0 = \left\{ -\frac{1}{G(\rho)}\left(G(\rho)v_0^{'}(\rho)\right)', v_0\Big|_{\rho=\pm1} = 0 \right\}.$$

Equation (35) coincides with the problem for the vortex solution of the inhomogeneous plate [2]. To determine α_1 and v_1 , we have

$$\left(Gv_1^{'}\right)' + \alpha_0^2 Gv_1 = (Gv_0)' - Gv_0^{'} - 2\alpha_0\alpha_1 Gv_0 - 2\alpha_0^2\rho Gv_0 \tag{36}$$

$$v_1\big|_{\rho=\pm1} = 0 \tag{37}$$

The solution for (36), (37) is in the form

$$v_{1n}(\rho) = \sum_{k=0}^{\infty} \gamma_{nk} v_{0k}(\rho) \tag{38}$$

Obviously, (38) satisfies boundary conditions (37). Substituting (38) into (36), we have

$$\alpha_{1n} = -\frac{1}{\alpha_{0n}}\left[\int_{-1}^{1} G(\rho)v_{0n}v_{0n}^{'}d\rho + \alpha_{0n}^2\int_{-1}^{1}\rho G(\rho)v_{0n}^2(\rho)d\rho\right],$$

$$\gamma_{nk} = \frac{1}{\alpha_{0k}^2 - \alpha_{0n}^2}\left[\int_{-1}^{1} G(\rho)v_{0n}v_{0k}^{'}d\rho + \int_{-1}^{1} G(\rho)v_{0k}v_{0n}^{'}d\rho + 2\alpha_{0n}^2\int_{-1}^{1}\rho G(\rho)v_{0n}\cdot v_{0k}d\rho\right] \quad ,(k \neq n)$$

$$\gamma_{nn} = -\int_{-1}^{1}\rho G(\rho)v_{0n}^2(\rho)d\rho \ .$$

So, the solutions corresponding to the third iteration process have the form:

$$u_\phi(\rho,\phi) = \sum_{n=1}^{\infty} v_n(\rho)m_n(\xi), \tag{39}$$

where

$$v_n(\rho) = v_{0n}(\rho) + \varepsilon\left\{-v_{0n}(\rho)\int_{-1}^{1}\rho G(\rho)v_{0n}^2(\rho)d\rho + \sum_{\substack{k=0\\k\neq n}}^{\infty}\frac{1}{\alpha_{0k}^2-\alpha_{0n}^2}\left[\int_{-1}^{1}G(\rho)(v_{0n}v_{0k}^{'} + v_{0k}v_{0n}^{'})d\rho + 2\alpha_{0n}^2\int_{-1}^{1}\rho G(\rho)v_{0n}v_{0k}d\rho\right]v_{0k}\right\} + O(\varepsilon^2),$$

$$m_n(\xi) = E_n e^{-\alpha_n\xi} + F_n e^{\alpha_n\xi}$$

E_n and F_n are arbitrary constants. The stresses $\sigma_{\rho\phi}, \sigma_{\phi\xi}$ are of the order $O(\varepsilon^{-1})$. The third asymptotic process determines solution (39), which has the character of a boundary layer. For $\sigma_{\phi\xi}$, we have

$$\sigma_{\phi\xi} = \sum_{n=1}^{\infty} G(\rho)v_n(\rho)m_n^{'}(\xi). \tag{40}$$

Substituting (40) into (13) was gotten

$$\sum_{n=1}^{\infty} G(\rho)v_n(\rho)m_n^{'}(\xi) = f_3^{\pm}(\rho). \tag{41}$$

Multiplying (41) by $v_k(\rho)e^{2\varepsilon\rho}$ and integration on the interval [-1; 1], using (33) we obtain the system with respect to E_n and F_n :

$$\left(\alpha_n e^{\alpha_n\xi}F_n - \alpha_n e^{-\alpha_n\xi}E_n\right)\Big|_{\xi=\pm l_0} = h_n^{\pm}, \tag{42}$$

where

$$h_n^{\pm} = \int_{-1}^{1} f_3^{\pm}(\rho)v_n(\rho)e^{2\varepsilon\rho}d\rho.$$

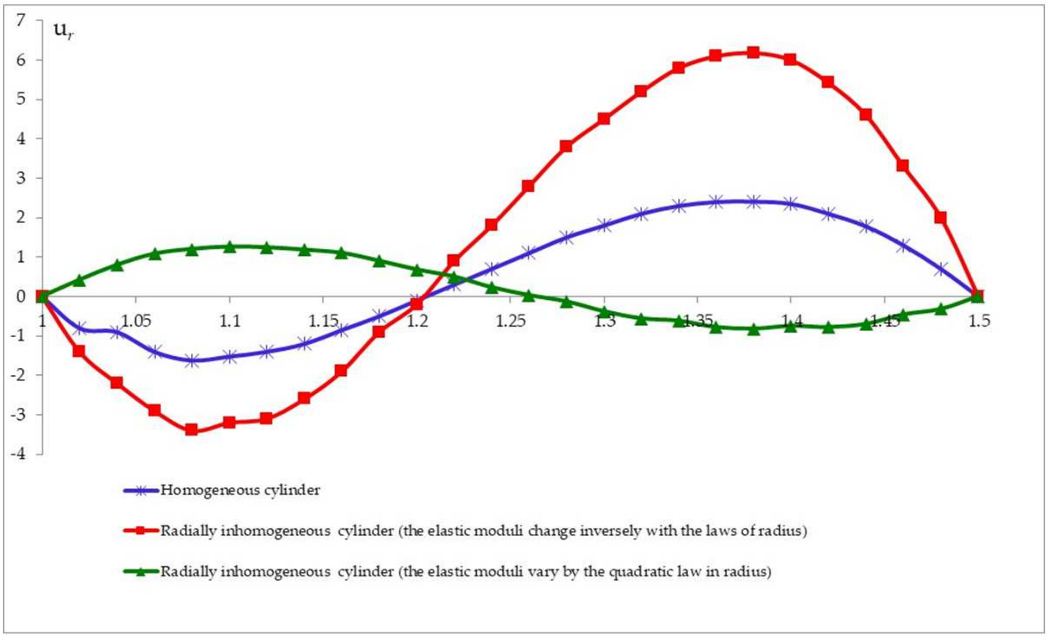

Figure 2. Distributions of u_r over the thickness of the cylinder in the case of $\varepsilon = 0.2$.

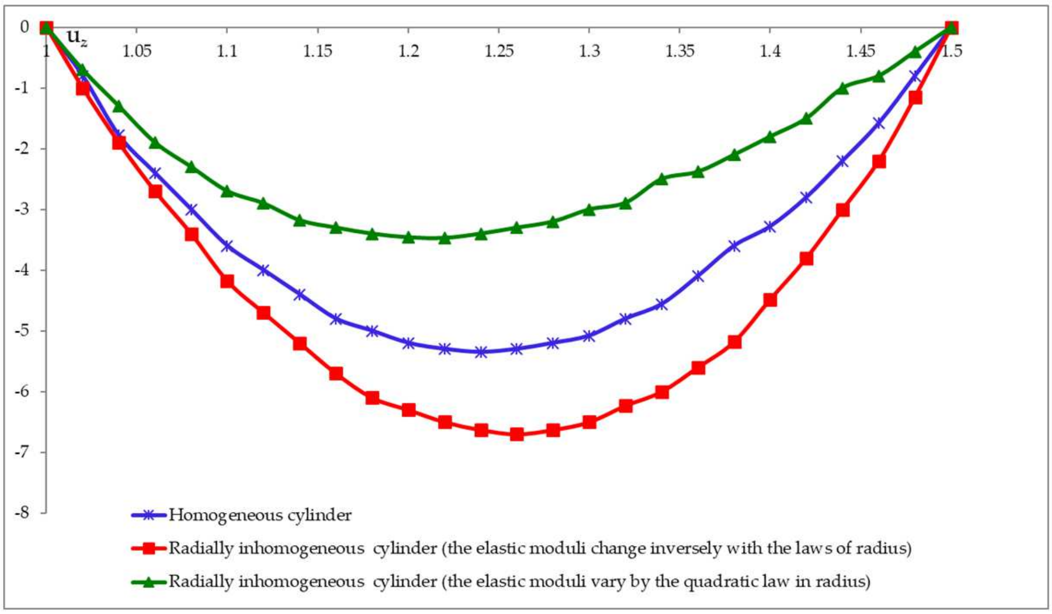

Figure 3. Distributions of u_z over the thickness of the cylinder in the case of $\varepsilon = 0.2$.

Solving (42) one can find unknown constants E_n and F_n :

$$E_n = \frac{h_n^+ e^{-\alpha_n l_0} - h_n^- e^{\alpha_n l_0}}{2\alpha_n sh(2\alpha_n l_0)}, \quad F_n = \frac{h_n^+ e^{\alpha_n l_0} - h_n^- e^{-\alpha_n l_0}}{2\alpha_n sh(2\alpha_n l_0)}.$$

5. Numerical Analysis

As an example the problem of the stress-strain state of the radially inhomogeneous and homogeneous cylinder of small thickness are considered.

It was assumed that the lateral part of the cylinder is fixed, and the boundary conditions are set at the ends of the cylinder:

$$\sigma_{rz} = Ar, \sigma_{zz} = A(2r^2 + 3r) \quad \text{at} \quad z = -1.5$$

$$\sigma_{rz} = Ar^2, \sigma_{zz} = A(r^2 + 4r) \quad \text{at} \quad z = 1.5.$$

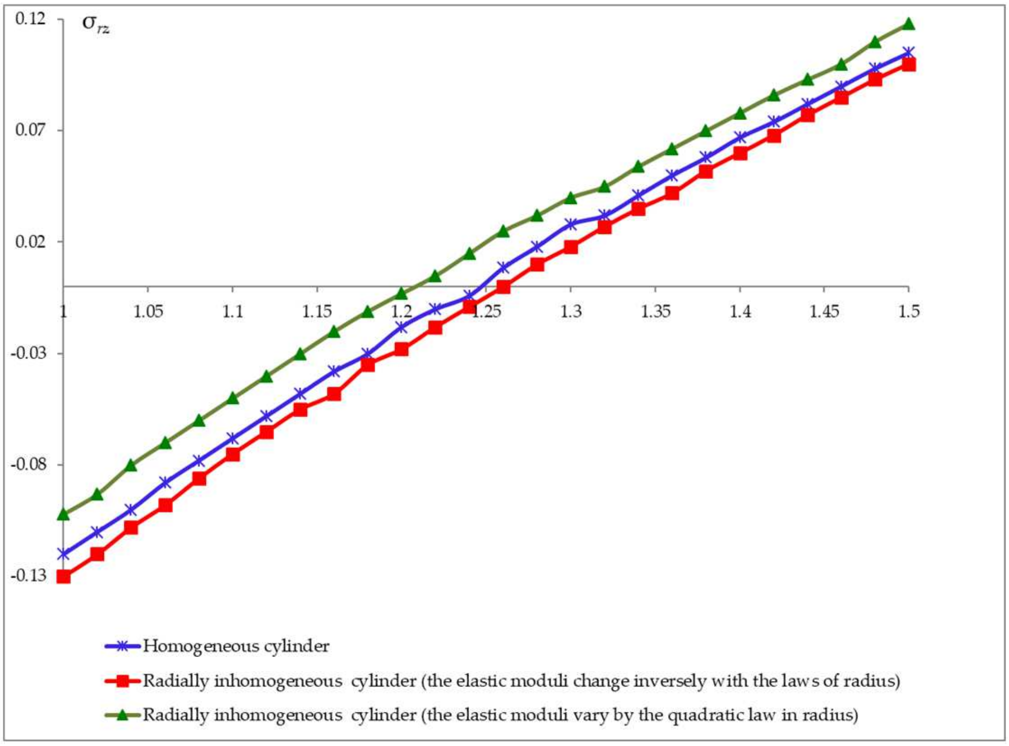

Figure 4. Distributions of σ_{rz} over the thickness of the cylinder in the case of $\varepsilon = 0.2$.

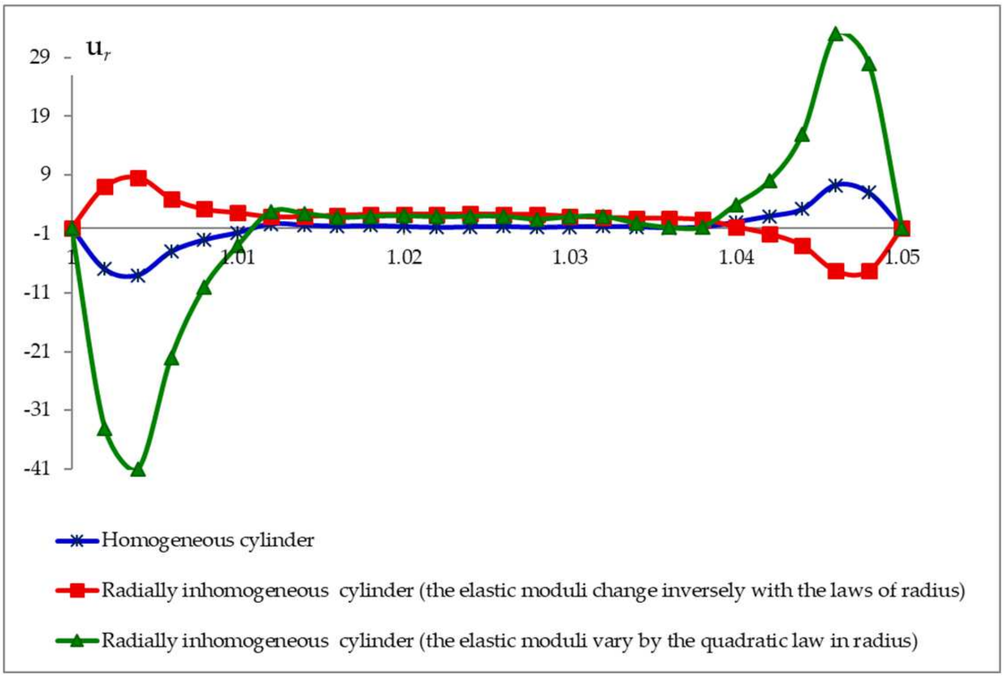

Figure 5. Distribution of u_r over the cylinder thickness for the case $\varepsilon = 0.02$.

The following two cases are studied:

1. The volume of the cylinder is $\Gamma = \{r \in [1;1.5],\ \phi \in [0;2\pi],\ z \in [-1.5;1.5]\}$.

The parameter characterizing the thickness of the cylinder is $\varepsilon = 0.2$.

Figures 2-4 show the thickness distributions (along the center line) of displacements u_r, u_z and stress σ_{rz} for the homogeneous and radially inhomogeneous cylinder.

For a radially inhomogeneous cylinder, the following cases are considered. The elastic moduli vary by the quadratic law in radius (increasing elastic moduli):

$$G = G_0 r^2, \lambda = \lambda_0 r^2.$$

The elastic moduli change inversely with the laws of radius (decreasing elastic moduli):

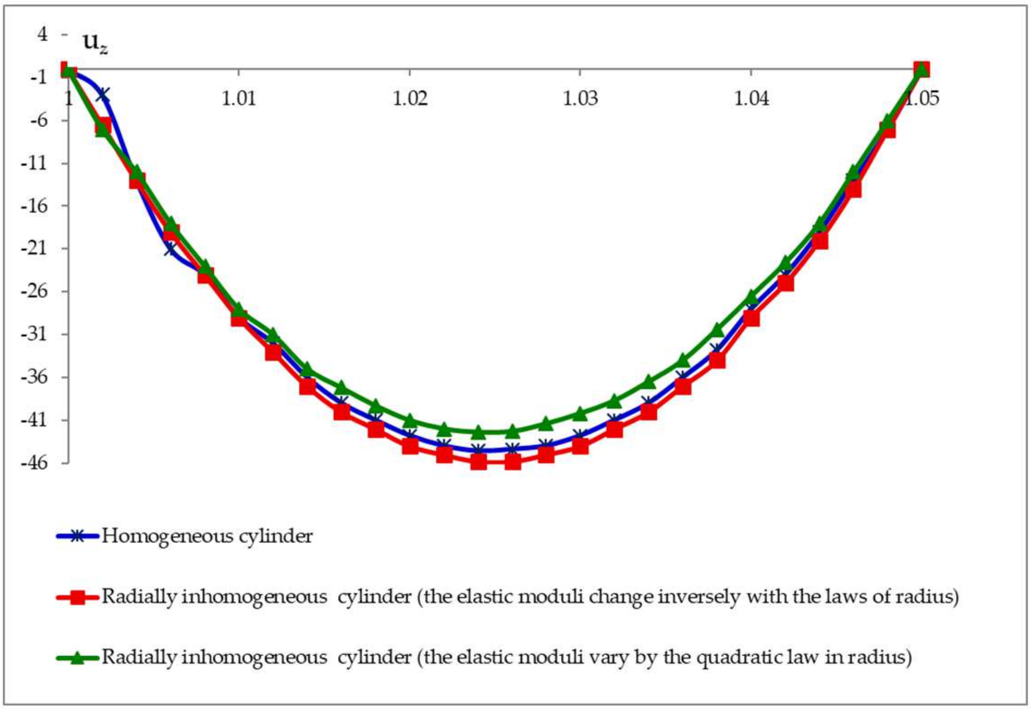

Figure 6. Distribution of u_z over the cylinder thickness for the case $\varepsilon = 0.02$.

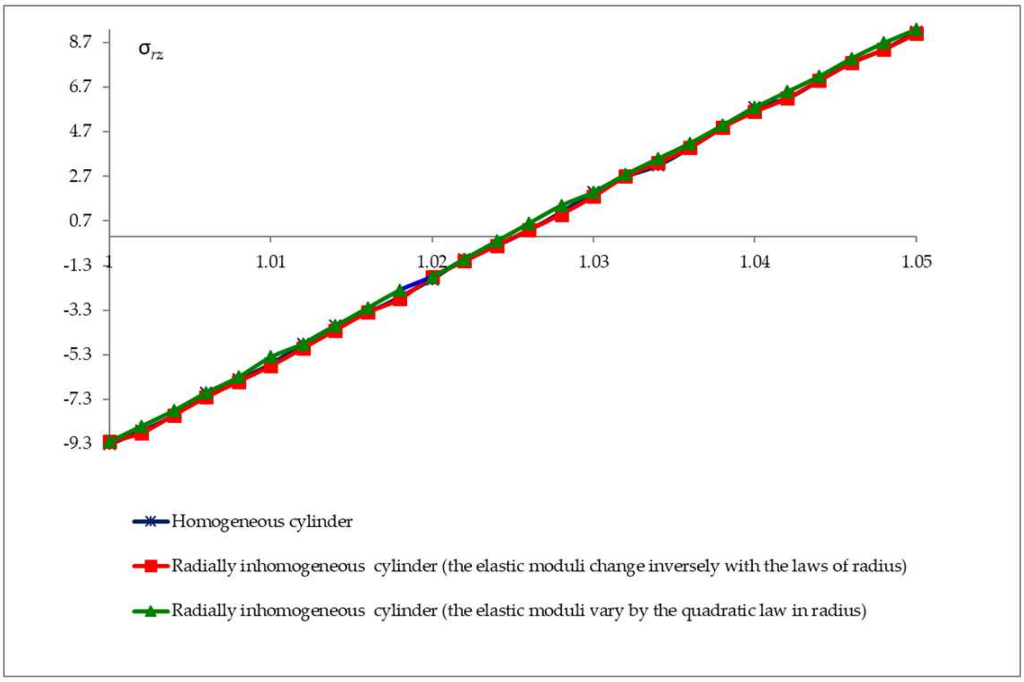

Figure 7. Distribution of σ_{rz} over the cylinder thickness for the case $\varepsilon = 0.02$.

$$G = \frac{G_0}{r}, \lambda = \frac{\lambda_0}{r}.$$

when the elastic moduli change by a quadratic law in radius at a distance of 0.1 from the inner surface, u_r takes on the greatest value. The distribution of u_r to the distance of 0.26 from the inner surface changes according to a quadratic law and the quadratic parabola is convex upward. At the distance of 0.26 from the inner surface, u_r changes almost according to the quadratic law and the parabola is convex down. At a distance of 0.12 from the outer surface u_r takes the smallest value.

Distributions of u_r in the thickness for the homogeneous and radially inhomogeneous cylinders, when the elastic moduli change inversely proportional to the radius law are qualitatively the same, but change quantitatively. The distribution of u_r in the thickness for the radially inhomogeneous cylinder, when the elastic moduli change by the quadratic law in radius, differs qualitatively from the homogeneous case (Fig. 2).

Distributions of u_z in the thickness for the homogeneous and radially inhomogeneous cylinder are qualitatively the same

(Fig. 3). The thickness distributions for σ_{rz} a homogeneous and radially inhomogeneous cylinder occur by the same laws (Fig. 4).

2. Consider the case when the volume of the cylinder is

$$\Gamma = \{r \in [1;1.05],\ \phi \in [0;2\pi],\ z \in [-1.5;1.5]\}.$$

The parameter characterizing the thickness of the cylinder is taken as $\varepsilon = 0.02$

Figures 5-7 show the distribution of u_r, u_z, σ_{rz} in thickness for a uniform and radially inhomogeneous cylinder $\varepsilon = 0.02$.

From $r = 1.012$ to $r = 1.038$ for the homogeneous and radially inhomogeneous cylinder, the distributions of u_r in thickness are close.

In the case of $\varepsilon = 0.02$ for the homogeneous and radially inhomogeneous cylinder, when the elasticity moduli change inversely proportional to the radius, the distributions of u_r within the thickness at the distance of 0.012 from the inner surface and at a distance of 0.012 from the outer one. Distributions of u_r within the thickness for the radially inhomogeneous cylinder, when the elasticity moduli change by the quadratic law in radius and for the homogeneous cylinder differ only quantitatively (Fig. 5).

Distributions of u_z in the radial direction for the homogeneous and radially inhomogeneous cylinder occur according to a quadratic law (Fig. 6).

The distributions of σ_{rz} in thickness for the homogeneous and radially inhomogeneous cylinders almost coincide (Fig. 7).

The results of the numerical analysis show that the change in radial displacement u_r over the thickness of the cylinder is significant and depends on the degree of heterogeneity of the material.

Distribution of u_z occurs according to a law close to quadratic and only quantitatively depends on the degree of heterogeneity of the material.

6. Discussion of the Results

Analysis of the stress-strain state of a radially inhomogeneous cylinder of small thickness basing on three-dimensional equations of the elasticity theory is reduced to the study of the boundary value problems for the systems of the linear differential equations of the second order in partial derivatives with variable coefficients. These coefficients include elasticity moduli, which are arbitrary positive continuous functions of radius, significantly complicating the construction of solutions to boundary value problems. In this general case, it is impossible to construct exact solutions of the boundary value problem under study. Considering that the formulated boundary value problems include a small parameter characterizing the thickness of the cylinder, we use the method of asymptotic integration of the equations of the elasticity theory to construct a solution. This is an effective method for studying the three-dimensional stress state of inhomogeneous bodies of finite dimensions.

In this paper the classical problem of the mathematical theory of elasticity for the radially inhomogeneous isotropic cylinder of small thickness is studied. The elasticity moduli are taken arbitrary positive continuous functions along the radius, the values of which vary within the same order. The axially symmetric problem splits into two independent problems. Firs, is the problem in which the sought quantities are $u_\rho(\rho,\xi)$, $u_\xi(\rho,\xi)$ and second is the torsion problem, the sought quantities of which are $u_\phi(\rho,\xi)$. There are two separate and independent of each other problems. After separating the variables, the boundary value problems are reduced to the spectral problems. Using the method of asymptotic integration, solutions of the spectral problems are constructed as the cylinder thickness parameter tends to zero.

Based on the asymptotic analysis, it was found that there are no solutions defining the internal stress-strain state of the radially inhomogeneous cylinder with a fixed lateral surface. Solutions are determined that have the character of a boundary layer and are localized at the ends of the cylinder. The first terms of its asymptotic expansion coincide with the Saint-Venant boundary effect on the theory of inhomogeneous plates.

Using the variational Lagrange principle, the question of satisfying the boundary conditions at the ends of the cylinder is investigated.

Further, the torsion problem was studied by the asymptotic method and it was found that the stress-strain state is formed only by the solution that has the character of a boundary layer.

7. Conclusions

The following concluding remarks can be extracted from the illustrated results of this research:

1) Features of the stress-strain state of the radially inhomogeneous cylinder of small thickness are disclosed. Based on the asymptotic analysis, it was found that when the lateral surface of the radially inhomogeneous cylinder is fixed, the stress-strain state is formed only by the solution that has the character of a boundary layer equivalent to the Saint-Venant boundary effect on the theory of inhomogeneous plates.
2) Asymptotic formulas for displacements and stresses are obtained that allow one to calculate a three-dimensional stress-strain state of the radially inhomogeneous cylinder of small thickness under various boundary conditions at the ends of the cylinder.
3) A numerical analysis is carried out, on the basis of which characteristic conclusions are made.

Conflict of Interest

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Nomenclature

$\sigma_{rr}, \sigma_{rz}, \sigma_{\phi\phi}, \sigma_{zz}, \sigma_{r\phi}, \sigma_{\phi z}$	Stress tensor components	ρ, ξ	New dimensionless variables
u_r, u_ϕ, u_z	Displacement vector components	ε	Small parameter characterizing the thickness of the cylinder
G, λ	Elastic moduli	μ, α	Spectral parameters
r, ϕ, z	Cylindrical coordinates	$\psi_k(\rho)$	Solutions of the generalized spectral Papkovich problem

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